Mathematics HL

First examinations 2008

DIPLOMA PROGRAMME

MATHEMATICS HL

First examinations 2008

International Baccalaureate Organization

Buenos Aires **Cardiff** **Geneva** **New York** **Singapore**

Diploma Programme
Mathematics HL

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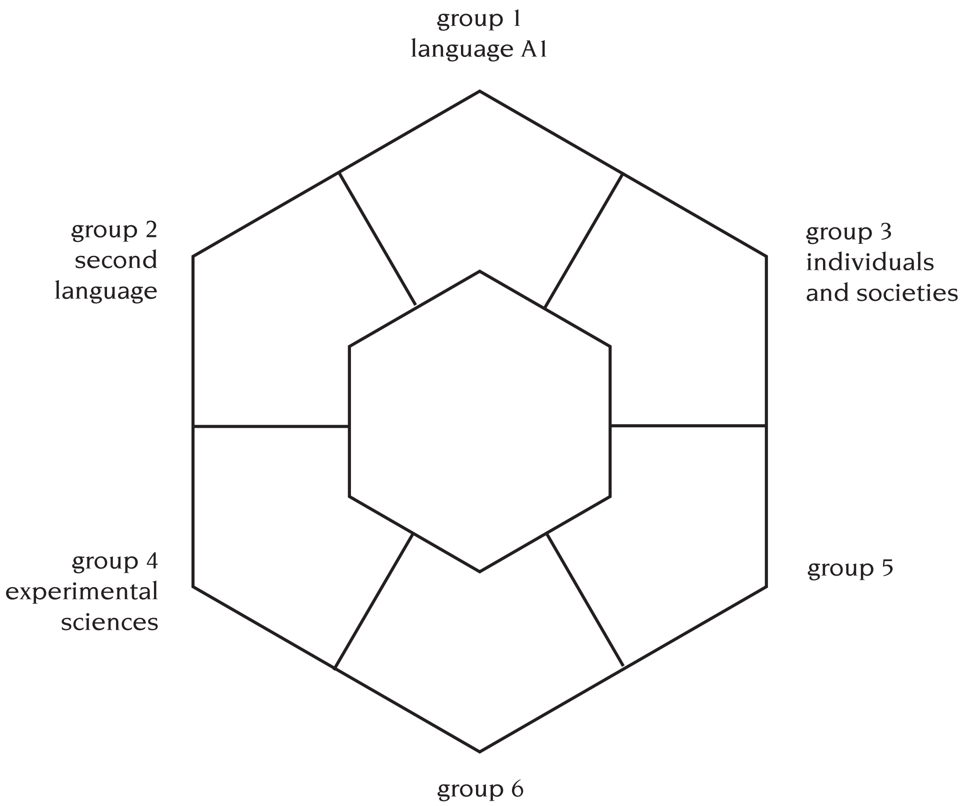
INTRODUCTION

The International Baccalaureate Diploma Programme (DP) is a rigorous pre-university course of studies, leading to examinations, that meets the needs of highly motivated secondary school students between the ages of 16 and 19 years. Designed as a comprehensive two-year curriculum that allows its graduates to fulfill requirements of various national education systems, the DP model is based on the pattern of no single country but incorporates the best elements of many. The DP is available in English, French and Spanish.

The programme model is displayed in the shape of a hexagon with six academic areas surrounding the core. Subjects are studied concurrently and students are exposed to the two great traditions of learning: the humanities and the sciences.



DP students are required to select one subject from each of the six subject groups. At least three and not more than four are taken at higher level (HL), the others at standard level (SL). HL courses represent 240 teaching hours; SL courses cover 150 hours. By arranging work in this fashion, students are able to explore some subjects in depth and some more broadly over the two-year period; this is a deliberate compromise between the early specialization preferred in some national systems and the breadth found in others.

Distribution requirements ensure that the science-orientated student is challenged to learn a foreign language and that the natural linguist becomes familiar with science laboratory procedures. While overall balance is maintained, flexibility in choosing HL concentrations allows the student to pursue areas of personal interest and to meet special requirements for university entrance.

