INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Monday 12 November 2001 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures, as appropriate.
- Write the make and model of your calculator on the front cover of your answer booklets *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

You are advised to start each new question on a new page. A correct answer with **no** *indication of the method used will usually receive* **no** *marks. You are therefore advised to show your working. In particular, where graphs from a graphic display calculator are being used to find solutions, you should sketch these graphs as part of your answer.*

SECTION A

Answer all **five** *questions from this section.*

1. *[Maximum mark: 11]*

Portable telephones are first sold in the country *Cellmania* in 1990. During 1990, the number of units sold is 160. In 1991, the number of units sold is 240 and in 1992, the number of units sold is 360.

In 1993 it was noticed that the annual sales formed a geometric sequence with first term 160, the 2nd and 3rd terms being 240 and 360 respectively.

(a) What is the common ratio of this sequence? *[1 mark]*

Assume that this trend in sales continues.

(b) How many units will be sold during 2002? *[3 marks]*

(c) In what year does the number of units sold first exceed 5000? *[4 marks]*

Between 1990 and 1992, the total number of units sold is 760.

(d) What is the total number of units sold between 1990 and 2002? *[2 marks]*

During this period, the total population of *Cellmania* remains approximately 80 000.

(e) Use this information to suggest a reason why the geometric growth in sales would not continue. *[1 mark]*

2. *[Maximum mark: 11]*

The speeds in km h^{-1} of cars passing a point on a highway are recorded in the following table.

Speed v	Number of cars
$v \le 60$	0
$60 < v \le 70$	7
$70 < v \le 80$	25
$80 < v \le 90$	63
$90 < v \le 100$	70
$100 < v \le 110$	71
$110 < v \le 120$	39
$120 < v \le 130$	20
$130 < v \le 140$	5
$v > 140$	0

(a) Calculate an estimate of the mean speed of the cars. *[2 marks]*

(b) The following table gives some of the cumulative frequencies for the information above.

Speed v	Cumulative frequency
$v \le 60$	0
$v \le 70$	7
$v \le 80$	32
$v \le 90$	95
$v \le 100$	a
$v \le 110$	236
$v \le 120$	b
$v \le 130$	295
$v \le 140$	300

(i) Write down the values of a and b .

(ii) On graph paper, construct a cumulative frequency **curve** to represent this information. Use a scale of 1 cm for 10 km h^{-1} on the horizontal axis and a scale of 1 cm for 20 cars on the vertical axis. *[5 marks]*

(c) Use your graph to determine

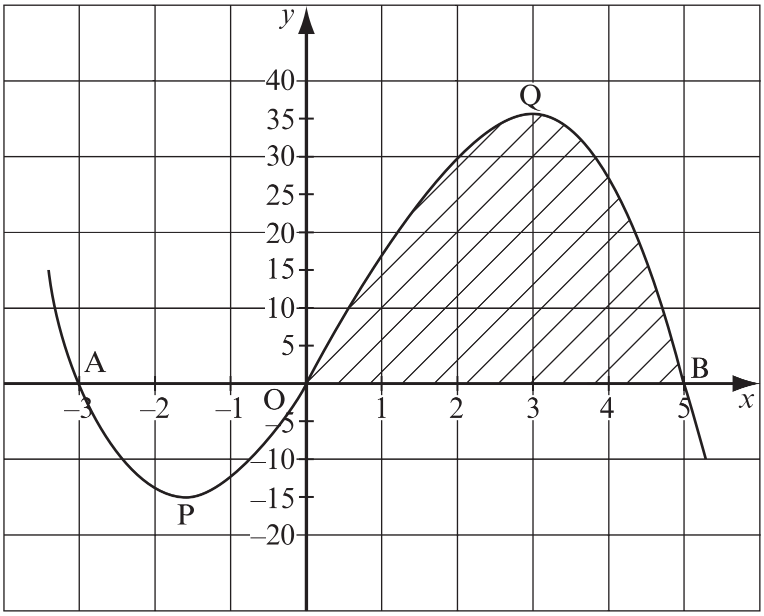
(i) the percentage of cars travelling at a speed in excess of 105 km h^{-1};

(ii) the speed which is exceeded by 15% of the cars. *[4 marks]*

3. *[Maximum mark: 15]*

The diagram below shows part of the graph of the function

$$f: x \mapsto -x^3 + 2x^2 + 15x.$$



The graph intercepts the x-axis at A(–3, 0), B(5, 0) and the origin, O. There is a minimum point at P and a maximum point at Q.

