IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Monday 11 November 2002 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
  Write the make and model of your calculator on the front cover of your answer booklets
  e.g. Casio *fx-9750G,* Sharp EL-9600, Texas Instruments TI-85.

*Please start each question on a new page. You are advised to show all working, where possible. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Incorrect answers with no working will normally receive **no** marks.*

SECTION A

*Answer all **five** questions from this section.*

1. *[Maximum mark: 15]*

In a suburb of a large city, 100 houses were sold in a three-month period. The following **cumulative frequency table** shows the distribution of selling prices (in thousands of dollars).

Selling price P (\$ 1000)	$P \le 100$	$P \le 200$	$P \le 300$	$P \le 400$	$P \le 500$
Total number of houses	12	58	87	94	100

(a) Represent this information on a cumulative frequency **curve**, using a scale of 1 cm to represent \$ 50000 on the horizontal axis and 1 cm to represent 5 houses on the vertical axis. *[4 marks]*

(b) Use your curve to find the interquartile range. *[3 marks]*

The information above is represented in the following frequency distribution.

Selling price P (\$ 1000)	$0 < P \le 100$	$100 < P \le 200$	$200 < P \le 300$	$300 < P \le 400$	$400 < P \le 500$
Number of houses	12	46	29	a	b

(c) Find the value of a and of b. *[2 marks]*

(d) Use mid-interval values to calculate an estimate for the mean selling price. *[2 marks]*

(This question continues on the following page)

(Question 1 continued)

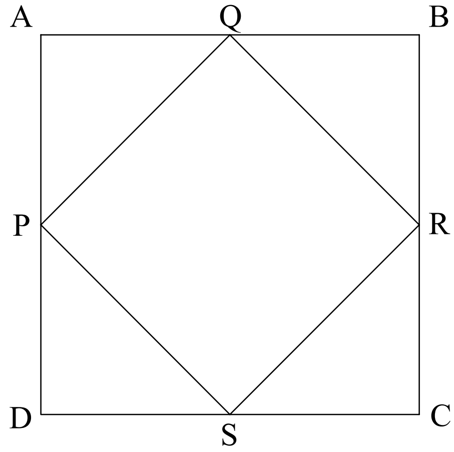
(e) Houses which sell for more than $ 350 000 are described as *De Luxe*.

(i) Use your graph to estimate the number of *De Luxe* houses sold. Give your answer to the nearest integer.

(ii) Two *De Luxe* houses are selected at random. Find the probability that **both** have a selling price of more than $ 400 000. *[4 marks]*

2. *[Maximum mark: 10]*

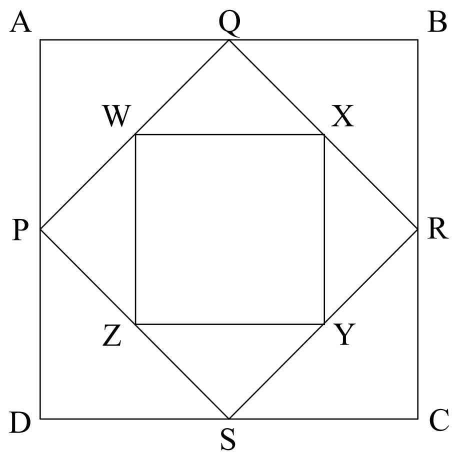
The diagram shows a square ABCD of side 4 cm. The midpoints P, Q, R, S of the sides are joined to form a **second** square.



(a) (i) Show that $PQ = 2\sqrt{2}$ cm.

(ii) Find the area of PQRS. *[3 marks]*

The midpoints W, X, Y, Z of the sides of PQRS are now joined to form a **third** square as shown.



(b) (i) Write down the area of the **third** square, WXYZ.

