INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Monday 12 November 2001 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures, as appropriate.
- Write the make and model of your calculator on the front cover of your answer booklets *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

You are advised to start each new question on a new page. A correct answer with **no** *indication of the method used will usually receive* **no** *marks. You are therefore advised to show your working. In particular, where graphs from a graphic display calculator are being used to find solutions, you should sketch these graphs as part of your answer.*

SECTION A

Answer all **five** *questions from this section.*

1. *[Maximum mark: 14]*

The function f is defined by $f : x \mapsto -0.5x^2 + 2x + 2.5$.

(a) (i) Determine $f'(x)$.

(ii) Evaluate $f'(0)$. *[2 marks]*

(b) Show that the equation of the line perpendicular to the tangent to the graph of f (i.e. the normal) at the point where the graph intercepts the y-axis may be written as $y = -0.5x + 2.5$. *[2 marks]*

The equation of the curve may be written as $y = f(x)$. The equation of the normal may be written as $y = g(x)$.

(c) Equate $f(x)$ and $g(x)$ and solve the resulting quadratic equation. *[3 marks]*

(d) Write down the coordinates of the other point of intersection of the normal and the curve. *[2 marks]*

(e) Write an expression involving an integral for the area enclosed between the curve and the normal. *[3 marks]*

(f) Evaluate the expression in part (e) . *[2 marks]*

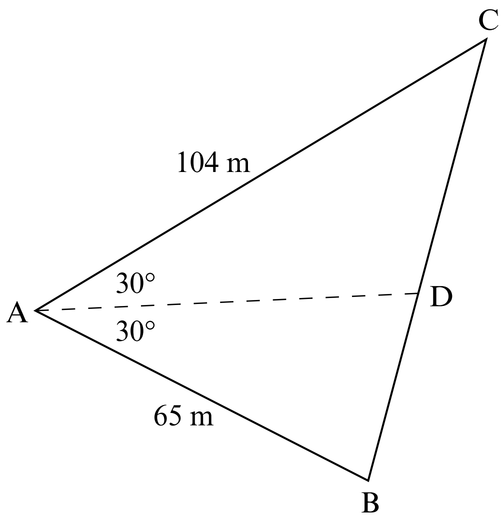
2. *[Maximum mark: 15]*

A farmer owns a triangular field ABC . One side of the triangle, [AC] , is 104 m in length, a second side, [AB] , is 65 m in length and the angle between these two sides is $60°$.

(a) Use the cosine rule to calculate the length of the third side of the field. *[3 marks]*

(b) Given that $\sin 60° = \frac{\sqrt{3}}{2}$, express the area of the field in the form $p\sqrt{3}$, where p is an integer. *[2 marks]*

The farmer divides the field into two parts by constructing a straight fence [AD] of length x m which bisects the $60°$ angle, as shown in the diagram.



(c) Show that the smaller area is given by $\frac{65x}{4}$ and obtain a similar expression for the larger area. *[3 marks]*

(d) **Hence**, determine the value of x in the form $q\sqrt{3}$, where q is an integer. *[3 marks]*

(e) (i) What can be said about $\sin \mathrm{A\widehat{D}C}$ and $\sin \mathrm{A\widehat{D}B}$?

(ii) Use the result of part (i) and the sine rule to prove that

$$\frac{\mathrm{BD}}{\mathrm{DC}} = \frac{5}{8}.$$

[4 marks]

3. *[Maximum mark: 12]*

The function f is defined by $f : x \mapsto x^3 e^{-x}$.

As x increases from 0, the graph of f rises to a maximum value and then decreases steadily, approaching a limiting value asymptotically.

(a) Sketch the graph of f for $0 \le x \le 10$, choosing a suitable scale for the y-axis. (There is no need to draw an accurate, scaled graph.) The y-intercept, the position of the maximum and the asymptotic behaviour should be clear from your graph. *[4 marks]*

(b) (i) Shade on your graph the area represented by

$$\int_0^5 x^3 e^{-x} \, dx .$$

(ii) Evaluate this integral. *[2 marks]*

(c) Give the equation of the horizontal asymptote to the graph of f. *[1 mark]*

(d) Consider the equation $f(x) = 1$.

(i) Draw a line on your graph illustrating why this equation has more than one solution.