(a) The function may also be written in the form $f: x \mapsto -x(x-a)(x-b)$, where $a < b$. Write down the value of

(i) a;

(ii) b. *[2 marks]*

(b) Find

(i) $f'(x)$;

(ii) the **exact** values of x at which $f'(x) = 0$;

(iii) the value of the function at Q. *[7 marks]*

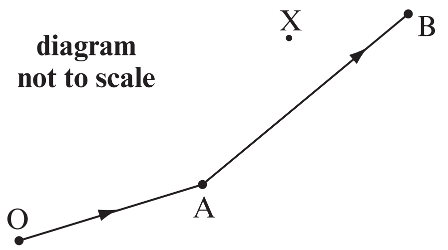
(c) (i) Find the equation of the tangent to the graph of f at O.

(ii) This tangent cuts the graph of f at another point. Give the x-coordinate of this point. *[4 marks]*

(d) Determine the area of the shaded region. *[2 marks]*

4. *[Maximum mark: 18]*

The diagram below shows the positions of towns O , A , B and X .



Town A is 240 km East and 70 km North of O .
Town B is 480 km East and 250 km North of O .
Town X is 339 km East and 238 km North of O .

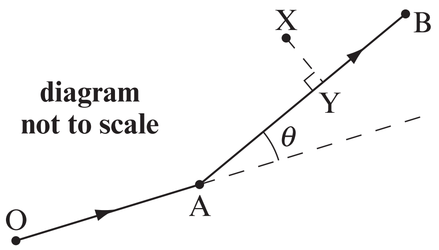
An airplane flies at a constant speed of 300 km h^{-1} from O towards A .

(a) (i) Show that a unit vector in the direction of $\overrightarrow{OA}$ is $\begin{pmatrix} 0.96 \\ 0.28 \end{pmatrix}$.

(ii) Write down the velocity vector for the airplane in the form $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$.

(iii) How long does it take for the airplane to reach A ? *[5 marks]*

At A the airplane changes direction so it now flies towards B . The angle between the original direction and the new direction is θ as shown in the following diagram. This diagram also shows the point Y , between A and B , where the airplane comes closest to X .



(b) Use the scalar product of two vectors to find the value of θ in degrees. *[4 marks]*

(c) (i) Write down the vector $\overrightarrow{AX}$.

(ii) Show that the vector $\boldsymbol{n} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$ is perpendicular to $\overrightarrow{AB}$.

(iii) By finding the projection of $\overrightarrow{AX}$ in the direction of $\boldsymbol{n}$, calculate the distance XY . *[6 marks]*

(d) How far is the airplane from A when it reaches Y ? *[3 marks]*

5. *[Maximum mark: 15]*

A ball is dropped vertically from a great height. Its velocity v is given by

$$v = 50 - 50e^{-0.2t}, \; t \geq 0$$

where v is in metres per second and t is in seconds.

(a) Find the value of v when

(i) $t = 0$;

(ii) $t = 10$. *[2 marks]*

(b) (i) Find an expression for the acceleration, a, as a function of t .

(ii) What is the value of a when $t = 0$? *[3 marks]*

(c) (i) As t becomes large, what value does v approach?

(ii) As t becomes large, what value does a approach?

(iii) Explain the relationship between the answers to parts (i) and (ii). *[3 marks]*

(d) Let y metres be the distance fallen after t seconds.

(i) Show that $y = 50t + 250e^{-0.2t} + k$, where k is a constant.

(ii) Given that $y = 0$ when $t = 0$, find the value of k .

(iii) Find the time required to fall 250 m, giving your answer correct to **four** significant figures. *[7 marks]*

SECTION B

Answer **one** *question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) Bags of cement are labelled 25 kg. The bags are filled by machine and the actual weights are normally distributed with mean 25.7 kg and standard deviation 0.50 kg.

