INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

SPEC/510/H(2)

MATHEMATICS

Higher Level

SPECIMEN

Paper 2

3 hours

This examination paper consists of 2 sections, Section A and Section B.
Section A consists of 5 questions.
Section B consists of 5 questions.
The maximum mark for Section A is 70.
The maximum mark for each question in Section B is 30.
The maximum mark for this paper is 100.

INSTRUCTIONS TO CANDIDATES

Do NOT open this examination paper until instructed to do so.

Answer all FIVE questions from Section A and ONE question from Section B.

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

EXAMINATION MATERIALS

Required:
IB Statistical Tables
Millimetre square graph paper
Graphic display calculator
Ruler and compasses

Allowed:
A simple translating dictionary for candidates not working in their own language

A correct answer with **no** *indication of the method used will normally receive* **no** *marks. You are therefore advised to show your working.*

SECTION A

Answer all FIVE questions from this section.

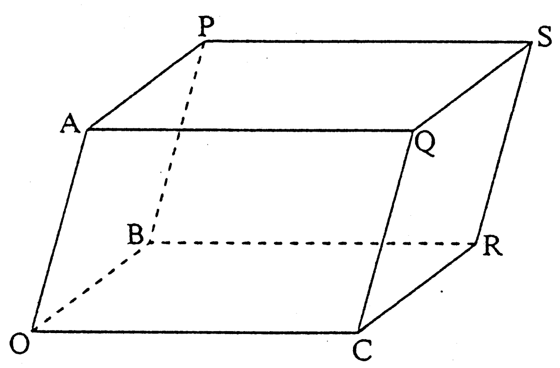
1. *[Maximum mark: 13]*

(i) For the vectors $\boldsymbol{a} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} - \boldsymbol{j} - \boldsymbol{k}$ and $\boldsymbol{c} = \boldsymbol{i} + 2\boldsymbol{j} + 2\boldsymbol{k}$, show that:

(a) $\boldsymbol{a} \times \boldsymbol{b} = -3\boldsymbol{i} - 2\boldsymbol{j} - 4\boldsymbol{k}$ *[2 marks]*

(b) $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = -(\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$ *[3 marks]*

(ii) Three points A, B and C have coordinates (2, 1, −2), (2, −1, −1) and (1, 2, 2) respectively. The vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$, where O is the origin, form three concurrent edges of a parallelepiped OAPBCQSR as shown in the following diagram.



(a) Find the coordinates of P, Q, R and S. *[4 marks]*

(b) Find an equation for the plane OAPB. *[2 marks]*

(c) Calculate the volume, V, of the parallelepiped given that

$$V = |\overrightarrow{OA} \times \overrightarrow{OB} \cdot \overrightarrow{OC}|.$$ *[2 marks]*

2. *[Maximum mark: 11]*

Let R be the matrix which represents a reflection of the plane in the line $y = \sqrt{3}x$. Let S be the matrix which represents an anticlockwise rotation about the origin through angle θ. Let T be the matrix which represents a reflection in the y-axis.

(a) Find the matrices R, S and T. *[4 marks]*

(b) If transformation T is equivalent to transformation R followed by transformation S

(i) write down the matrix equation connecting the matrices R, S and T;

(ii) express S in terms of R, T and/or their inverses and hence find the value of θ, $0 \le \theta < 2\pi$. *[7 marks]*

3. *[Maximum mark: 12]*

(a) Sketch and label the graphs of $f(x) = e^{-x^2}$ and $g(x) = e^{x^2} - 1$ for $0 \le x \le 1$, and shade the region A which is bounded by the graphs and the y-axis. *[3 marks]*

(b) Let the x-coordinate of the point of intersection of the curves $y = f(x)$ and $y = g(x)$ be p.

