MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Tuesday 6 May 2003 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Write the make and model of your calculator in the appropriate box on your cover sheet *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

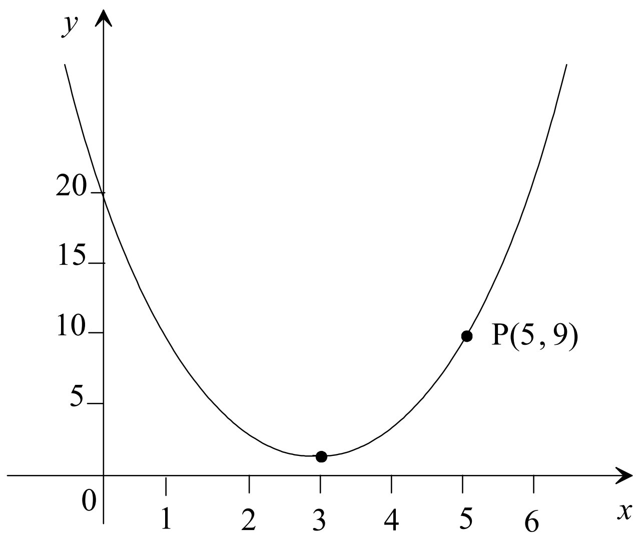
Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

*Answer all **five** questions from this section.*

1. *[Maximum mark: 10]*

The diagram shows part of the graph of the curve $y = a(x-h)^2 + k$, where $a, h, k \in \mathbb{Z}$.



(a) The vertex is at the point $(3, 1)$. Write down the value of h and of k. *[2 marks]*

(b) The point P(5, 9) is on the graph. Show that $a = 2$. *[3 marks]*

(c) Hence show that the equation of the curve can be written as

$$y = 2x^2 - 12x + 19.$$ *[1 mark]*

(This question continues on the following page)

(Question 1 continued)

(d) (i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

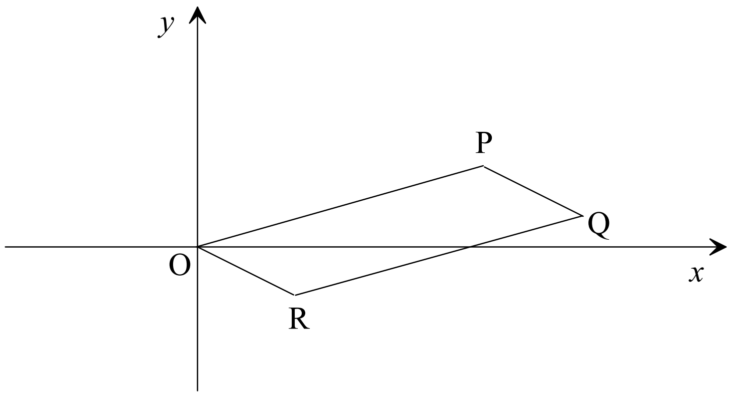
A tangent is drawn to the curve at $\mathrm{P}(5, 9)$.

(ii) Calculate the gradient of this tangent.

(iii) Find the equation of this tangent. *[4 marks]*

2. *[Maximum mark: 14]*

The diagram shows a parallelogram OPQR in which $\vec{OP} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$, $\vec{OQ} = \begin{pmatrix} 10 \\ 1 \end{pmatrix}$.



(a) Find the vector $\vec{OR}$. *[3 marks]*

(b) Use the scalar product of two vectors to show that $\cos O\hat{P}Q = -\frac{15}{\sqrt{754}}$. *[4 marks]*

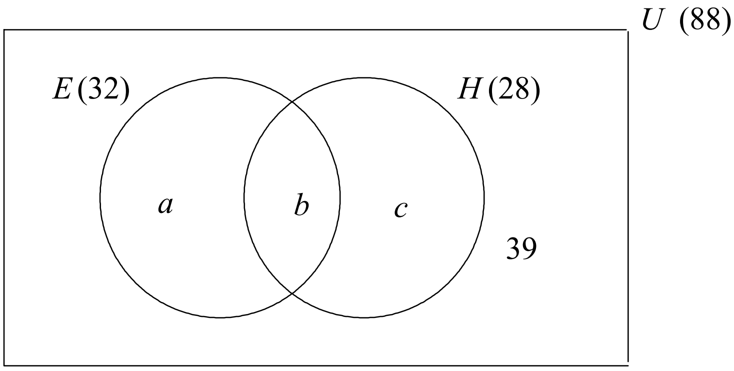
(c) (i) Explain why $\cos P\hat{Q}R = -\cos O\hat{P}Q$.