(a) What is the probability a bag selected at random will weigh less than 25.0 kg? *[2 marks]*

In order to reduce the number of underweight bags (bags weighing less than 25 kg) to 2.5% of the total, the mean is increased without changing the standard deviation.

(b) Show that the increased mean is 26.0 kg. *[3 marks]*

It is decided to purchase a more accurate machine for filling the bags. The requirements for this machine are that only 2.5% of bags be under 25 kg and that only 2.5% of bags be over 26 kg.

(c) Calculate the mean and standard deviation that satisfy these requirements. *[3 marks]*

The cost of the new machine is \$5000 . Cement sells for \$0.80 per kg.

(d) Compared to the cost of operating with a 26 kg mean, how many bags must be filled in order to recover the cost of the new equipment? *[3 marks]*

(ii) It is claimed that the heights of seventeen-year-old males in a particular country are normally distributed with mean 175 cm and standard deviation 12.0 cm. In order to investigate this claim, the heights of a random sample of 36 seventeen-year-old males from the country are measured and the mean height of this sample is found to be 178.9 cm. The stated population standard deviation of 12.0 cm is assumed to be accurate.

Hypothesis testing is used. The alternative hypothesis, H_1, is that the mean is greater than 175 cm.

(a) (i) State the null hypothesis, H_0.

(ii) State whether a one-tailed or a two-tailed test is being used. *[2 marks]*

(b) At the 5% level of significance, how large would the sample mean (to nearest 0.1 cm) need to be to reject H_0? *[4 marks]*

(c) State your conclusions from this investigation. *[2 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) In a school history examination, a particular essay is marked out of 25. Owing to the large number of candidates, not all essays are marked by the same teacher. In order to make sure that all teachers are marking to the same standard, the Principal re-marks a sample of 8 essays from each teacher's marking.

The following table shows the mark (x) awarded by Teacher A, and the mark (y) awarded by the Principal for the same essay.

x	5	7	10	13	15	16	18	20
y	6	8	12	15	17	19	22	24

(a) Calculate the equation of the least-squares regression line of y on x in the form $y = px + q$. *[3 marks]*

(b) Determine the coefficient of linear correlation, r, between y and x. *[2 marks]*

The values of p, q and r for Teachers B and C are given in the following table.

	p	q	r
Teacher B	1.02	–3.08	0.997
Teacher C	1.00	0	0.890

The final mark a candidate receives is calculated by substituting the mark x given by the teacher into the equation of the regression line for that teacher.

(c) A final mark of 16 will be awarded to a candidate awarded 16 by Teacher C. What final mark (to the nearest integer) will be awarded to a candidate awarded 16 by

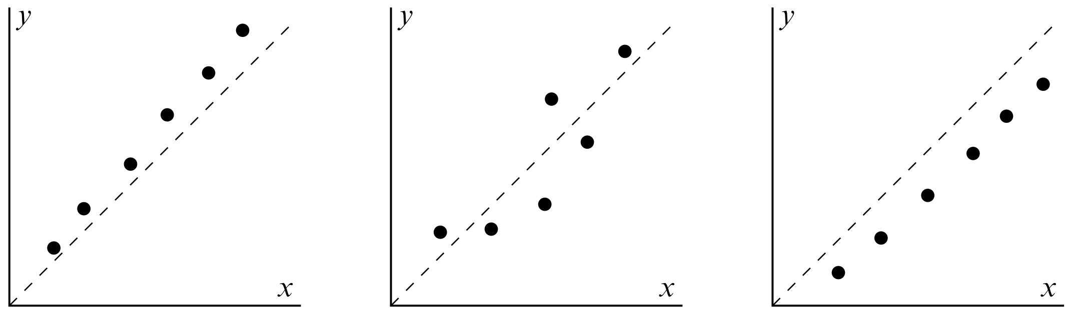
(i) Teacher A ?

(ii) Teacher B ? *[2 marks]*

(This question continues on the following page)

(Question 6(iii) continued)

The graphs below are scatter diagrams that are obtained when y is plotted against x for each of the three teachers. The line with equation $y = x$ is indicated as a dotted line in each case.