Without finding the value of p, show that

$$\frac{p}{2} < \text{area of region } A < p.$$

[4 marks]

(c) Find the value of p correct to four decimal places. *[2 marks]*

(d) Express the area of region A as a definite integral and calculate its value. *[3 marks]*

4. *[Maximum mark: 16]*

(a) Sketch the graph of the function

$$C(x) = \cos x + \tfrac{1}{2}\cos 2x$$

for $-2\pi \le x \le 2\pi$. *[5 marks]*

(b) Prove that the function $C(x)$ is periodic and state its period. *[3 marks]*

(c) For what values of x, $-2\pi \le x \le 2\pi$, is $C(x)$ a maximum? *[2 marks]*

(d) Let $x = x_0$ be the smallest positive value of x for which $C(x) = 0$. Find an approximate value of x_0 which is correct to two significant figures. *[2 marks]*

(e) (i) Prove that $C(x) = C(-x)$ for all x. *[2 marks]*

(ii) Let $x = x_1$ be that value of x, $\pi < x < 2\pi$, for which $C(x) = 0$. Find the value of x_1 in terms of x_0. *[2 marks]*

5. *[Maximum mark: 18]*

(a) The function f is defined by

$$f : x \mapsto e^x - 1 - x .$$

(i) Find the minimum value of f. *[2 marks]*

(ii) Prove that $e^x \ge 1 + x$ for all real values of x. *[3 marks]*

(b) Use the principle of mathematical induction to prove that

$$(1+1)(1+\tfrac{1}{2})(1+\tfrac{1}{3})\ldots(1+\tfrac{1}{n}) = n+1$$

for all integers $n \ge 1$. *[6 marks]*

(c) Use the results of parts (a) and (b) to prove that

$$e^{(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n})} > n .$$ *[4 marks]*

(d) Find a value of n for which

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n} > 100 .$$ *[3 marks]*

SECTION B

Answer ONE question from this section.

Statistics

6. *[Maximum mark: 30]*

(i) In a reforested area of pine trees, heights of trees planted in a specific year seem to follow a normal distribution. A sample of 100 such trees is selected to test the validity of this hypothesis. The results of measuring tree heights, to the nearest centimetre, are recorded in the first two columns of the table below.

Height of tree	Observed frequency	Expected frequency
$15 \le h < 45$	6	6
$45 \le h < 75$	11	a
$75 \le h < 105$	15	16
$105 \le h < 135$	20	20
$135 \le h < 165$	18	b
$165 \le h < 195$	14	15
$195 \le h < 225$	10	c
$225 \le h < 255$	6	5
Total	100	100

(a) Describe what is meant by

(i) a goodness of fit test (a complete explanation required); *[3 marks]*

(ii) the level of significance of a hypothesis test. *[1 mark]*

(b) Find the mean and standard deviation of the sample data in the table above. Show how you arrived at your answers. *[4 marks]*

(c) Most of the expected frequencies have been calculated in the third column. (Frequencies have been rounded to the nearest integer, and frequencies in the first and last classes have been extended to include the rest of the data beyond 15 and 225 . Find the values of a , b and c and show how you arrived at your answers. *[4 marks]*

(d) In order to test for the goodness of fit, the test statistic was calculated to be 1.0847 . Show how this was done. *[3 marks]*

(e) State your hypotheses, critical number, decision rule and conclusion (using a 5% level of significance). *[5 marks]*

(This question continues on the following page)

(*Question 6 continued*)

(ii) A language institute runs a summer school for high school students preparing for German examinations. At the beginning of the course, students are given a test of understanding of spoken German. After four weeks, the students are given an identical test.

The table below gives the pretest and post-test scores for 12 students who attended this summer school. The maximum possible score is 50 .