Successful DP students meet three requirements in addition to the six subjects. The interdisciplinary theory of knowledge (TOK) course is designed to develop a coherent approach to learning that transcends and unifies the academic areas and encourages appreciation of other cultural perspectives. The extended essay of some 4,000 words offers the opportunity to investigate a topic of special interest and acquaints students with the independent research and writing skills expected at university. Participation in the creativity, action, service (CAS) requirement encourages students to be involved in creative pursuits, physical activities and service projects in the local, national and international contexts.

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NATURE OF THE SUBJECT

Introduction

The nature of mathematics can be summarized in a number of ways: for example, it can be seen as a well-defined body of knowledge, as an abstract system of ideas, or as a useful tool. For many people it is probably a combination of these, but there is no doubt that mathematical knowledge provides an important key to understanding the world in which we live. Mathematics can enter our lives in a number of ways: we buy produce in the market, consult a timetable, read a newspaper, time a process or estimate a length. Mathematics, for most of us, also extends into our chosen profession: artists need to learn about perspective; musicians need to appreciate the mathematical relationships within and between different rhythms; economists need to recognize trends in financial dealings; and engineers need to take account of stress patterns in physical materials. Scientists view mathematics as a language that is central to our understanding of events that occur in the natural world. Some people enjoy the challenges offered by the logical methods of mathematics and the adventure in reason that mathematical proof has to offer. Others appreciate mathematics as an aesthetic experience or even as a cornerstone of philosophy. This prevalence of mathematics in our lives provides a clear and sufficient rationale for making the study of this subject compulsory within the DP.

Summary of courses available

Because individual students have different needs, interests and abilities, there are four different courses in mathematics. These courses are designed for different types of students: those who wish to study mathematics in depth, either as a subject in its own right or to pursue their interests in areas related to mathematics; those who wish to gain a degree of understanding and competence better to understand their approach to other subjects; and those who may not as yet be aware how mathematics may be relevant to their studies and in their daily lives. Each course is designed to meet the needs of a particular group of students. Therefore, great care should be taken to select the course that is most appropriate for an individual student.

In making this selection, individual students should be advised to take account of the following types of factor.

- Their own abilities in mathematics and the type of mathematics in which they can be successful
- Their own interest in mathematics, and those particular areas of the subject that may hold the most interest for them
- Their other choices of subjects within the framework of the DP
- Their academic plans, in particular the subjects they wish to study in future
- Their choice of career

Teachers are expected to assist with the selection process and to offer advice to students about how to choose the most appropriate course from the four mathematics courses available.

Mathematical studies SL

This course is available at SL only. It caters for students with varied backgrounds and abilities. More specifically, it is designed to build confidence and encourage an appreciation of mathematics in students who do not anticipate a need for mathematics in their future studies. Students taking this course need to be already equipped with fundamental skills and a rudimentary knowledge of basic processes.

Mathematics SL

This course caters for students who already possess knowledge of basic mathematical concepts, and who are equipped with the skills needed to apply simple mathematical techniques correctly. The majority of these students will expect to need a sound mathematical background as they prepare for future studies in subjects such as chemistry, economics, psychology and business administration.

Mathematics HL

This course caters for students with a good background in mathematics who are competent in a range of analytical and technical skills. The majority of these students will be expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology. Others may take this subject because they have a strong interest in mathematics and enjoy meeting its challenges and engaging with its problems.

Further mathematics SL

This course is available at SL only. It caters for students with a good background in mathematics who have attained a high degree of competence in a range of analytical and technical skills, and who display considerable interest in the subject. Most of these students will intend to study mathematics at university, either as a subject in its own right or as a major component of a related subject. The course is designed specifically to allow students to learn about a variety of branches of mathematics in depth and also to appreciate practical applications.

Mathematics HL—course details

This course caters for students with a good background in mathematics who are competent in a range of analytical and technical skills. The majority of these students will be expecting to include mathematics as a major component of their university studies, either as a subject in its own right or within courses such as physics, engineering and technology. Others may take this subject because they have a strong interest in mathematics and enjoy meeting its challenges and engaging with its problems.