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(d) Which scatter diagram would correspond to

(i) Teacher B ?

(ii) Teacher C ? *[2 marks]*

(e) The marks awarded by Teacher C would not be changed by this process. In spite of this fact, explain why Teacher C's marking is the least reliable. *[2 marks]*

Further Calculus

Radian measure is used, where appropriate, throughout question 7.

7. *[Maximum mark: 30]*

(i) Consider the function $y = \dfrac{3x-2}{2x+5}$.

The graph of this function has a vertical and a horizontal asymptote.

(a) Write down the equation of

(i) the vertical asymptote;

(ii) the horizontal asymptote. *[2 marks]*

(b) Find $\dfrac{dy}{dx}$, simplifying the answer as much as possible. *[3 marks]*

(c) How many points of inflexion does the graph of this function have? *[1 mark]*

(ii) (a) Use the substitution $u = \tan 3x$ to obtain an expression in terms of u for the indefinite integral

$$\int \tan^3(3x)\left(\frac{1}{\cos^2(3x)}\right)dx.$$

[3 marks]

(b) Hence obtain an **exact** value for $\displaystyle\int_0^{\frac{\pi}{9}} \tan^3(3x)\left(\frac{1}{\cos^2(3x)}\right)dx$. *[3 marks]*

(iii) Consider the graph of $y = x\sin x$ between $x = 0$ and $x = \pi$.

(a) Find $\dfrac{dy}{dx}$. *[1 mark]*

(b) Show that the second derivative $\dfrac{d^2y}{dx^2} = -x\sin x + 2\cos x$. *[2 marks]*

(This question continues on the following page)

(Question 7(iii) continued)

This graph has a point of inflexion near $x = 1$.

(c) Show that an equation that must be satisfied at this point of inflexion is

$$x = \frac{2}{\tan x}.$$

[2 marks]

An alternative equation that must also be satisfied at this point of inflexion is

$$x = \arctan\left(\frac{2}{x}\right).$$

Let $g(x) = \dfrac{2}{\tan x}$ and let $h(x) = \arctan\left(\dfrac{2}{x}\right)$.

Fixed-point iteration is to be used to find the x-coordinate of the point of inflexion. The starting value $x_0 = 1$ is to be used.

(d) Using $x_{n+1} = g(x_n)$, find correct to **four** decimal places

(i) x_1;

(ii) x_2. *[2 marks]*

(e) Using $x_{n+1} = h(x_n)$, find correct to **four** decimal places

(i) x_1;

(ii) x_2. *[2 marks]*

(f) Give the x-coordinate of the point of inflexion correct to **six** decimal places. *[2 marks]*

The graph of $y = x \sin x$ has a local maximum near $x = 2$.

(g) Explain why the x-coordinate of this local maximum must satisfy the equation $f(x) = 0$, where $f(x) = \sin x + x \cos x$. *[1 mark]*

(h) The solution to $f(x) = 0$ is to be found using the Newton-Raphson method. Starting with $x_0 = 2$, find

(i) x_1, correct to **four** decimal places;

(ii) the solution, correct to **six** decimal places. *[6 marks]*

Further Geometry

8. *[Maximum mark: 30]*

(i) The transformation $\boldsymbol{R}$ is represented by the matrix $\boldsymbol{R} = \begin{pmatrix} 0.28 & 0.96 \\ 0.96 & -0.28 \end{pmatrix}$.

(a) Calculate the determinant of this matrix. *[2 marks]*

(b) Show that the matrix is its own inverse. *[2 marks]*

(c) Determine the image of the following points under this transformation

(i) $(0, 0)$;

(ii) $(4, 3)$. *[2 marks]*

(d) Hence, give a full geometric description of the transformation $\boldsymbol{R}$. *[2 marks]*

Another transformation $\boldsymbol{S}$ is represented by the matrix

$$\boldsymbol{S} = \begin{pmatrix} -0.96 & 0.28 \\ 0.28 & 0.96 \end{pmatrix}.$$

The composite transformation $\boldsymbol{T}$ is the transformation $\boldsymbol{R}$ followed by the transformation $\boldsymbol{S}$.