(ii) Show that the areas of ABCD, PQRS, and WXYZ form a geometric sequence. Find the common ratio of this sequence. *[3 marks]*

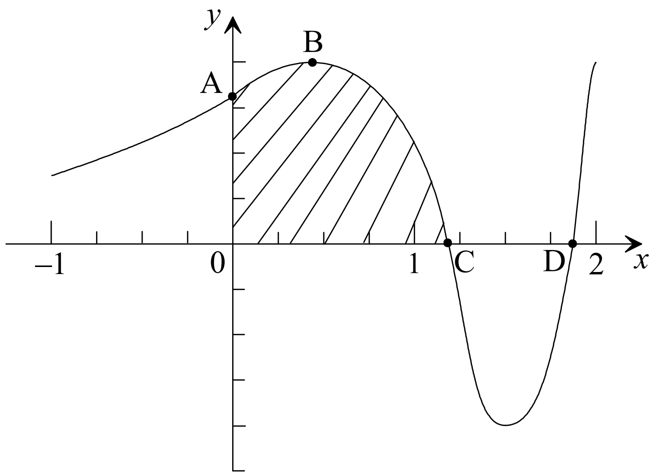
The process of forming smaller and smaller squares (by joining the midpoints) is **continued indefinitely**.

(c) (i) Find the area of the 11^{th} square.

(ii) Calculate the sum of the areas of **all** the squares. *[4 marks]*

3. *[Maximum mark: 18]*

The diagram below shows a sketch of the graph of the function $y = \sin\left(e^x\right)$ where $-1 \le x \le 2$, and x is in **radians**. The graph cuts the y-axis at A, and the x-axis at C and D. It has a maximum point at B.



(a) Find the coordinates of A. *[2 marks]*

(b) The coordinates of C may be written as $(\ln k, 0)$. Find the **exact** value of k. *[2 marks]*

(c) (i) Write down the y-coordinate of B.

(ii) Find $\frac{dy}{dx}$.

(iii) Hence, show that at B, $x = \ln\frac{\pi}{2}$. *[6 marks]*

(d) (i) Write down the integral which represents the shaded area.

(ii) Evaluate this integral. *[5 marks]*

(e) (i) Copy the above diagram into your answer booklet. (There is no need to copy the shading.) On your diagram, sketch the graph of $y = x^3$.

(ii) The two graphs intersect at the point P. Find the x-coordinate of P. *[3 marks]*

4. *[Maximum mark: 12]*

The following diagram shows the point O with coordinates $(0, 0)$, the point A with position vector $\boldsymbol{a} = 12\boldsymbol{i} + 5\boldsymbol{j}$, and the point B with position vector $\boldsymbol{b} = 6\boldsymbol{i} + 8\boldsymbol{j}$. The angle between (OA) and (OB) is θ.

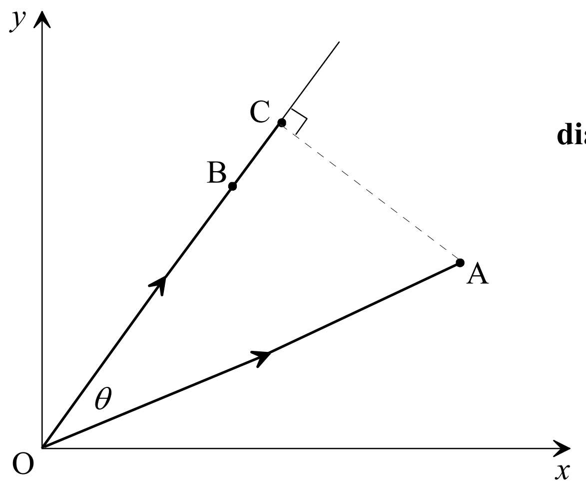


diagram not to scale

Find

(a) (i) $|\boldsymbol{a}|$;

(ii) a unit vector in the direction of $\boldsymbol{b}$;

(iii) the **exact** value of $\cos\theta$ in the form $\frac{p}{q}$, where $p, q \in \mathbb{Z}$. *[6 marks]*

The point C is the foot of the perpendicular from A to (OB). The position vector of C is denoted by $\boldsymbol{c}$.

(b) (i) Find the scalar projection of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$.

(ii) Find $\boldsymbol{c}$, expressing your answer in the form $m\boldsymbol{i} + n\boldsymbol{j}$, where m and n are to be determined.

(iii) Calculate $|\mathbf{AC}|$. *[6 marks]*

5. *[Maximum mark: 15]*

In this question, s represents displacement in metres, and t represents time in seconds.

(a) The velocity $v\,\text{ms}^{-1}$ of a moving body may be written as $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 30 - at$, where a is a constant. Given that $s = 0$ when $t = 0$, find an expression for s in terms of a and t. *[5 marks]*

Trains approaching a station start to slow down when they pass a signal which is 200 m from the station.

