IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Friday 7 May 2004 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Write the make and model of your calculator on the appropriate box on your cover sheet *e.g.* Casio *fx-9750G,* Sharp EL-9600, Texas Instruments TI-85.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 15]*

The points A and B have the position vectors $\begin{pmatrix} 4 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ respectively.

(a) (i) Find the vector $\overrightarrow{\text{AB}}$.

(ii) Find $|\overrightarrow{\text{AB}}|$. *[4 marks]*

The point D has position vector $\begin{pmatrix} d \\ 22 \end{pmatrix}$.

(b) Find the vector $\overrightarrow{\text{AD}}$ in terms of *d*. *[2 marks]*

The angle $\text{B}\hat{\text{A}}\text{D}$ is 90°.

(c) (i) Show that $d = 9$.

(ii) Write down the position vector of the point D. *[3 marks]*

The quadrilateral ABCD is a rectangle.

(d) Find the position vector of the point C. *[4 marks]*

(e) Find the area of the rectangle ABCD. *[2 marks]*

2. *[Maximum mark: 14]*

The derivative of the function f is given by $f'(x) = \mathrm{e}^x + x - 5$. The point $(1, \mathrm{e} - 2)$ lies on the graph of $f(x)$.

(a) Show that $f(x) = \mathrm{e}^x + \frac{1}{2}x^2 - 5x + 2.5$. *[7 marks]*

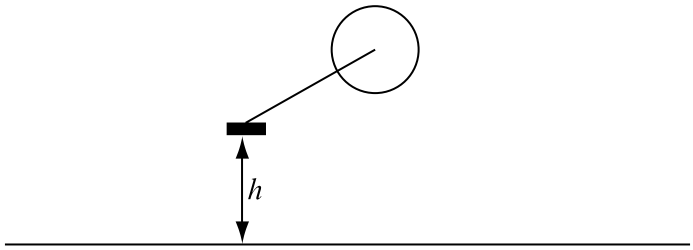
(b) Sketch the graph of $y = f(x)$ for $-3 < x < 3$. *[2 marks]*

(c) Find the minimum value of $f(x)$. *[2 marks]*

(d) Find the area enclosed by the graph $y = f(x)$, the axes and the line $x = 2$. *[3 marks]*

3. *[Maximum mark: 10]*

The diagram below shows a bicycle pedal.



The height h cm, of the bicycle pedal above the ground after t seconds is given by $h = 24 - 14\sin 2t$.

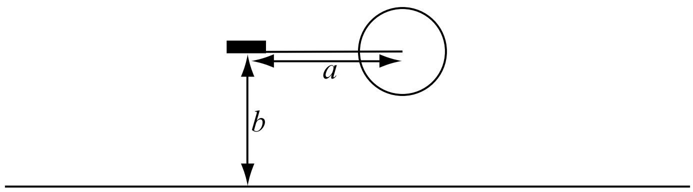
(a) Find the height when $t = 0$. *[2 marks]*

(b) Find the maximum height of the pedal above the ground. *[1 mark]*

(c) Find the first time at which this occurs. *[1 mark]*

(d) How long does one revolution of the pedal take? *[2 marks]*

The diagram shows the pedal in its starting position.

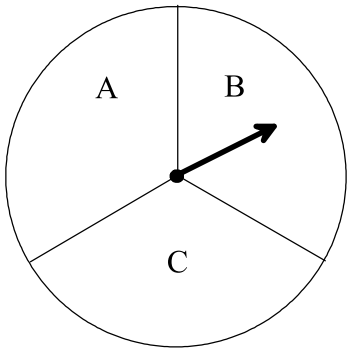


(e) Write down the lengths a and b. *[2 marks]*

(f) Write down a formula for the height of the other pedal above the ground at time t seconds. *[2 marks]*

4. *[Maximum mark: 16]*

(i) A disc is divided into three equal sectors A, B and C as shown in the diagram. The arrow is spun. It cannot land on the lines between the sectors.



The arrow is spun twice. Each time, the letter of the sector that the arrow points to is recorded.

(a) List the sample space. *[1 mark]*

(b) Calculate the probability that

(i) both letters are the same;

(ii) at least one letter is an A;

(iii) at least one letter is an A and both letters are the same;