(e) (i) Find the matrix representing $\boldsymbol{T}$.

(ii) Give a full geometric description of the transformation $\boldsymbol{T}$. *[3 marks]*

(This question continues on the following page)

(Question 8 continued)

(ii) *In this part,* ***position vectors*** *of points should be represented in* ***column*** *form.*

The point P has coordinates (x, y) .

(a) Under a particular translation, the image of the point $(5, 2)$ is the origin $(0, 0)$. The point Q is the image of P under this translation. Write down the position vector of Q . *[1 mark]*

(b) The point U is the image of Q under a rotation of 90° about the origin. Write down the position vector of U . *[2 marks]*

(c) Under a second translation, the image of the origin is the point $(5, 2)$. V is the image of U under this translation. Write down the position vector of V . *[1 mark]*

(d) The transformation $\boldsymbol{W}$ may be represented by

$$\boldsymbol{W}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} p \\ q \end{pmatrix} .$$

Under $\boldsymbol{W}$, the image of P is V .

(i) Determine the value of

(a) p ;

(b) q . *[3 marks]*

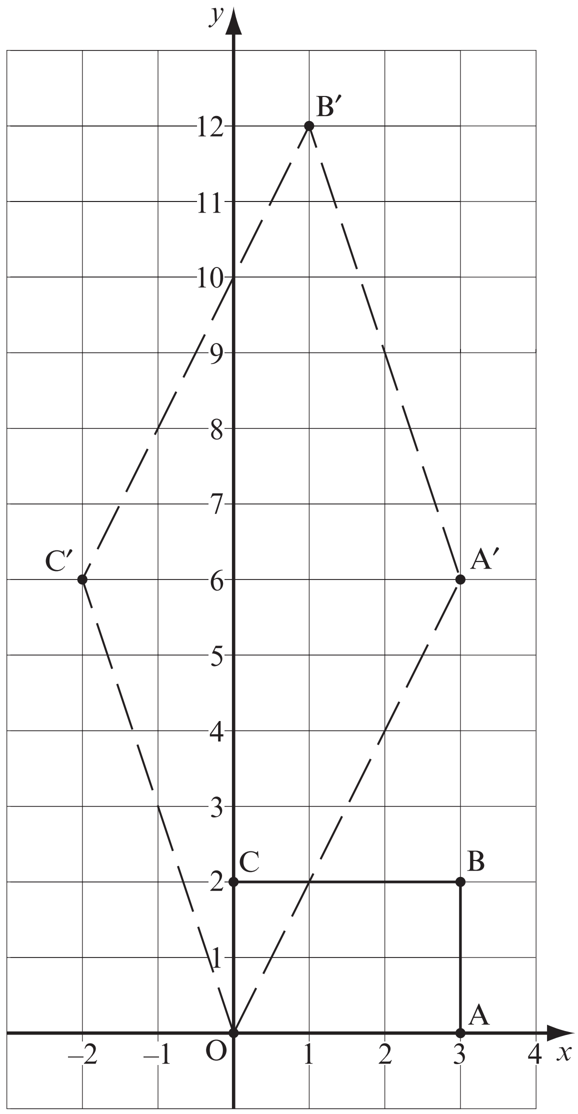
(ii) Show that the only invariant point of transformation $\boldsymbol{W}$ is the point $(5, 2)$. *[2 marks]*

(This question continues on the following page)

(Question 8 continued)

(iii) The following diagram shows the rectangle OABC and the parallelogram OA′B′C′, where OABC is transformed into OA′B′C′ by the transformation represented by the matrix

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}.$$



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(Question 8(iii) continued)

(a) Find the area of OA′B′C′. *[2 marks]*

An angle, θ, exists, such that

$$\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

where θ is the angle of rotation about $(0,0)$ for the transformation represented by the middle matrix.

(b) Show that $\theta = \arctan 2$. *[3 marks]*

(c) Give a full geometric description of the transformations represented by

(i) $\begin{pmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{5} \end{pmatrix}$;

(ii) $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. *[4 marks]*

(d) Which of these two transformations described in part (c) is applied first? *[1 mark]*