(b) The velocity of Train 1 t seconds after passing the signal is given by $v = 30 - 5t$.

(i) Write down its velocity as it passes the signal.

(ii) Show that it will stop before reaching the station. *[5 marks]*

(c) Train 2 slows down so that it stops at the station. Its velocity is given by $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 30 - at$, where a is a constant.

(i) Find, in terms of a, the time taken to stop.

(ii) Use your solutions to parts (a) and (c)(i) to find the value of a. *[5 marks]*

SECTION B

*Answer **one** question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) In a country called *Tallopia*, the height of adults is normally distributed with a mean of 187.5 cm and a standard deviation of 9.5 cm.

(a) What percentage of adults in *Tallopia* have a height greater than 197 cm? *[3 marks]*

(b) A standard doorway in *Tallopia* is designed so that $99\,\%$ of adults have a space of at least 17 cm over their heads when going through a doorway. Find the height of a standard doorway in *Tallopia*. Give your answer to the nearest cm. *[4 marks]*

(c) The height (in cm) of six-year-old children in *Tallopia* has a standard deviation of 4.5. Let $x_1, x_2, \ldots x_{20}$ be the heights (in cm) of 20 six-year-old *Tallopian* children selected at random. It is found that $\sum_{i=1}^{20} x_i = 2412$.

(i) Find a $95\,\%$ confidence interval for the mean height of all six-year-old *Tallopian* children. Give the limits **correct to the nearest 0.1 cm**.

(ii) What is the probability that the interval found in part (i) does not contain the mean? *[5 marks]*

(ii) A sample of 10 pairs of values $(x_1, y_1), (x_2, y_2) \ldots (x_{10}, y_{10})$ is described by the following statistics.

$\bar{x} = 56.45,\ \bar{y} = 63.12,\ s_x = 10.25,\ s_y = 12.14,\ s_{xy} = 74.66$

The product-moment correlation coefficient between the variables x and y is $r = 0.6$.

(a) If the value of x increases, is the value of y expected to increase, decrease, or remain the same? *[1 mark]*

(b) Find the equation of the least squares regression line of y on x, expressing your answer in the form $y = mx + c$. *[3 marks]*

(c) Predict the value of y for $x = 76$. *[2 marks]*

(d) Using a simple diagram, explain what is meant by the least squares regression line of y on x. *[4 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) A survey on a proposal to ban bear hunting resulted in the following data.

	In favour	Against
Urban resident	390	120
Rural resident	160	30

(a) Assuming that the attitude towards the ban is independent of whether the person lives in an urban or rural area, a table of expected frequencies is calculated for the above data. Part of this table is given below.

	In favour	Against
Urban resident	400.7	a
Rural resident	b	c

Calculate the values of a, b and c. *[3 marks]*

(b) Calculate the χ^2 statistic. *[2 marks]*

(c) At the 5% level of significance, test the claim that the attitude towards the ban is independent of whether the person lives in an urban or rural area.

(i) State your conclusion.

(ii) Give the reasons for your conclusion. *[3 marks]*

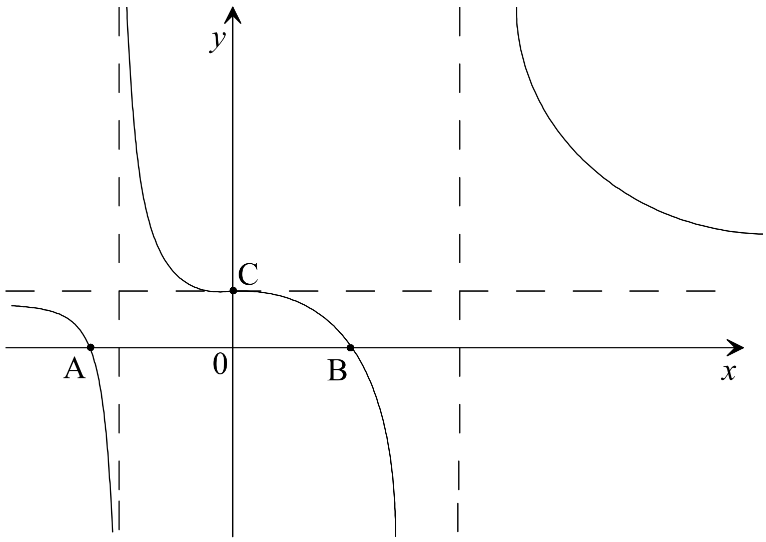
Further Calculus

7. *[Maximum mark: 30]*

(i) Let $g(x) = x^4 - 2x^3 + x^2 - 2$.