(iv) at least one letter is an A or both letters are the same. *[4 marks]*

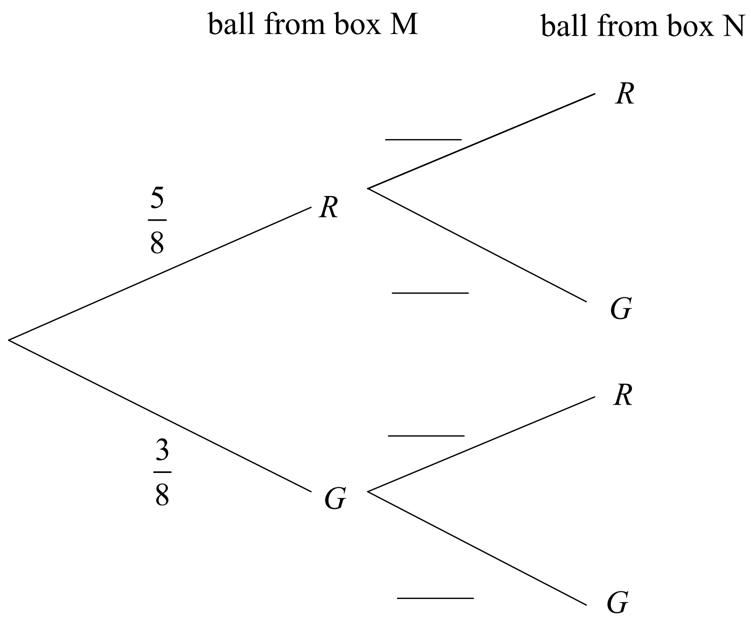
(This question continues on the following page)

(Question 4 continued)

(ii) Two boxes M and N contain red (R) and green (G) balls.
Box M contains five red balls and three green balls.
Box N contains four red balls and six green balls.

A ball is taken at random from box M and moved into box N. A ball is then taken at random from box N.

(a) Copy and complete the tree diagram.



[4 marks]

(b) Calculate the probability that the ball taken from box N is green. *[3 marks]*

(c) Given that the ball taken from box N is green, find the probability that the ball taken from box M is red. *[4 marks]*

5. *[Maximum mark: 15]*

There were 1420 doctors working in a city on 1 January 1994. After n years the number of doctors, D, working in the city is given by

$$D = 1420 + 100n \, .$$

(a) (i) How many doctors were there working in the city at the start of 2004?

(ii) In what year were there first more than 2000 doctors working in the city? *[3 marks]*

At the beginning of 1994 the city had a population of 1.2 million. After n years, the population, P, of the city is given by

$$P = 1\,200\,000\,(1.025)^n \, .$$

(b) (i) Find the population P at the beginning of 2004.

(ii) Calculate the percentage growth in population between 1 January 1994 and 1 January 2004.

(iii) In what year will the population first become greater than 2 million? *[7 marks]*

(c) (i) What was the average number of people per doctor at the beginning of 1994?

(ii) After how many **complete** years will the number of people per doctor first fall below 600? *[5 marks]*

SECTION B

*Answer **one** question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) The volume of liquid in a can is normally distributed with a mean of 379 ml and a standard deviation of d ml.

(a) If $d = 2.7$, find the probability that a can will contain less than 375 ml. *[3 marks]*

(b) When $d = 3.6$, the volume of liquid in 90 % of the cans exceeds x ml. Find the value of x, giving your answer to one decimal place. *[3 marks]*

(c) The proportion of cans in which the volume of liquid is less than 370 ml is 0.01. Find the value of d. *[4 marks]*

(ii) Each day, a factory recorded the number (x) of boxes it produces and the total production cost (y) dollars. The results for nine days are shown in the following table.

x	28	45	60	48	51	33	67	40	56
y	460	580	770	600	640	490	830	570	730

(a) Write down the equation of the least squares regression line of y on x. *[3 marks]*

(b) In this situation, interpret the meaning of

(i) the gradient;

(ii) the y-intercept. *[2 marks]*

(c) Use the equation found in part (a) to answer the following.

(i) Estimate the cost of producing 55 boxes.

(ii) The factory sells the boxes for \$ 13.20 each. Find the least number of boxes which the factory should produce in one day in order to make a profit. *[6 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) In a forest, the circumference of trees at their base is normally distributed with mean 70 cm and standard deviation 8 cm.

(a) The circumferences of a sample of 30 trees are measured. Find the probability that the mean circumference is greater than 72 cm. *[3 marks]*

(b) Another sample is to be selected so that the probability that its mean lies between 68 and 72 is greater than 0.95. Find the minimum size of this sample. *[6 marks]*

Further Calculus

7. *[Maximum mark: 30]*

(i) The function g is defined as $g(x) = \frac{e^x}{x}, -2 \le x \le 3$.

(a) Sketch the graph of $y = g(x)$, with $-4 \le y \le 6$. *[2 marks]*

(b) (i) Find $g'(x)$.

(ii) Hence calculate the exact values of the coordinates of the minimum point on the graph of g. *[6 marks]*

(c) The second derivative of g is $g''(x) = \frac{e^x(x^2 - 2x + 2)}{x^3}$.

Use this result to explain why the graph of g can have no inflexion point. *[4 marks]*

(ii) ***Note: Radians are used throughout this part of the question.***

Consider the function $f(x) = \sin x + 2x - 1$.