Student	Pretest scores	Post-test scores	Differences in scores
1	44	43	−1
2	41	44	3
3	43	46	3
4	40	44	4
5	32	30	−2
6	43	41	−2
7	47	45	−2
8	29	34	5
9	41	47	6
10	32	39	7
11	41	46	5
12	42	42	0

(a) State why it would not be appropriate to work with the difference between the **means** of these two sets of scores. Hence determine the 90% confidence interval for the mean difference in score as a result of attending the summer school. *[6 marks]*

(b) It is hoped that those attending the summer school improve their listening skills. Carry out an appropriate test on the difference in scores, at the 5% level of significance, to determine whether this is the case. *[4 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) Let $S = \{(x, y) \mid x, y \in \mathbb{R}\}$, and let $(a, b), (c, d) \in S$. Define the relation Δ on S as follows:

$$(a, b)\,\Delta\,(c, d) \Leftrightarrow a^2 + b^2 = c^2 + d^2 .$$

(a) Show that Δ is an equivalence relation. *[4 marks]*

(b) Find all ordered pairs (x, y) where $(x, y)\,\Delta\,(1, 2)$. *[2 marks]*

(c) Describe the partition created by this relation on the (x, y) plane. *[1 mark]*

(ii) Define the function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that

$$f(x, y) = (2y - x, x + y) .$$

(a) Show that f is injective. *[4 marks]*

(b) Show that f is surjective. *[3 marks]*

(c) Show that f has an inverse function. Find this inverse and verify your result. *[5 marks]*

(iii) Consider the following sets:

$$A = \{3^n (\text{mod } 10) \mid n \in \mathbb{N}\}; \quad B = \left\{z_k = \cos\frac{k\pi}{2} + \text{i}\sin\frac{k\pi}{2} \mid k \in \{0, 1, 2, 3\} \text{ and } \text{i} = \sqrt{-1}\right\}$$

(a) Show that B is a group under normal multiplication. *[4 marks]*

(b) Write down the multiplication table for $A(\text{mod}10)$. *[2 marks]*

(c) Find the order of each element of A. *[3 marks]*

(d) Hence, or otherwise, show that the two groups are isomorphic. Find this isomorphism. *[2 marks]*

Discrete Mathematics

8. *[Maximum mark: 30]*

(i) Consider the following homogeneous difference equation:

$$a_n = 5a_{n-1} - 6a_{n-2}\ ,\ n \geq 2\ ,\ a_0 = 1\ ,\ a_1 = 4\ .$$

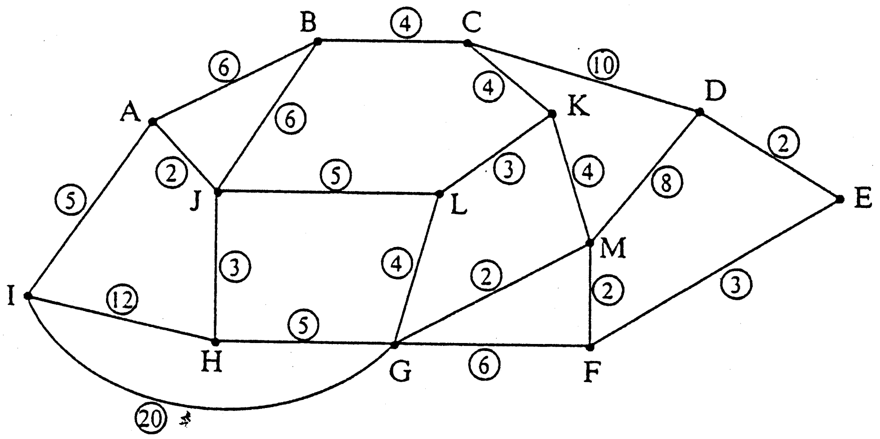
(a) Write down the first five terms in the sequence, starting with a_0. *[2 marks]*

(b) Write down the characteristic polynomial and find its solutions. *[3 marks]*

(c) Hence, find the general solution to the difference equation. *[5 marks]*

(ii) Find two integers x and y such that the greatest common divisor of $(242, 165) = 242x + 165y$. *[5 marks]*

(iii) The diagram shows a weighted graph.



Use Kruskal's algorithm to find a minimum spanning tree as follows:

(a) state the major steps in the algorithm; *[2 marks]*

(b) execute the algorithm showing the steps in your work; *[4 marks]*

(c) sketch the minimum spanning tree found, and write down its weight. *[2 marks]*

(This question continues on the following page)

(*Question 8 continued*)

(iv) Scheduling examinations requires that 'conflict' situations be avoided.

