INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICAL METHODS
STANDARD LEVEL
PAPER 2

Friday 3 November 2000 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures, as appropriate.

- Write the make and model of your calculator on the front cover of your answer booklets *e.g.* Casio *fx-7400G*, Sharp EL-9400, Texas Instruments TI-80.

You are advised to start each new question on a new page. A correct answer with **no** *indication of the method used will usually receive* **no** *marks. You are therefore advised to show your working. In particular, where graphs from a graphic display calculator are being used to find solutions, you should sketch these graphs as part of your answer.*

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 10]*

A survey is carried out to find the waiting times for 100 customers at a supermarket.

waiting time (seconds)	number of customers
0 – 30	5
30 – 60	15
60 – 90	33
90 – 120	21
120 – 150	11
150 – 180	7
180 – 210	5
210 – 240	3

(a) Calculate an estimate for the mean of the waiting times, by using an appropriate approximation to represent each interval. *[2 marks]*

(b) Construct a cumulative frequency table for these data. *[1 mark]*

(c) Use the cumulative frequency table to draw, on graph paper, a cumulative frequency graph, using a scale of 1 cm per 20 seconds waiting time for the horizontal axis and 1 cm per 10 customers for the vertical axis. *[4 marks]*

(d) Use the cumulative frequency graph to find estimates for the median and the lower and upper quartiles. *[3 marks]*

2. *[Maximum mark: 15]*

A rock-climber slips off a rock-face and falls vertically. At first he falls freely, but after 2 seconds a safety rope slows him down. The height h metres of the rock-climber after t seconds of the fall is given by:

$$h = 50 - 5t^2, \quad 0 \le t \le 2$$
$$h = 90 - 40t + 5t^2, \quad 2 \le t \le 5$$

(a) Find the height of the rock-climber when $t = 2$. *[1 mark]*

(b) Sketch a graph of h against t for $0 \le t \le 5$. *[4 marks]*

(c) Find $\dfrac{\mathrm{d}h}{\mathrm{d}t}$ for:

(i) $0 \le t \le 2$

(ii) $2 \le t \le 5$ *[2 marks]*

(d) Find the velocity of the rock-climber when $t = 2$. *[2 marks]*

(e) Find the times when the velocity of the rock-climber is zero. *[3 marks]*

(f) Find the minimum height of the rock-climber for $0 \le t \le 5$. *[3 marks]*

3. *[Maximum mark: 15]*

In this question you should note that radians are used throughout.

(a) (i) Sketch the graph of $y = x^2 \cos x$, for $0 \le x \le 2$ making clear the approximate positions of the positive x-intercept, the maximum point and the end-points.

(ii) Write down the **approximate** coordinates of the positive x-intercept, the maximum point and the end-points. *[7 marks]*

(b) Find the **exact value** of the positive x-intercept for $0 \le x \le 2$. *[2 marks]*

Let R be the region in the first quadrant enclosed by the graph and the x-axis.

(c) (i) Shade R on your diagram.

(ii) Write down an integral which represents the area of R. *[3 marks]*

(d) Evaluate the integral in part (c)(ii), either by using a graphic display calculator, or by using the following information.

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^2 \sin x + 2x \cos x - 2 \sin x) = x^2 \cos x.$$ *[3 marks]*

4. *[Maximum mark: 20]*

In this question the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ km represents a displacement due east, and the vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ km represents a displacement due north.

The point (0 , 0) is the position of *Shipple Airport*. The position vector $\boldsymbol{r}_1$ of an aircraft *Air One* is given by

$$\boldsymbol{r}_1 = \begin{pmatrix} 16 \\ 12 \end{pmatrix} + t\begin{pmatrix} 12 \\ -5 \end{pmatrix},$$

where t is the time in minutes **since** 12:00 .