(ii) Write down the solutions of $f(x) = 1$. *[3 marks]*

(e) Give the coordinates of the maximum point on the graph of f. *[2 marks]*

4. *[Maximum mark: 12]*

Bag A contains 2 red balls and 3 green balls. Bag B contains 4 red balls and 2 green balls. A person reaches into one of the bags and takes out two balls. If bag A is chosen, the probabilities for the different outcomes are

$$\text{P(2 red balls)} = \frac{1}{10}$$

$$\text{P(2 green balls)} = \frac{3}{10}$$

$$\text{P(1 red ball and 1 green ball)} = \frac{6}{10}$$

(a) Calculate the probabilities for the same three outcomes if bag B is chosen. *[5 marks]*

In order to decide which bag to choose, a standard die with six faces is rolled. If a 1 or 6 is rolled, bag A is chosen. If a 2 , 3 , 4 or 5 is rolled, bag B is chosen.

The die is rolled and then two balls are drawn from the selected bag.

(b) Calculate the probability that two red balls will be selected. *[3 marks]*

(c) Given that two red balls are obtained, what is the conditional probability that a 1 or 6 was rolled on the die? *[4 marks]*

5. *[Maximum mark: 17]*

A trapezium is a quadrilateral with two parallel sides. A trapezium ABCD is defined by four lines with the following equations

$$\text{Line 1: (AB)} \quad \boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \end{pmatrix} + p\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

$$\text{Line 2: (AD)} \quad \boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -8 \\ 3 \end{pmatrix} + q\begin{pmatrix} 3 \\ 11 \end{pmatrix}$$

$$\text{Line 3: (CD)} \quad \boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 23 \end{pmatrix} + s\begin{pmatrix} -4 \\ -3 \end{pmatrix}$$

$$\text{Line 4: (BC)} \quad \boldsymbol{r} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 23 \end{pmatrix} + t\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Lines 1 and 2 intersect at the point A(–8 , 3) , and lines 1 and 4 at the point B(12 , 18) . Lines 3 and 4 intersect at C , and lines 2 and 3 at D .

(a) Which **two** lines are parallel? *[1 mark]*

(b) Using the intervals $-10 \le x \le 15$ and $0 \le y \le 25$, draw a neat sketch illustrating the four lines and the trapezium formed by them. Label the vertices. *[4 marks]*

(c) (i) Write down the coordinates of C .

(ii) Show algebraically that the coordinates of D are (–5 , 14) . *[5 marks]*

(d) The vector $-3\boldsymbol{i} + 4\boldsymbol{j}$ is perpendicular to $\overrightarrow{\mathrm{AB}}$. Use this fact to find the projection of $\overrightarrow{\mathrm{AD}}$ in the direction of $-3\boldsymbol{i} + 4\boldsymbol{j}$. *[4 marks]*

(e) Calculate the area of the trapezium ABCD . *[3 marks]*

SECTION B

Answer **one** *question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) The enrolment in physics classes at 3 high schools, A, B and C is investigated. The table below summarizes the results.

	A	B	C
Take physics	47	95	58
Don't take physics	103	145	152

The hypothesis that **the enrolment in physics classes is independent of the high school attended** is to be examined using a χ^2 test.

(a) Construct a table similar to the one above showing the expected frequencies, if the hypothesis is valid. *[3 marks]*

(b) Show how a χ^2 value of 7.574 is calculated using the two tables. *[3 marks]*

(c) Comment on the validity of the hypothesis using levels of significance of both 5% and 1%. *[4 marks]*

(ii) A normally distributed population has a mean of 65 and a standard deviation of 12. Random samples of size 25 are drawn from this population. The means of these samples are calculated.

(a) What is the expected value for the sample mean? *[1 mark]*

(b) Show that the standard error for the mean (i.e. standard deviation for the population of sample means) will be 2.4. *[2 marks]*

The probability that a sample mean lies between the symmetric limits a and b is 0.95.