(ii) Hence show that $\sin P\hat{Q}R = \frac{23}{\sqrt{754}}$.

(iii) Calculate the area of the parallelogram OPQR, giving your answer as an integer. *[7 marks]*

3. *[Maximum mark: 12]*

In a school of 88 boys, 32 study economics (E), 28 study history (H) and 39 do not study either subject. This information is represented in the following Venn diagram.



(a) Calculate the values a, b, c. *[4 marks]*

(b) A student is selected at random.

(i) Calculate the probability that he studies **both** economics and history.

(ii) Given that he studies economics, calculate the probability that he does **not** study history. *[3 marks]*

(c) A group of three students is selected at random from the school.

(i) Calculate the probability that none of these students studies economics.

(ii) Calculate the probability that at least one of these students studies economics. *[5 marks]*

4. *[Maximum mark: 18]*

An aircraft lands on a runway. Its velocity $v\,\text{ms}^{-1}$ at time t seconds after landing is given by the equation $v = 50 + 50e^{-0.5t}$, where $0 \le t \le 4$.

(a) Find the velocity of the aircraft

(i) when it lands;

(ii) when $t = 4$. *[4 marks]*

(b) Write down an integral which represents the distance travelled in the first four seconds. *[3 marks]*

(c) Calculate the distance travelled in the first four seconds. *[2 marks]*

After four seconds, the aircraft slows down (decelerates) **at a constant rate** and comes to rest when $t = 11$.

(d) **Sketch** a graph of velocity against time for $0 \le t \le 11$. Clearly label the axes and mark on the graph the point where $t = 4$. *[5 marks]*

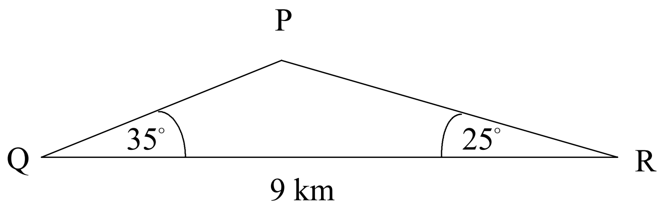
(e) Find the constant rate at which the aircraft is slowing down (decelerating) between $t = 4$ and $t = 11$. *[2 marks]*

(f) Calculate the distance travelled by the aircraft between $t = 4$ and $t = 11$. *[2 marks]*

5. *[Maximum mark: 16]*

The points P, Q, R are three markers on level ground, joined by straight paths PQ, QR, PR as shown in the diagram. $QR = 9\text{ km}$, $P\hat{Q}R = 35^\circ$, $P\hat{R}Q = 25^\circ$.

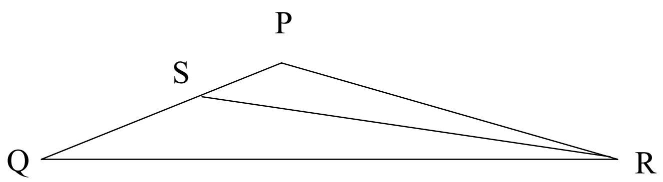
diagram not to scale



(a) Find the length PR. *[3 marks]*

(b) Tom sets out to walk from Q to P at a steady speed of 8 km h^{-1}. At the same time, Alan sets out to jog from R to P at a steady speed of $a\text{ km h}^{-1}$. They reach P at the same time. Calculate the value of a. *[7 marks]*

(c) The point S is on [PQ], such that $RS = 2QS$, as shown in the diagram.



Find the length QS. *[6 marks]*

SECTION B

*Answer **one** question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) A company manufactures television sets. They claim that the lifetime of a set is normally distributed with a mean of 80 months and standard deviation of 8 months.

(a) What proportion of television sets break down in less than 72 months? *[2 marks]*

(b) (i) Calculate the proportion of sets which have a lifetime between 72 months and 90 months.

(ii) Illustrate this proportion by appropriate shading in a sketch of a normal distribution curve. *[5 marks]*

(c) If a set breaks down in less than x months, the company replace it free of charge. They replace 4 % of the sets. Find the value of x. *[3 marks]*

(d) A newspaper claims that the mean lifetime of a set is less than 80 months. To test this claim, they take a random sample of 100 sets and find that the mean lifetime is 78.5 months.