(a) Show that the *Air One* aircraft

(i) is 20 km from *Shipple Airport* at 12:00 ;

(ii) has a speed of 13 km/min. *[4 marks]*

(b) Show that a cartesian equation of the path of *Air One* is:

$$5x + 12y = 224 .$$ *[3 marks]*

The position vector $\boldsymbol{r}_2$ of an aircraft *Air Two* is given by

$$\boldsymbol{r}_2 = \begin{pmatrix} 23 \\ -5 \end{pmatrix} + t\begin{pmatrix} 2.5 \\ 6 \end{pmatrix},$$

where t is the time in minutes **since** 12:00 .

(c) Find the angle between the paths of the two aircraft. *[4 marks]*

(d) (i) Find a cartesian equation for the path of *Air Two*.

(ii) Hence find the coordinates of the point where the two paths cross. *[5 marks]*

(e) Given that the two aircraft are flying at the same height, show that they do not collide. *[4 marks]*

5. *[Maximum mark: 10]*

Initially a tank contains 10 000 litres of liquid. At the time $t = 0$ minutes a tap is opened, and liquid then flows out of the tank. The volume of liquid, V litres, which remains in the tank after t minutes is given by

$$V = 10\,000\,(0.933^t)\,.$$

(a) Find the value of V after 5 minutes. *[1 mark]*

(b) Find how long, to the nearest second, it takes for half of the initial amount of liquid to flow out of the tank. *[3 marks]*

(c) The tank is regarded as effectively empty when 95% of the liquid has flowed out. Show that it takes almost three-quarters of an hour for this to happen. *[3 marks]*

(d) (i) Find the value of $10\,000 - V$ when $t = 0.001$ minutes.

(ii) Hence or otherwise, estimate the initial flow rate of the liquid. Give your answer in litres per minute, correct to two significant figures. *[3 marks]*

SECTION B

*Answer **one** question from this section.*

Statistical Methods

6. *[Maximum mark: 30]*

(i) An urban highway has a speed limit of 50 km h^{-1}. It is known that the speeds of vehicles travelling on the highway are normally distributed, with a standard deviation of 10 km h^{-1}, and that 30% of the vehicles using the highway exceed the speed limit.

(a) Show that the mean speed of the vehicles is approximately 44.8 km h^{-1}. *[3 marks]*

The police conduct a 'Safer Driving' campaign intended to encourage slower driving, and want to know whether the campaign has been effective. It is found that a sample of 25 vehicles has a mean speed of 41.3 km h^{-1}.

(b) Given that the null hypothesis is

H_0 : the mean speed has been unaffected by the campaign

state H_1 , the alternative hypothesis. *[1 mark]*

(c) State whether a one-tailed or two-tailed test is appropriate for these hypotheses, and explain why. *[2 marks]*

(d) Has the campaign had significant effect at the 5% level? *[4 marks]*

(This question continues on the following page)

(Question 6 continued)

(ii) A group of 100 people have a certain virus which can develop into a fatal disease. Out of the 100 people, 50 are given treatment with a new drug. Of the untreated people 31 develop the disease, but of the treated people only 20 develop the disease. The results of this treatment are summarised in the following table.

	Develop disease	Do not develop disease
Untreated	31	19
Treated	20	30

(a) Assuming that the new drug has no effect, construct a table of expected frequencies for this group. *[2 marks]*

(b) Apply the Yates continuity correction to the original data. Present your answer in the form of a table. *[2 marks]*

(c) Calculate χ^2_{calc} for these data. *[3 marks]*

(d) Do these data indicate that the administration of the drug significantly affects, at the 5% level, the development of the disease? *[3 marks]*

(iii) Sophie keeps a record of her marks for her weekly assignments in Mathematical Methods.

time in weeks, (x)	1	2	3	4
percentage mark obtained, (y)	55	57	56	59

(a) Write down the values of the mean and standard deviation for x and y. *[2 marks]*

(b) Show that the mean of xy is 143.25. *[2 marks]*

(c) Show that the product-moment correlation coefficient is 0.83 (to two decimal places). *[2 marks]*

(d) Use the method of least squares to find the y on x regression line $y = a + bx$ which best fits these observations. *[4 marks]*