(c) Find the values of a and b. *[4 marks]*

(d) Let $d = a - b$. The sample size is increased so that d is decreased. Find the minimum sample size required so that $d = 5$. *[4 marks]*

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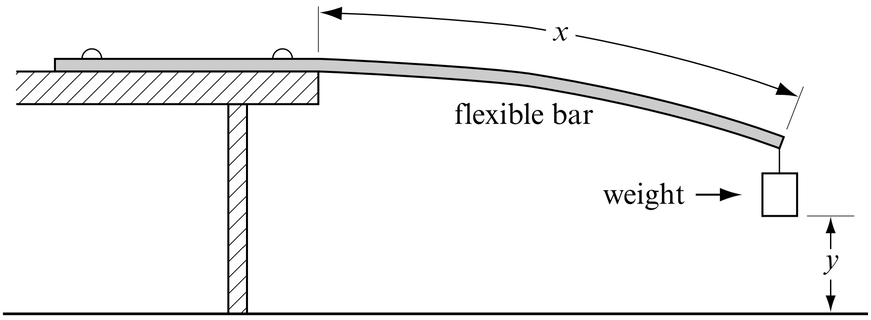
(Question 6 continued)

(iii)



One end of a flexible bar is attached to a table. The other end of the bar extends a distance, x metres, beyond the edge of the table. A weight is then attached to the end of the bar and the distance, y metres, from the bottom of the weight to the floor is measured.

(a) The point where the bar is attached is varied, and x and y are measured, with the following results.

x	1	2	3	4	5
y	1.99	1.91	1.72	1.34	0.77

The value of the coefficient of linear correlation between x and y is calculated as $r_x = -0.950$.

What does the negative value of r_x indicate? *[1 mark]*

(b) In order to seek greater correlation, the square of the projecting length, denoted by u , is calculated, i.e. $u = x^2$. The values of u and y are:

u	1	4	9	16	25
y	1.99	1.91	1.72	1.34	0.77

(i) Calculate the value of the correlation coefficient, r_u , between u and y .

(ii) What is indicated by the fact that the value of r_u lies between -0.950 and -1 ? *[3 marks]*

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(Question 6(iii) continued)

(c) To see if a still greater correlation can be found, the cube of the projecting length, denoted by v, is calculated. The coefficient of linear correlation between v and y is almost exactly equal to –1 .

(i) What may be concluded from this last result?

(ii) Obtain the least squares regression line relating y and v in the form $y = av + b$.

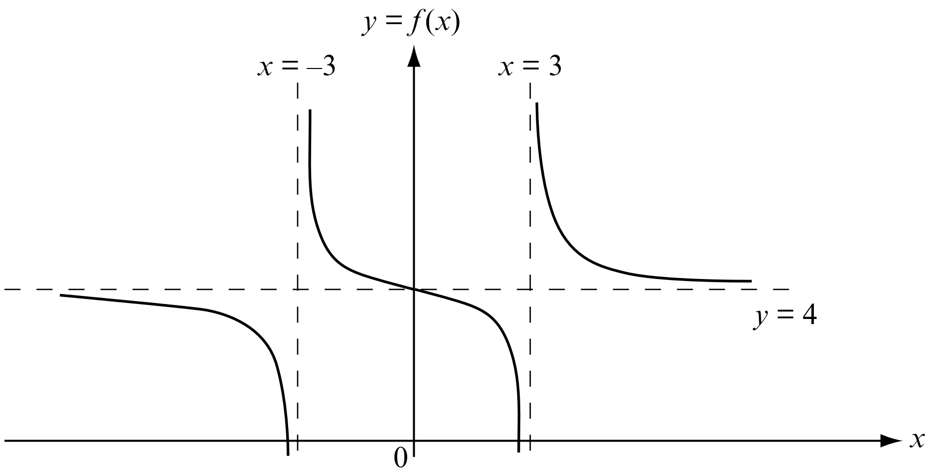
(iii) What minimum value of x would be required for the weight to touch the floor?

[5 marks]

Further Calculus

7. *[Maximum mark: 30]*

(i) The graph of a function of the form $f: x \mapsto a + \dfrac{2x}{x^2 - b^2}$ is given below. The equations of the asymptotes are marked on the graph.



(a) Write down the values of a and b. *[2 marks]*

(b) Show that $f'(x) = -\dfrac{2(x^2+9)}{(x^2-9)^2}$. *[3 marks]*

(c) Use the result of part (b) to explain why the graph of f has no stationary points, (ie points where the derivative is zero). *[2 marks]*

(d) The second derivative of the function f equals 0 at one point only. Use the graph to give the coordinates of this point. (Do not obtain an expression for the second derivative.) *[1 mark]*

(e) Find a function $F(x)$ such that $F'(x) = f(x)$. *[3 marks]*

(f) **Hence**, evaluate $\int_4^{11} f(x)\,\mathrm{d}x$ expressing your answer in the form $p + q \ln 2$, where $p, q \in \mathbb{N}$. *[3 marks]*

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(Question 7 continued)

(ii) The equation $2x^3 - 9x^2 + 11 = 0$ has a solution which lies between $x = 4$ and $x = 4.5$.