(i) State the null hypothesis and the alternative hypothesis.

(ii) At the 5 % level of significance, show that this is sufficient evidence to support the newspaper's claim. *[5 marks]*

(ii) A furniture manufacturer makes chairs to sell to shops.

Over a six-week period, the cost $\$y$ of producing x chairs is given in the following table.

	Week 1	Week 2	Week 3	Week 4	Week 5	Week 6
Number of chairs x	22	40	32	28	46	44
Production cost $\$y$	3 200	4 600	3 800	3 700	5 100	5 000

(a) Find the equation of the regression line of y on x for this data. *[2 marks]*

(b) The chairs are sold at $120 each. Find the least number of chairs which the factory must sell each week in order to make a profit. *[5 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) An Australian golfer plays 30 tournaments in a year, 17 in Australia and 13 in Europe. The number of tournaments in which he finished in the top ten is given in the following table:

	Australia	**Europe**
Top Ten finish	6	6
Outside Top Ten	11	7

(a) Construct a table of expected frequencies based on the assumption that the golfer's performance is independent of where he is playing. (Do not apply Yates' continuity correction.) *[2 marks]*

(b) Find χ^2 for this data. *[2 marks]*

(c) (i) Determine whether the data provides evidence at the 5 % significance level that the golfer performs better in one continent than in the other.

(ii) Illustrate your answer, using a diagram. *[4 marks]*

Further Calculus

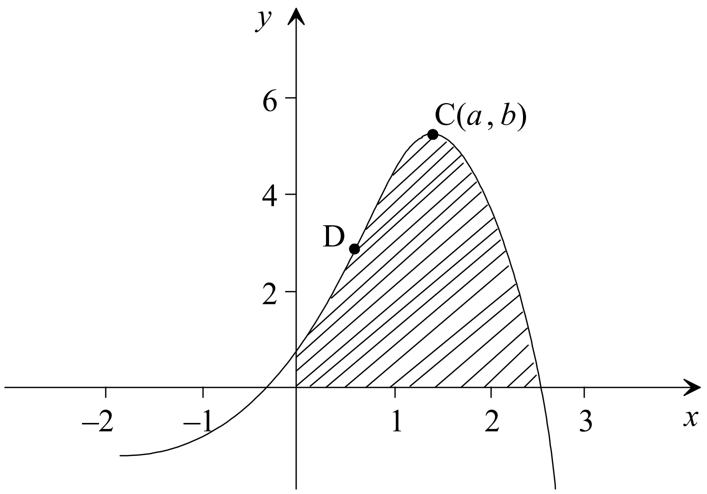
7. *[Maximum mark: 30]*

(i) Consider the function $f(x) = \cos x + \sin x$.

(a) (i) Show that $f(-\frac{\pi}{4}) = 0$.

(ii) Find in terms of π, the smallest **positive** value of x which satisfies $f(x) = 0$. *[3 marks]*

The diagram shows the graph of $y = e^{x}(\cos x + \sin x), -2 \le x \le 3$. The graph has a maximum turning point at $C(a, b)$ and a point of inflexion at D.



(b) Find $\frac{dy}{dx}$. *[3 marks]*

(c) Find the **exact** value of a and of b. *[4 marks]*

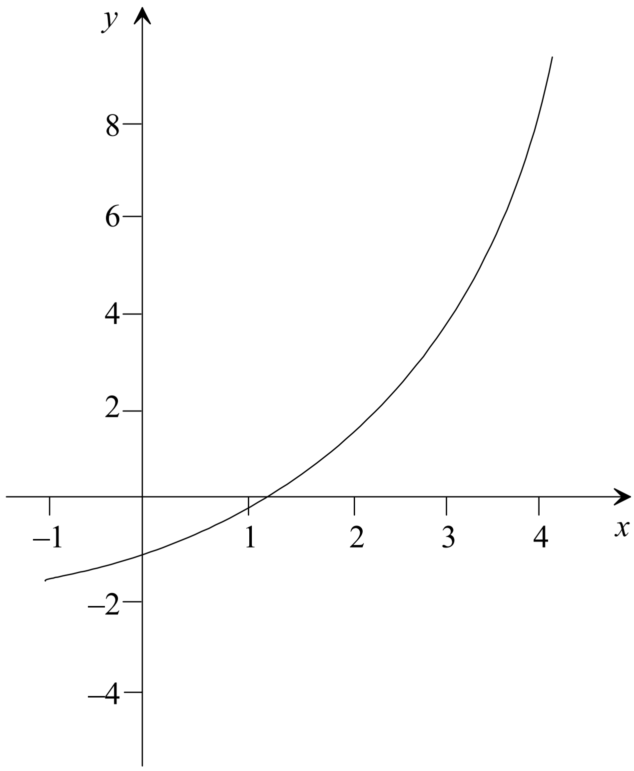
(d) Show that at D, $y = \sqrt{2}e^{\frac{\pi}{4}}$. *[5 marks]*