Further Calculus

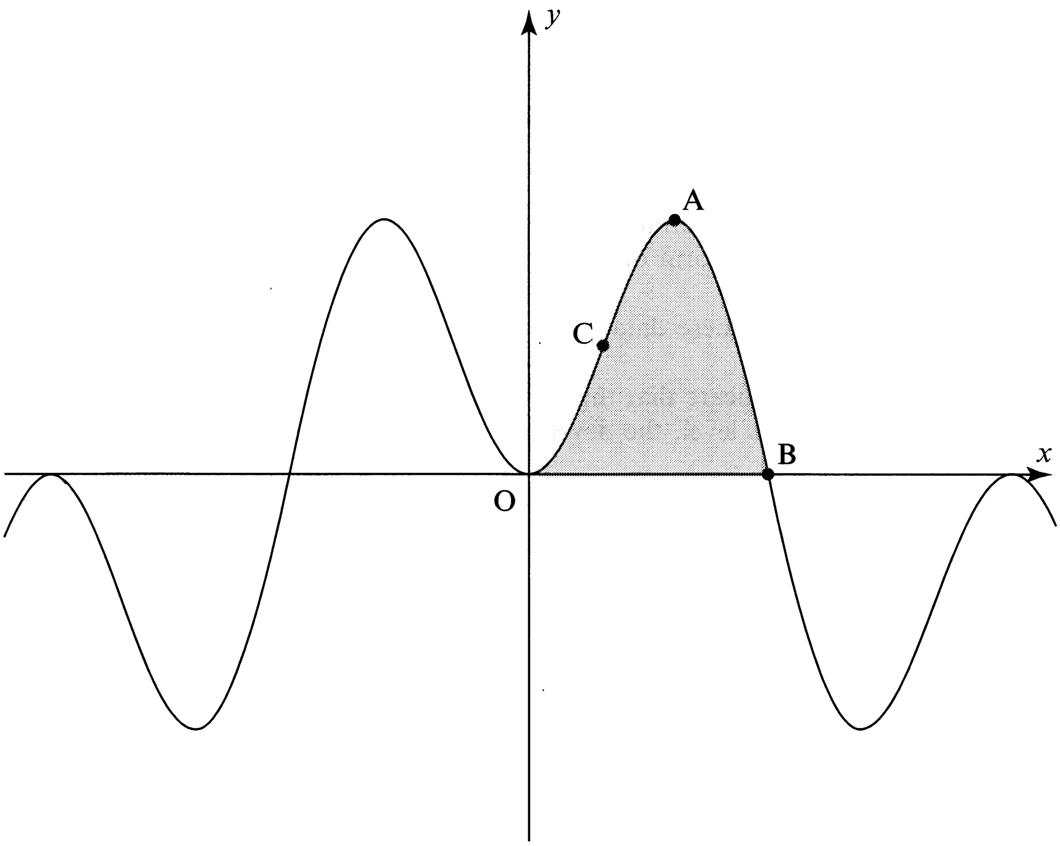
7. *[Maximum mark: 30]*

(i) **In this part of the question, radians are used throughout.**

The function f is given by

$$f(x) = (\sin x)^2 \cos x .$$

The following diagram shows part of the graph of $y = f(x)$.



The point A is a maximum point, the point B lies on the x-axis, and the point C is a point of inflexion.

(a) Give the period of f. *[1 mark]*

(b) From consideration of the graph of $y = f(x)$, find **to an accuracy of one significant figure** the range of f. *[1 mark]*

(c) (i) Find $f'(x)$.

(ii) Hence show that at the point A , $\cos x = \sqrt{\frac{1}{3}}$.

(iii) Find the exact maximum value. *[9 marks]*

(This question continues on the following page)

(Question 7 continued)

(d) Find the exact value of the x-coordinate at the point B . *[1 mark]*

(e) (i) Find $\int f(x)\,dx$.