(a) Solve $g(x) = 0$. *[2 marks]*

Let $f(x) = \dfrac{2x^3}{g(x)} + 1$. A part of the graph of $f(x)$ is shown below.



(b) The graph has vertical asymptotes with equations $x = a$ and $x = b$ where $a < b$. Write down the values of

(i) a;

(ii) b. *[2 marks]*

(c) The graph has a horizontal asymptote with equation $y = 1$. Explain why the value of $f(x)$ approaches 1 as x becomes very large. *[2 marks]*

(d) The graph intersects the x-axis at the points A and B. Write down the **exact** value of the x-coordinate at

(i) A;

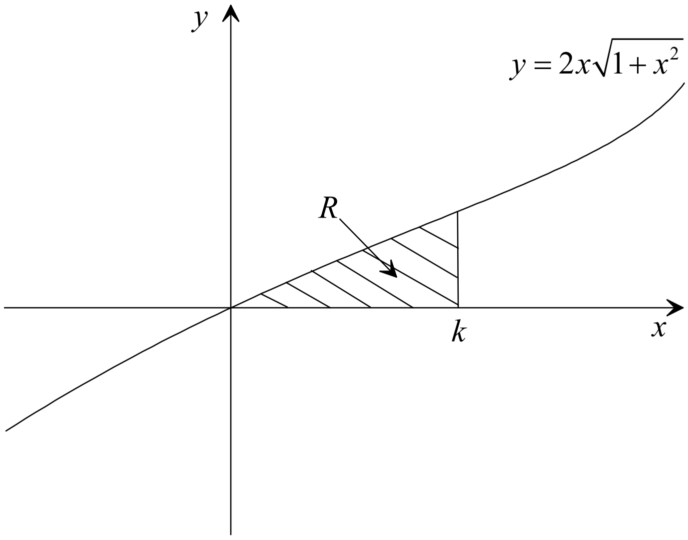
(ii) B. *[2 marks]*

(e) The curve intersects the y-axis at C. **Use the graph** to explain why the values of $f'(x)$ and $f''(x)$ are zero at C. *[2 marks]*

(This question continues on the following page)

(Question 7 continued)

(ii) The diagram below shows the shaded region R enclosed by the graph of $y = 2x\sqrt{1+x^2}$, the x-axis, and the vertical line $x = k$.



(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$. *[3 marks]*

(b) Using the substitution $u = 1 + x^2$ or otherwise, show that

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + c.$$ *[3 marks]*

(c) Given that the area of R equals 1, find the value of k. *[3 marks]*

(This question continues on the following page)

(Question 7 continued)

(iii) Consider the function $h(x) = x^{\frac{1}{5}}$.

(a) (i) Find the equation of the tangent to the graph of h at the point where $x = a, (a \neq 0)$. Write the equation in the form $y = mx + c$.

(ii) Show that this tangent intersects the x-axis at the point $(-4a, 0)$. *[5 marks]*

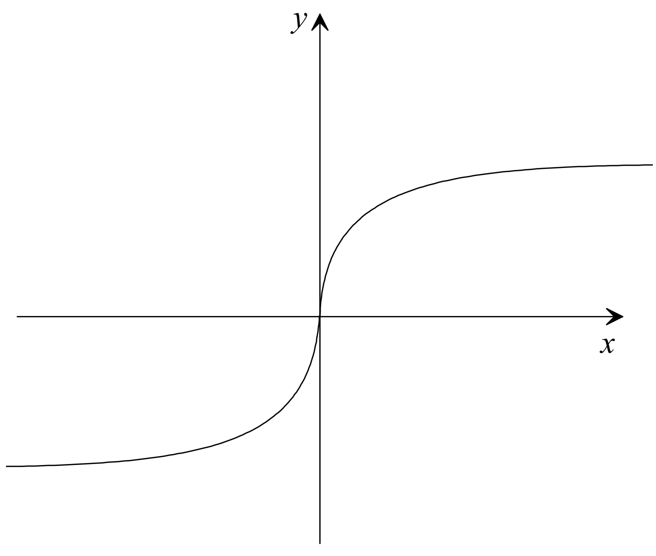
A student fails to notice the obvious solution of $x = 0$, and attempts to solve the equation $h(x) = 0$ using the Newton-Raphson method.

