INTERNATIONAL BACCALAUREATE

MATHEMATICS

Higher Level

Specimen Paper
(corresponding to examinations from May 1995 onwards)

Paper 2

2 hours 30 minutes

This examination paper consists of two sections, Section A and Section B.

Section A consists of four questions.

Section B consists of four questions.

The maximum total mark for Section A is 80.

The maximum mark for each question in Section B is 40.

The maximum mark for this paper is 120.

This examination paper consists of 11 pages.

INSTRUCTIONS TO CANDIDATES

DO NOT open this examination paper until instructed to do so.

Answer all FOUR questions from Section A and ONE question from Section B.

Unless otherwise stated in the question, all numerical answers which are not exact should be given correct to three significant figures.

EXAMINATION MATERIALS

Required/Essential:

- IB Statistical Tables
- Millimetre square graph paper
- Electronic calculator
- Ruler and compasses

Allowed/Optional:

- A simple translating dictionary for candidates not working in their own language

FORMULAE

Trigonometrical identities :

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\text{If } \tan\frac{\theta}{2} = t \text{ then } \sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}$$

Integration by parts :

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Standard integrals :

$$\int \frac{\mathrm{d}x}{x^2 + a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2 - x^2}} = \arcsin\frac{x}{a} + c \quad (|x| < a)$$

Statistics : If $(x_1, x_2, \ldots, x_n)$ occur with frequencies $(f_1, f_2, \ldots, f_n)$ then the mean m and standard derivation s are given by

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i} \qquad s = \sqrt{\frac{\Sigma f_i (x_i - m)^2}{\Sigma f_i}}, \quad i = 1, 2, \ldots n$$

Binomial distribution :

$$p_x = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \ldots n$$

A correct answer with **no** *indication of the method used will normally receive* **no** *marks. You are therefore advised to show your working.*

SECTION A

Answer all FOUR questions from this section.

1. *[Maximum mark: 18]*

(i) Consider the lines

$$L: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 11 \\ -2 \\ 3 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \\ -1 \end{pmatrix} \quad \text{and } M: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 5 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$$

(a) Find the acute angle between a vector in the direction of L and a vector in the direction of M. *[3 marks]*

(b) Find a vector $\vec{n}$ that is perpendicular to both lines. *[2 marks]*

(c) Find the equation of the plane P that contains the line L and is perpendicular to $\vec{n}$. *[2 marks]*

(d) Show that a vector $\overrightarrow{AB}$, where $A \in L$ and $B \in M$, is of the form

$$\begin{pmatrix} -11 - 4s + 2t \\ -1 + s + t \\ 2 + s + 2t \end{pmatrix}.$$

[2 marks]

(e) Find the values of s and t such that $\overrightarrow{AB}$ is parallel to $\vec{n}$ and hence, or otherwise, find the distance between the lines L and M. *[4 marks]*

(ii) Given that (k, l) lies on the line $y = -2x$,

(a) find l in terms of k; *[1 mark]*

(b) find the equation of the line containing the points (w, z), where

$$\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -6 & 0 \end{pmatrix}\begin{pmatrix} k \\ l \end{pmatrix}.$$

[4 marks]

2. *[Maximum mark: 24]*

(i) (a) Sketch the curve given by $cy = c^2x^2 - 6$, where c is a positive real number. On the same axes sketch the lines $y = x$ and $y = -x$. Clearly identify the coordinates of points of intersection between the curve, the lines and the axes.

[4 marks]

(b) Find the area of the region above the curve $cy = c^2x^2 - 6$, below the line $y = x$ and above the line $y = -x$. How does this area behave as c gets larger?

[4 marks]

(c) Find the volume of the solid that is formed if that part of the region which is above the x axis is rotated about the y **axis**.

[6 marks]

(ii) At a particular moment a light aircraft is flying west at 360 kmh^{-1} and passes directly above a bus travelling southwest at 80 kmh^{-1} on a level road. The aircraft is flying at an altitude of 2 km.

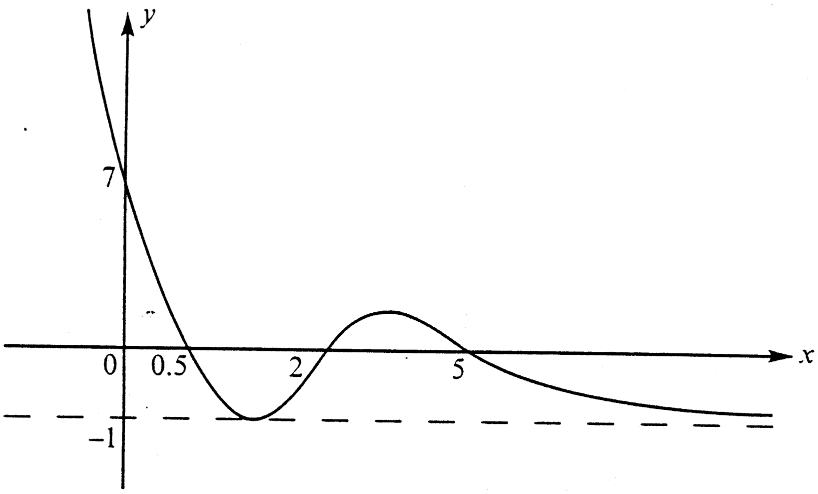
How fast, to the nearest km, is the distance between the aircraft and the bus changing 30 seconds later. (A clear diagram will be very useful in solving this problem.)