The nature of the subject is such that it focuses on developing important mathematical concepts in a comprehensible, coherent and rigorous way. This is achieved by means of a carefully balanced approach. Students are encouraged to apply their mathematical knowledge to solving problems set in a variety of meaningful contexts. Development of each topic should feature justification and proof of results. Students embarking on this course should expect to develop insight into mathematical form and structure, and should be intellectually equipped to appreciate the links between concepts in different topic areas. They should also be encouraged to develop the skills needed to continue their mathematical growth in other learning environments.

The internally assessed component, the portfolio, offers students a framework for developing independence in their mathematical learning through engaging in mathematical investigation and mathematical modelling. Students will be provided with opportunities to take a considered approach to these activities, and to explore different ways of approaching a problem. The portfolio also allows students to work without the time constraints of a written examination and to develop skills in communicating mathematical ideas.

This course is a demanding one, requiring students to study a broad range of mathematical topics through a number of different approaches and to varying degrees of depth. Students wishing to study mathematics in a less rigorous environment should therefore opt for one of the standard level courses, mathematics SL or mathematical studies SL.

AIMS

The aims of all courses in group 5 are to enable students to:

- appreciate the multicultural and historical perspectives of all group 5 courses
- enjoy the courses and develop an appreciation of the elegance, power and usefulness of the subjects
- develop logical, critical and creative thinking
- develop an understanding of the principles and nature of the subject
- employ and refine their powers of abstraction and generalization
- develop patience and persistence in problem solving
- appreciate the consequences arising from technological developments
- transfer skills to alternative situations and to future developments
- communicate clearly and confidently in a variety of contexts.

Internationalism

One of the aims of this course is to enable students to appreciate the multiplicity of cultural and historical perspectives of mathematics. This includes the international dimension of mathematics. Teachers can exploit opportunities to achieve this aim by discussing relevant issues as they arise and making reference to appropriate background information. For example, it may be appropriate to encourage students to discuss:

- differences in notation
- the lives of mathematicians set in a historical and/or social context
- the cultural context of mathematical discoveries
- the ways in which specific mathematical discoveries were made and the techniques used to make them
- how the attitudes of different societies towards specific areas of mathematics are demonstrated
- the universality of mathematics as a means of communication.

OBJECTIVES

Having followed any one of the mathematics courses in group 5, students are expected to know and use mathematical concepts and principles. In particular, students must be able to:

- read, interpret and solve a given problem using appropriate mathematical terms
- organize and present information and data in tabular, graphical and/or diagrammatic forms
- know and use appropriate notation and terminology
- formulate a mathematical argument and communicate it clearly
- select and use appropriate mathematical strategies and techniques
- demonstrate an understanding of both the significance and the reasonableness of results
- recognize patterns and structures in a variety of situations, and make generalizations
- recognize and demonstrate an understanding of the practical applications of mathematics
- use appropriate technological devices as mathematical tools
- demonstrate an understanding of and the appropriate use of mathematical modelling.

SYLLABUS OUTLINE

Mathematics HL

The course consists of the study of seven core topics and one option topic. **Total 240 hrs**

Core syllabus content 190 hrs

Requirements

All topics in the core are compulsory. Students must study all the sub-topics in each of the topics in the syllabus as listed in this guide. Students are also required to be familiar with the topics listed as presumed knowledge (PK).

Topic 1—Algebra	20 hrs
Topic 2—Functions and equations	26 hrs
Topic 3—Circular functions and trigonometry	22 hrs
Topic 4—Matrices	12 hrs
Topic 5—Vectors	22 hrs
Topic 6—Statistics and probability	40 hrs
Topic 7—Calculus	48 hrs

Option syllabus content 40 hrs

Requirements

Students must study all the sub-topics in one of the following options as listed in the syllabus details.