(b) Taking $x_1 = 0.5$ and using the results of part (a), or otherwise, calculate

(i) x_2;

(ii) x_5. *[3 marks]*

(c) Copy the following graph of $h(x)$.



By drawing appropriate tangents on your graph, indicate clearly the positions of x_2 and x_3. *[3 marks]*

Further Geometry

8. *[Maximum mark: 30]*

(i) (a) Write down the matrices representing the following transformations.

(i) $\boldsymbol{P}$: a reflection in the x-axis;

(ii) $\boldsymbol{Q}$: a one-way stretch of scale factor 5 parallel to the y-axis. *[2 marks]*

(b) The elementary transformation $\boldsymbol{E}_1$ is given by $\boldsymbol{E}_1 = \boldsymbol{PQ}$.
The transformation $\boldsymbol{M}$ is the composition of the elementary transformations $\boldsymbol{E}_3$, $\boldsymbol{E}_2$ and $\boldsymbol{E}_1$, where $\boldsymbol{M} = \boldsymbol{E}_3\boldsymbol{E}_2\boldsymbol{E}_1$,

$$\boldsymbol{M} = \begin{pmatrix} 1 & 3 \\ 3 & 4 \end{pmatrix} \text{ and } \boldsymbol{E}_3 = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}.$$

(i) Show that $\boldsymbol{E}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$.

(ii) Find $\boldsymbol{E}_2$. *[6 marks]*

The triangle ABC is defined by $\mathrm{A}(1, 0)$, $\mathrm{B}(1, 2)$ and $\mathrm{C}(3, 0)$. The triangle $\mathrm{A'B'C'}$ is the image of triangle ABC under the transformation $\boldsymbol{M}$.

(c) (i) Calculate the coordinates of $\mathrm{A'}$, $\mathrm{B'}$ and $\mathrm{C'}$.

(ii) On graph paper, using 1 cm to represent 1 unit on both axes, draw and label triangle $\mathrm{A'B'C'}$.

(iii) Find the area of triangle $\mathrm{A'B'C'}$. *[6 marks]*

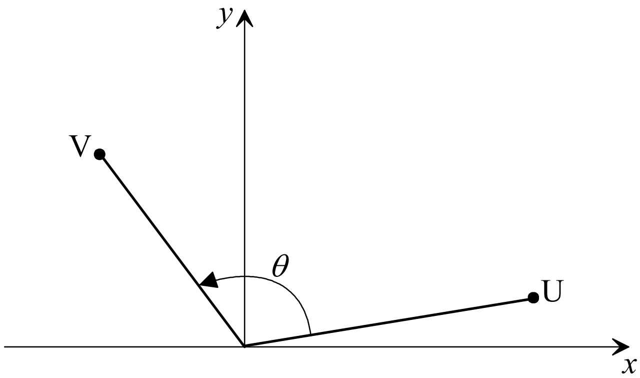
(d) Find the image of the line with equation $y = -x + 3$ under $\boldsymbol{M}$, expressing your answer in the form $y = mx + c$. *[4 marks]*

(This question continues on the following page)

(Question 8 continued)

(ii) The points U and V have position vectors $\boldsymbol{u} = 7\boldsymbol{i} + \boldsymbol{j}$ and $\boldsymbol{v} = -5\boldsymbol{i} + 5\boldsymbol{j}$ respectively.

(a) A rotation $\boldsymbol{R}$ through angle θ about (0, 0) transforms U into V, as shown in the following diagram.



(i) Represent this information as a matrix equation.

(ii) Find the matrix $\boldsymbol{R}$ which represents this transformation. *[6 marks]*

(b) Another rotation $\boldsymbol{Q}$, through 90° about the point (h, k) also transforms U into V. The rotation $\boldsymbol{Q}$ may be represented by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \boldsymbol{T}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix},$$

where $\boldsymbol{T}$ denotes a rotation through 90° about (0, 0).

Calculate

(i) the value of a and of b;

(ii) the value of h and of k. *[6 marks]*