(a) Show that this equation may be rearranged in the form $x = g(x)$ where

$$g(x) = \sqrt[3]{\left(\frac{9x^2 - 11}{2}\right)} .$$

[2 marks]

(b) Using the iterative process $x_{n+1} = g(x_n)$ with $x_0 = 4.25$, write down the values of x_1 and x_2 to the accuracy displayed on your calculator. *[2 marks]*

(c) Write down the solution, giving your answer correct to **six** significant figures. *[2 marks]*

(iii) (a) Use the fact that $\int 3^x \mathrm{d}x = \frac{3^x}{\ln 3} + C$, where C is a constant, to show that $\frac{8}{\ln 3}$ is the **exact** value of $\int_0^2 3^x \mathrm{d}x$. *[2 marks]*

(b) The above integral is approximated using the trapezium rule with two intervals (three ordinates). What is the value of this approximation? *[3 marks]*

(c) Calculate the error (the difference from the exact value of $\frac{8}{\ln 3}$) when the approximation in part (b) is used. Give your answer correct to **two** significant figures. *[2 marks]*

(d) When the trapezium rule with four intervals is used, the error is approximately 0.18 . The error is reduced by the same factor each time the number of intervals is doubled. The number of intervals is restricted to the set $\{2 , 4 , 8 , 16 , 32 , \ldots\}$. Use this information to calculate the minimum number of intervals required for the error to be less than 0.001 . *[3 marks]*

Further Geometry

8. *[Maximum mark: 30]*

(i) The unit square is defined by the origin, O, and the vertices A $(1, 0)$, B $(1, 1)$ and C $(0, 1)$. The transformation represented by the matrix $\boldsymbol{M}$ transforms OABC to OA′B′C′, where $\boldsymbol{M} = \begin{pmatrix} -3 & -4 \\ 4 & -3 \end{pmatrix}$.

(a) Write down the coordinates of A′ and C′. *[2 marks]*

(b) Determine

(i) $|\overrightarrow{OA'}|$;

(ii) $|\overrightarrow{OC'}|$;

(iii) $\overrightarrow{OA'} \cdot \overrightarrow{OC'}$. *[4 marks]*

(c) Give a geometric description of the transformed figure OA′B′C′. *[2 marks]*

The transformation represented by the matrix $\boldsymbol{M}$ may be considered as an enlargement followed by a rotation, both transformations having $(0, 0)$ as their centre.

(d) Write down the matrix $\boldsymbol{P}$ representing the enlargement. *[1 mark]*

(e) Using the fact that $\boldsymbol{RP} = \boldsymbol{M}$, write down the matrix $\boldsymbol{R}$ representing the rotation. *[2 marks]*

(f) Given that the angle of rotation for $\boldsymbol{R}$ is θ, give the value of θ accurate to the nearest $0.1°$. *[2 marks]*

(ii) An unknown 2×2 matrix, $\boldsymbol{X}$, satisfies the matrix equation

$$\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 4 & -4 \\ -1 & 12 \end{pmatrix}.$$

(a) Write down the inverse of the matrix $\begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix}$. *[2 marks]*

(b) Hence, express $\boldsymbol{X}$ as a product of two matrices. *[2 marks]*

(c) Evaluate $\boldsymbol{X}$. *[2 marks]*

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(Question 8 continued)

(iii) A transformation, $\boldsymbol{S}$, is described by

$$\mathbf{S}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix}.$$

(a) Find the image under $\boldsymbol{S}$ of

(i) $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$;

(ii) $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$. *[4 marks]*

(b) Given that $\boldsymbol{S}$ is a reflection, give the equation of the line of reflection in the form: $ax + by + c = 0$, where $a, b, c \in \mathbb{Z}$. *[3 marks]*

(iv) The matrix $\boldsymbol{N}$ represents an anticlockwise rotation of 90° about $(0, 0)$.

(a) Find the matrix $\boldsymbol{N}$. *[1 mark]*

(b) Transformation $\boldsymbol{Q}$ is an anticlockwise rotation of 90° about the point $(3, 2)$. $\boldsymbol{Q}$ may be written in the form

$$\boldsymbol{Q}: \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \boldsymbol{N}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h \\ k \end{pmatrix}.$$

Find the values of h and k. *[3 marks]*