Topic 8—Statistics and probability	40 hrs
Topic 9—Sets, relations and groups	40 hrs
Topic 10—Series and differential equations	40 hrs
Topic 11—Discrete mathematics	40 hrs

Portfolio 10 hrs

Two pieces of work, based on different areas of the syllabus, representing the following two types of tasks:

- mathematical investigation
- mathematical modelling.

SYLLABUS DETAILS

Format of the syllabus

The syllabus to be taught is presented as three columns.

- **Content**: the first column lists, under each topic, the sub-topics to be covered.
- **Amplifications/inclusions**: the second column contains more explicit information on specific sub-topics listed in the first column. This helps to define what is required in terms of preparing for the examination.
- **Exclusions**: the third column contains information about what is not required in terms of preparing for the examination.

Although the mathematics HL course is similar in content to parts of the mathematics SL course, there are differences. In particular, students and teachers are expected to take a more sophisticated approach for mathematics HL, during the course and in the examinations. Where appropriate, guidelines are provided in the second and third columns of the syllabus details (as indicated by the phrase "See SL guide").

Teaching notes and calculator suggestions linked to the syllabus content are contained in a separate publication.

Course of study

Teachers are required to teach all the sub-topics listed for the seven topics in the core, together with all the sub-topics in the chosen option.

The topics in the syllabus do not need to be taught in the order in which they appear in this guide. Similarly, it is not necessary to teach all the topics in the core before starting to teach an option. Teachers should therefore construct a course of study that is tailored to the needs of their students and that integrates the areas covered by the syllabus, and, where necessary, the presumed knowledge (PK).

Integration of portfolio assignments

The two pieces of work for the portfolio, based on the two types of tasks (mathematical investigation and mathematical modelling), should be incorporated into the course of study, and should relate directly to topics in the syllabus. Full details of how to do this are given in the section on internal assessment.

Time allocation

The recommended teaching time for higher level courses is 240 hours. For mathematics HL, it is expected that 10 hours will be spent on work for the portfolio. The time allocations given in this guide are approximate, and are intended to suggest how the remaining 230 hours allowed for the teaching of the syllabus might be allocated. However, the exact time spent on each topic depends on a number of factors, including the background knowledge and level of preparedness of each student. Teachers should therefore adjust these timings to correspond to the needs of their students.

Use of calculators

Students are expected to have access to a graphic display calculator (GDC) at all times during the course. The minimum requirements are reviewed as technology advances, and updated information will be provided to schools. It is expected that teachers and schools monitor calculator use with reference to the calculator policy. Regulations covering the types of calculator allowed are provided in the *Vade Mecum*. Further information and advice is provided in the teacher support material.

There are specific requirements for calculators used by students studying the statistics and probability option.

Mathematics HL information booklet

Because each student is required to have access to a clean copy of this booklet during the examination, it is recommended that teachers ensure students are familiar with the contents of this document from the beginning of the course. The booklet is provided by the IBO and is published separately.

Teacher support materials

A variety of teacher support materials will accompany this guide. These materials will include suggestions to help teachers integrate the use of GDCs into their teaching, guidance for teachers on the marking of portfolios, and specimen examination papers and markschemes. These will be distributed to all schools.

External assessment guidelines

It is recommended that teachers familiarize themselves with the section on external assessment guidelines, as this contains important information about the examination papers. In particular, students need to be familiar with notation the IBO uses and the command terms, as these will be used without explanation in the examination papers.

Presumed knowledge

General

Students are not required to be familiar with all the topics listed as PK **before** they start this course. However, they should be familiar with these topics before they take the **examinations**, because questions assume knowledge of them.

Teachers must therefore ensure that any topics designated as PK that are unknown to their students at the start of the course are included at an early stage. They should also take into account the existing mathematical knowledge of their students to design an appropriate course of study for mathematics HL.

This list of topics is not designed to represent the outline of a course that might lead to the mathematics HL course. Instead, it lists the knowledge, together with the syllabus content, that is essential to successful completion of the mathematics HL course

Students must be familiar with SI (*Système International*) units of length, mass and time, and their derived units.

Topics

Number and algebra

Routine use of addition, subtraction, multiplication and division using integers, decimals and fractions, including order of operations.