(a) Use the Newton-Raphson method to solve $f(x) = 0$, starting with $x_0 = 1$. Give your answer to seven significant figures. *[3 marks]*

(b) The equation $f(x) = 0$ may also be solved using fixed-point iteration, with $x_{n+1} = \frac{1 - \sin x_n}{2}$.

(i) Starting with $x_0 = 1$, write down the value of x_1 and of x_2 giving your answers to seven decimal places.

(ii) Explain why this iteration will always converge, no matter what starting value is chosen. *[5 marks]*

(This question continues on the following page)

(Question 7 continued)

(iii) (a) Use the trapezium rule with two sub-intervals to find $\int_0^2 9x^2\left(\sqrt{x^3+1}\right)\mathrm{d}x$. *[4 marks]*

(b) (i) Use the substitution $u = x^3 + 1$ to show that

$$\int 9x^2\left(\sqrt{x^3+1}\right)\mathrm{d}x = 2(x^3+1)^{\frac{3}{2}} + c.$$

(ii) Hence, solve for k $\int_0^k 9x^2\left(\sqrt{x^3+1}\right)\mathrm{d}x = 594$. *[6 marks]*

Further Geometry

8. *[Maximum mark: 30]*

(i) (a) Write down the matrices representing the following transformations.

(i) $\boldsymbol{H}$, shear of scale factor 2 in the direction of the x-axis.

(ii) $\boldsymbol{S}$, stretch of scale factor 2 in the direction of the y-axis.

(iii) $\boldsymbol{R}$, reflection in the x-axis. *[3 marks]*

(b) Give a full geometric description for each one of the transformations represented by $\boldsymbol{H}^{-1}, \boldsymbol{S}^{-1}, \boldsymbol{R}^{-1}$. *[3 marks]*

(c) The transformation $\boldsymbol{M}$ is $\boldsymbol{R}$, followed by $\boldsymbol{S}$, followed by $\boldsymbol{H}$.

(i) Express the matrix $\boldsymbol{M}$ as a product of $\boldsymbol{R}$, $\boldsymbol{S}$ and $\boldsymbol{H}$.

This gives $\boldsymbol{M} = \begin{pmatrix} 1 & -4 \\ 0 & -2 \end{pmatrix}$.

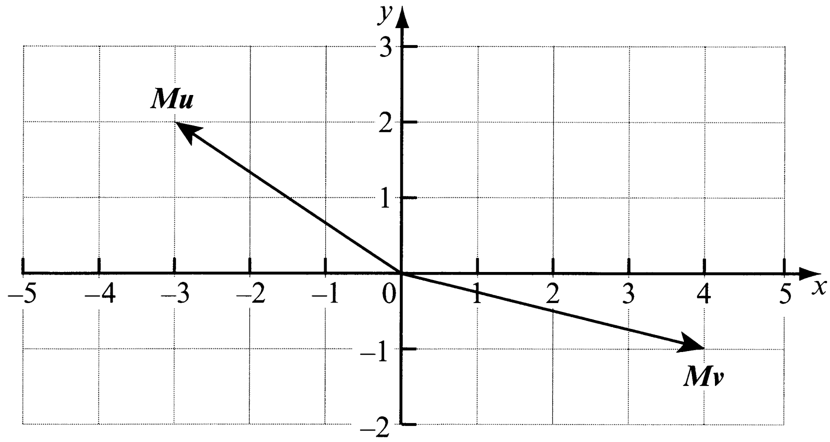
(ii) Find all points that are invariant under $\boldsymbol{M}$.

(iii) Find the image of the line $y = -x + 2$ under $\boldsymbol{M}$. *[8 marks]*

(This question continues on the following page)

(Question 8 (i) continued)

(d) The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are transformed by $\boldsymbol{M}$. Their images $\boldsymbol{Mu}$ and $\boldsymbol{Mv}$ are shown below.



(i) Write down the column vectors $\boldsymbol{Mu}$ and $\boldsymbol{Mv}$.

(ii) Let $\boldsymbol{w} = 3\boldsymbol{u} - 2\boldsymbol{v}$. Its image under $\boldsymbol{M}$ is $\begin{pmatrix} a \\ b \end{pmatrix}$. Find a and b. *[4 marks]*

(ii) The line L has equation $y = \frac{\sqrt{3}}{3}x + 2$. The transformation $\boldsymbol{T}$ is the reflection in L and it may be expressed as

$$\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{F}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}.$$

(a) (i) Find the matrix $\boldsymbol{F}$.

(ii) Find the vector $\begin{pmatrix} h \\ k \end{pmatrix}$. *[8 marks]*

(b) (i) Show that the image of the origin $(0, 0)$ under $\boldsymbol{T}$ is $(-\sqrt{3}, 3)$.

(ii) Hence calculate the perpendicular distance d from the origin to L. *[4 marks]*