[10 marks]

3. *[Maximum mark 22]*

(i) The curve of $y = f(x)$, $x \in \mathbb{R}$, is sketched below.

The intercepts on each axis are marked; also, $(1, -1)$ is a local minimum, $(3, 0.6)$ is a local maximum, and $f(x) \geqslant -1$ for all $x \in \mathbb{R}$.



(a) How many points of inflection are there? *[1 mark]*

(b) State the number of real solutions of the equation $f(x) = k$, $k \in \mathbb{R}$, in the following cases:

(i) $k < -1$

(ii) $k = -1$

(iii) $-1 < k < 0.6$

(iv) $k > 0.6$ *[4 marks]*

(c) For the graph of $y = \dfrac{1}{f(x)}$, state the y-intercept, and the equations of **all** asymptotes. *[4 marks]*

(d) Hence sketch the graph of $y = \dfrac{1}{f(x)}$. *[4 marks]*

(ii) (a) Write $2 \sin x + 3 \cos x$ in the form $R \sin (x + \alpha)$. *[3 marks]*

(b) Hence state, in terms of e, the maximum and minimum values of $e^{2 \sin x + 3 \cos x}$. *[2 marks]*

(c) Solve the equation $e^{2 \sin x + 3 \cos x} = 1$ for $0° < x < 360°$. *[4 marks]*

4. *[Maximum mark: 16]*

(i) A fair die with faces numbered 1 to 6 is tossed 144 times. Write down, but do not evaluate, an expression for the probability that the number 5 will appear on the uppermost face 20 times or fewer. Find an approximation to this probability, explaining clearly how you do this.

If repeated samples of 144 throws were made, in how many samples, approximately, would the number 5 appear 20 times or fewer? *[8 marks]*

(ii) (a) A box contains g green counters and y yellow counters. A counter is drawn from the box, the colour is noted, and the counter is then replaced in the box. This procedure is repeated n times. What is the probability, in terms of n, g and y, that x green counters are drawn during the process?

(b) What is the probability that x green counters were drawn during the above process if the counter is not replaced each time? Explain your answer clearly.

(c) An inspector checks boxes of light bulbs, each box containing twenty five bulbs, by selecting four bulbs at random from each box. If all four bulbs work the box is accepted, but if one fails to work the whole box is checked. What is the probability that a box will be accepted without further investigation even though it contains seven faulty bulbs? *[8 marks]*

SECTION B

Answer ONE question from this section.

Abstract Algebra

5. *[Maximum mark: 40]*

Let M denote the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ in which the entries are members of $\mathbb{Z}_7$, the set of integers modulo 7, and $a \neq 0$.

(a) Given that matrix multiplication is associative, show that M is a group with respect to matrix multiplication modulo 7. You may state, without proof, any properties of $\mathbb{Z}_7$ that you use. *[12 marks]*

(b) What is the order of the two members of M,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}?$$

[8 marks]

(c) Answer the following questions about the set M, in each case showing clearly your reasoning.

(i) How many elements are in the set M?

(ii) Are there any possible proper subgroups of M?

(iii) Is the operation of matrix multiplication commutative in the set M? *[10 marks]*

(d) Let C be the subset of M where $a = 1$. Prove that C is cyclic and calculate its order. What are the subgroups of C. *[10 marks]*

Graphs and Trees

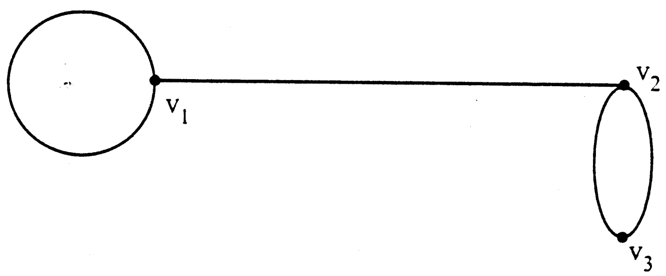
6. *[Maximum mark: 40]*

(i) Given any graph prove that

(a) the sum of the degrees of the vertices equals twice the number of edges;

(b) the number of vertices of odd degree is even. *[10 marks]*

(ii) Given the graph below, write down the adjacency matrix and calculate the total number of walks of length 2 between vertices.