(e) Find the area of the shaded region. *[2 marks]*

(This question continues on the following page)

(Question 7 continued)

(ii) Consider the function $g(x) = \frac{x^3}{3} - \frac{x^2}{2} + x - 1$. A sketch of part of the graph of g is shown below.



The Newton-Raphson method is used to solve $g(x) = 0$. The starting value is $x_0 = 2$.

(a) Calculate x_1 and x_2. *[3 marks]*

(b) Draw a diagram to illustrate how the Newton-Raphson method is used to find x_1. *[4 marks]*

(c) How many iterations are required to give the solution correct to **six** significant figures? *[1 mark]*

(This question continues on the following page)

(Question 7 continued)

The solution to $g(x) = 0$ can also be found using fixed-point iteration.

(d) Show that one possible iteration is

$$x_{n+1} = \sqrt[3]{3\left(\frac{x_n^2}{2} - x_n + 1\right)}.$$

[1 mark]

(e) Use the iteration given in part (d) to answer the following.

(i) Using $x_0 = 2$, find x_1 and x_2.

(ii) How many iterations are required to give the solution correct to **six** significant figures?

[4 marks]

Further Geometry

8. *[Maximum mark: 30]*

(i) The matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{X}$ are given by

$$\boldsymbol{A}=\begin{pmatrix}3 & 1\\ -5 & 6\end{pmatrix},\ \boldsymbol{B}=\begin{pmatrix}4 & 8\\ 0 & -3\end{pmatrix},\ \boldsymbol{X}=\begin{pmatrix}a & b\\ c & d\end{pmatrix},\text{ where } a, b, c, d \in \mathbb{Q}.$$

Given that $\boldsymbol{AX}+\boldsymbol{X}=\boldsymbol{B}$, find the **exact** values of *a, b, c* and *d*. *[8 marks]*

(ii) A linear transformation is represented by the matrix $\boldsymbol{P}=\begin{pmatrix}5 & 3\\ 2 & 6\end{pmatrix}$.

(a) Write down $\boldsymbol{P}\begin{pmatrix}0\\ 1\end{pmatrix}$. *[1 mark]*

(b) Find

(i) $\boldsymbol{P}\begin{pmatrix}1\\ 1\end{pmatrix}$;

(ii) $\boldsymbol{P}\begin{pmatrix}3\\ -2\end{pmatrix}$. *[2 marks]*

(c) Hence find the equations of the two lines which are invariant under the transformation $\boldsymbol{P}$. *[5 marks]*

(d) The origin is an invariant point under the transformation $\boldsymbol{P}$. Use your answer to part (c) to explain why it is the **only** invariant point. *[1 mark]*

(This question continues on the following page)

(Question 8 continued)

(iii) The matrix $\boldsymbol{R}$ represents a rotation through θ about the origin, where θ is acute and $\tan\theta = \frac{12}{5}$.

(a) Find the matrix $\boldsymbol{R}$, using exact values. *[4 marks]*

A rotation through θ about the point $\mathrm{P}(5, 1)$ is represented by $\boldsymbol{T}$.

(b) Write down the image of $\mathrm{P}(5, 1)$ under $\boldsymbol{T}$. *[1 mark]*

(c) $\boldsymbol{T}$ may be written in the form $\boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{R}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix}$. Find the value of h and the value of k. *[4 marks]*

(d) $\boldsymbol{T}$ may be expressed as the composition of two isometries $\boldsymbol{S}$ and $\boldsymbol{R}$ *i.e.* $\boldsymbol{T} = \boldsymbol{S} \circ \boldsymbol{R}$. Give a complete geometrical description of the transformation $\boldsymbol{S}$. *[2 marks]*

(e) The point Q with coordinates $(17, -4)$ maps onto Q' under the transformation $\boldsymbol{T}$. Find the length PQ'. *[2 marks]*