A school has the following 7 examinations to be scheduled. The ✓ means that the two courses have students in common.

	English	French	Biology	Physics	Chemistry	Economics
Mathematics	✓	✓		✓		✓
English		✓	✓	✓		✓
French				✓	✓	✓
Biology				✓	✓	✓
Physics					✓	
Chemistry						✓

(a) Produce a graph of the situation. *[2 marks]*

(b) Find the chromatic number. *[3 marks]*

(c) Produce a possible schedule. *[2 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

(i) Consider the functions $f(x) = x^2 - 1$ and $g(x) = (x-1)e^{x-1}$.

(a) Find the first three derivatives of $g(x)$. *[2 marks]*

(b) There is one point of intersection on the graphs of $f(x)$ and $g(x)$ whose x-coordinate can be found exactly. What is it? *[3 marks]*

(c) Show that $g^{(n)}(1) = n$, for all natural numbers. *[5 marks]*

(d) Find Taylor's expansion for $g(x)$ centred at $x = 1$, up to the term in x^3 . *[3 marks]*

(e) Use the Newton-Raphson method to find the other positive point of intersection of the graphs of the two functions to three decimal places. *[4 marks]*

(f) Estimate the negative point of intersection to three decimal places.

Hence show that the area enclosed between the two curves is approximately 1.116 . *[4 marks]*

(g) Using your answer to part (d), find the polynomial expansion for $g(x) - f(x)$ and then use integration to evaluate the area enclosed between the resulting curve and the x-axis. Compare the answer to part (f) and explain the discrepancy between the two. *[4 marks]*

(ii) Test whether the following is a convergent series:

$$\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{1}{n!} \right).$$

If it is, then find an approximation for the sum to two decimal places; if it is not, explain why this is so. *[5 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) You are given a hyperbola with focus F(3, 0), the y-axis as the directrix and eccentricity 2.

(a) Show that the equation of this hyperbola is $3x^2 + 6x - y^2 = 9$. *[3 marks]*

Let M and N be the points of intersection of the line with equation $3x - 2y + 3 = 0$ and the hyperbola.

(b) Prove that the coordinates of M are (3, 6) and those of N are (–5, –6). *[3 marks]*

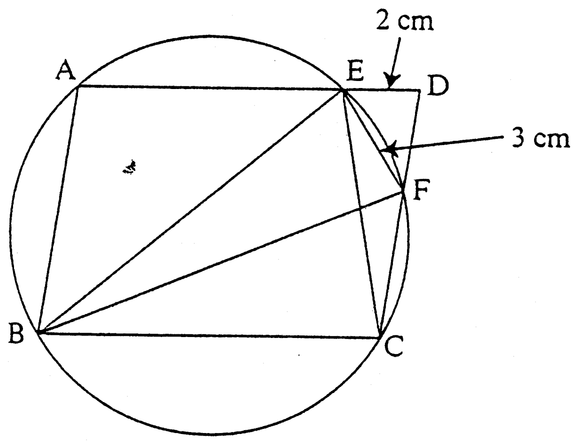
(c) The line (MN) intersects the y-axis at point I. Show that $\frac{\text{NI}}{\text{MI}} = \frac{5}{3}$. *[2 marks]*

(d) Show that $\frac{\text{NF}}{\text{MF}} = \frac{5}{3}$. *[1 mark]*

(e) Hence, or otherwise, show that (FI) bisects $\text{N}\hat{\text{F}}\text{M}$. *[3 marks]*

(f) Generalise the previous result to any conic section with focus F, directrix D and eccentricity e. Prove your conjecture. *[7 marks]*

(ii) ABCD is a parallelogram with ED = 2 cm and EF = 3 cm.



(a) Show that $\triangle$CED is an isosceles triangle. *[4 marks]*

(b) Show that $\frac{\text{ED}}{\text{EF}} = \frac{\text{EC}}{\text{EB}}$. *[4 marks]*

(c) Given that the area of $\triangle$CED is k, find the area of $\triangle$BEF. *[3 marks]*