(ii) Find the area of the shaded region in the diagram. *[4 marks]*

(f) Given that $f''(x) = 9(\cos x)^3 - 7\cos x$, find the x-coordinate at the point C . *[4 marks]*

(ii) The equation $e^{-x} = x^2$ has a positive solution.

(a) Show that this solution lies between $x = 0.5$ and $x = 1$. *[2 marks]*

This solution is to be found using fixed-point iteration. To do this the equation is re-arranged in the form $x = g(x)$, where $g(x) = \sqrt{e^{-x}}$.

(b) (i) Use the iterative formula

$$x_{n+1} = g(x_n)$$

with $x_0 = 0.7$, to obtain the first four decimal digits of x_1 , x_2 , and x_3 .

(ii) Find the solution that lies between 0.5 and 1 to an accuracy of six significant figures. *[4 marks]*

There is another possible re-arrangement of this equation in the form $x = g^{-1}(x)$.

(c) (i) Find this re-arrangement, and write down the corresponding iterative formula.

(ii) Use this iterative formula with $x_0 = 0.7$ to obtain the first four decimal digits of x_1 , x_2 and x_3 .

(iii) Comment on the behaviour of this iteration process. *[4 marks]*

Further Geometry

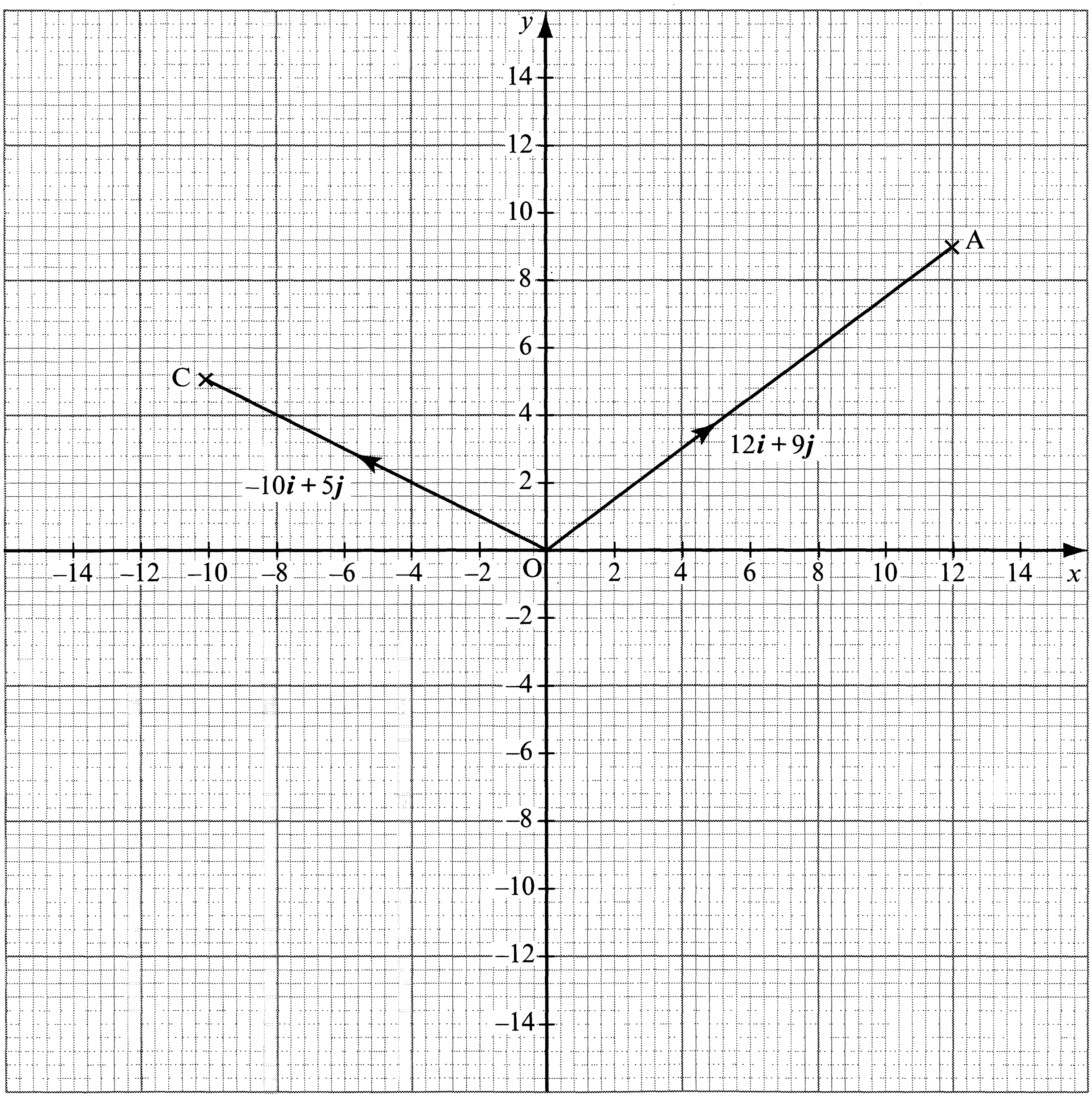
8. *[Maximum mark: 30]*

(i) Transformation $\boldsymbol{P}$ is reflection in the line $x + y = 0$, and transformation $\boldsymbol{Q}$ is rotation of $+90°$ about the origin $(0, 0)$.

(a) Describe fully the single transformation equivalent to $\boldsymbol{PQ}$. *[2 marks]*

(b) The transformation $\boldsymbol{X}$ satisfies $\boldsymbol{XQ} = \boldsymbol{P}$. Describe $\boldsymbol{X}$ fully. *[3 marks]*

(ii) The diagram shows the point A with position vector $\overrightarrow{OA} = 12\boldsymbol{i} + 9\boldsymbol{j}$, and the point C with position vector $\overrightarrow{OC} = -10\boldsymbol{i} + 5\boldsymbol{j}$.