[6 marks]

(iii) (a) Prove, or disprove, that a simple graph with 6 vertices and 11 edges cannot have any isolated vertices. *[4 marks]*

(b) Calculate the maximum number of edges that a simple graph with n vertices can have and yet not be connected. *[4 marks]*

(iv) (a) Define a planar graph. Given that the degree of a face of a planar graph is the number of edges in a walk around the boundary of the face, deduce that the sum of the degrees of all the faces in a planar graph is equal to twice the number of edges in the graph.

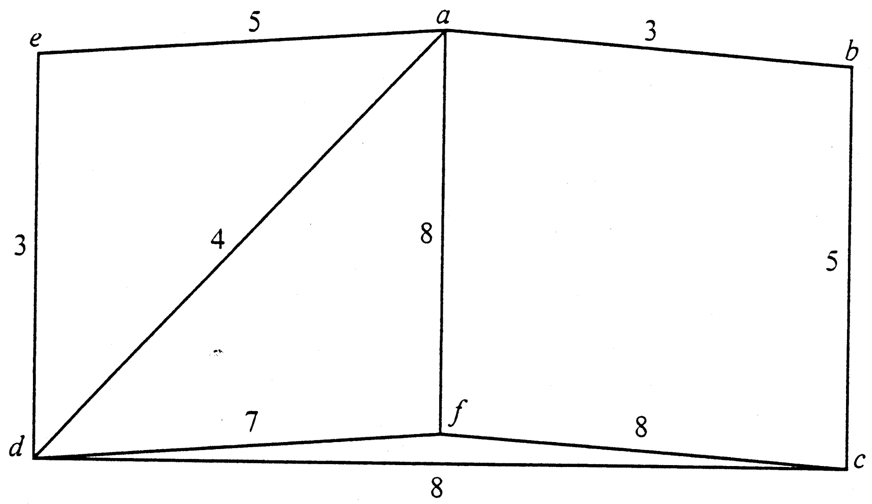
(b) Let G be a planar simple graph with n vertices, where n is at least 3, and m edges. Show that $m \leqslant 3n - 6$.

(c) Deduce that a graph that is obtained by joining every vertex of a regular pentagon to every other vertex cannot be planar. *[10 marks]*

(This question continues on page 9)

(Question 6 continued)

(v) Find a minimal spanning tree, and its associated numeric value, for the weighted graph below.



[6 marks]

Statistics

7. *[Maximum mark: 40]*

(a) The number X of breakdown calls received per week by a small garage has a Poisson distribution with mean 5.

Calculate

(i) $p(X = 5)$;

(ii) $p(X \geqslant 7)$. *[6 marks]*

(b) Assuming the numbers of calls in different weeks are independent, what are the mean and variance of the number of calls received in two weeks? *[6 marks]*

(c) For a sample of 100 calls, 57 were classified as being due to electrical faults. Carefully outlining any assumptions made and distributional results used, obtain a 95% confidence interval for the proportion of breakdowns due to electrical faults. *[12 marks]*

(d) The garage uses a computer system for processing the paperwork for breakdowns. A sample of 16 breakdowns had a mean of 5.3 minutes and a standard deviation of 1.1 minutes for processing the paperwork. The installers of the computer system claim that the average processing time is 4.5 minutes for such paperwork. Is there reason to doubt the claim? Again provide all the relevant assumptions and distributional results used in the derivation of your answer. Provide a 95% confidence interval for the mean processing time based on the above sample data. *[16 marks]*

Analysis and Approximation

8. *[Maximum mark: 40]*

(i) (a) Given that the function $f(x)$ can be differentiated $(n + 1)$ times, write down the first $n + 1$ terms of the Taylor series for $f(x)$ about the point a, together with the corresponding remainder term.

Hence obtain the first four terms of the Taylor series of $\sin x$ about $x = \dfrac{\pi}{3}$ and the first four terms of the Maclaurin series for $\sin x$.

What is the maximum error in each case if these quadratics are used to approximate $\sin x$ in the interval $1 \leqslant x \leqslant 2$? *[12 marks]*

(b) Using the case of $n = 0$ in the Taylor series for $f(x)$ required in part (a), obtain the mean value theorem. Interpret this result geometrically.

Use the mean value theorem to prove that

$$|\sin u - \sin v| \leqslant |u - v|$$

for all real values of u and v. *[10 marks]*

(ii) (a) Show the equation $2x = e^{-x^2}$ has exactly one solution between $x = 0.4$ and $x = 0.5$. *[5 marks]*

(b) Use an iterative process of the form $x_{n+1} = f(x_n)$, with $x_0 = 0.4$, to find the solution in part (ii) (a) to three significant figures. Show all the steps in your solution. *[8 marks]*

(c) Draw a suitable sketch to indicate geometrically how the iterations x_{n+1} and x_n obtained in your answer to part (ii) (b) are related. *[5 marks]*