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(Question 8 continued)

(a) Given that quadrilateral OABC is a parallelogram, use the diagram to find the position vector of B . *[1 mark]*

A shear transformation with matrix $\boldsymbol{M}$ has invariant line with equation $4y - 3x = 0$. Under this transformation, the image of C is C′ , whose coordinates are (–6 , 8).

(b) (i) Explain why A is an invariant point of this transformation.

(ii) Show that [CC′] is parallel to the invariant line. *[3 marks]*

(c) (i) Use the results of part (b) to obtain the equation

$$\boldsymbol{M}\begin{pmatrix} 12 & -10 \\ 9 & 5 \end{pmatrix} = \begin{pmatrix} 12 & -6 \\ 9 & 8 \end{pmatrix}.$$

(ii) Hence find $\boldsymbol{M}$.

(iii) Show that det $\boldsymbol{M}$ is 1 , and give a geometric explanation for this. *[8 marks]*

(d) Given that [OC′] is perpendicular to [OA], find the area of the parallelogram OABC . *[3 marks]*

(iii) Transformations $\boldsymbol{R}$ and $\boldsymbol{S}$ are defined as follows:

$\boldsymbol{R}$: reflection in the line with equation $x + y = 6$
$\boldsymbol{S}$: reflection in the line with equation $y = 1$.

(a) Show that $\boldsymbol{S} : (x , y) \mapsto (x , 2 - y)$. *[4 marks]*

(b) Given that $\boldsymbol{R} : (x , y) \mapsto (6 - y , 6 - x)$, show that

$$\boldsymbol{SR} : (x , y) \mapsto (6 - y , x - 4) .$$ *[2 marks]*

(c) The invariant point of the transformation $\boldsymbol{SR}$ has coordinates (x_0 , y_0) . Using the fact that $\boldsymbol{SR} : (x_0 , y_0) \mapsto (x_0 , y_0)$, find the values of x_0 and y_0 . *[2 marks]*

(d) Hence describe $\boldsymbol{SR}$ fully as a single rotation. *[2 marks]*