Example: $2(3+4\times7)=62$.

Simple positive exponents.

Examples: $2^3=8$; $(-3)^3=-27$; $(-2)^4=16$.

Simplification of expressions involving roots (surds or radicals).

Examples: $\sqrt{27}+\sqrt{75}=8\sqrt{3}$; $\sqrt{3}\times\sqrt{5}=\sqrt{15}$.

Prime numbers and factors, including greatest common factors and least common multiples.

Simple applications of ratio, percentage and proportion, linked to similarity.

Definition and elementary treatment of absolute value (modulus), $|a|$.

Rounding, decimal approximations and significant figures, including appreciation of errors.

Expression of numbers in standard form (scientific notation), that is, $a\times10^k$, $1\le a<10$, $k\in\mathbb{Z}$.

Concept and notation of sets, elements, universal (reference) set, empty (null) set, complement, subset, equality of sets, disjoint sets. Operations on sets: union and intersection. Commutative, associative and distributive properties. Venn diagrams.

Number systems: natural numbers; integers, $\mathbb{Z}$; rationals, $\mathbb{Q}$, and irrationals; real numbers, $\mathbb{R}$.

Intervals on the real number line using set notation and using inequalities. Expressing the solution set of a linear inequality on the number line and in set notation.

The concept of a relation between the elements of one set and between the elements of one set and those of another set. Mappings of the elements of one set onto or into another, or the same, set. Illustration by means of tables, diagrams and graphs.

Basic manipulation of simple algebraic expressions involving factorization and expansion.

Examples: $ab+ac=a(b+c)$; $(a\pm b)^2=a^2+b^2\pm2ab$; $a^2-b^2=(a-b)(a+b)$;
$3x^2+5x+2=(3x+2)(x+1)$; $xa-2a+xb-2b=(x-2)(a+b)$.

Rearrangement, evaluation and combination of simple formulae. Examples from other subject areas, particularly the sciences, should be included.

The linear function $x\mapsto ax+b$ and its graph, gradient and y-intercept.

Addition and subtraction of algebraic fractions with denominators of the form $ax+b$.

Example: $\frac{2x}{3x-1}+\frac{3x+1}{2x+4}$.

The properties of order relations: $<, \le, >, \ge$.

Examples: $a>b, c>0 \Rightarrow ac>bc$; $a>b, c<0 \Rightarrow ac<bc$.

Solution of equations and inequalities in one variable, including cases with rational coefficients.

Example: $\frac{3}{7}-\frac{2x}{5}=\frac{1}{2}(1-x)\Rightarrow x=\frac{5}{7}$.

Solution of simultaneous equations in two variables.

Geometry

Elementary geometry of the plane including the concepts of dimension for point, line, plane and space. Parallel and perpendicular lines, including $m_1 = m_2$, and $m_1 m_2 = -1$. Geometry of simple plane figures. The function $x \mapsto ax + b$: its graph, gradient and y-intercept.

Angle measurement in degrees. Compass directions and bearings. Right-angle trigonometry. Simple applications for solving triangles.

Pythagoras' theorem and its converse.

The Cartesian plane: ordered pairs (x, y), origin, axes. Mid-point of a line segment and distance between two points in the Cartesian plane.

Simple geometric transformations: translation, reflection, rotation, enlargement. Congruence and similarity, including the concept of scale factor of an enlargement.

The circle, its centre and radius, area and circumference. The terms "arc", "sector", "chord", "tangent" and "segment".

Perimeter and area of plane figures. Triangles and quadrilaterals, including parallelograms, rhombuses, rectangles, squares, kites and trapeziums (trapezoids); compound shapes.

Statistics

Descriptive statistics: collection of raw data, display of data in pictorial and diagrammatic forms (for example, pie charts, pictograms, stem and leaf diagrams, bar graphs and line graphs).

Calculation of simple statistics from discrete data, including mean, median and mode.

Core syllabus content

Topic 1—Core: Algebra 20 hrs

Aims

The aim of this section is to introduce students to some basic algebraic concepts and applications.

Details

	Content	Amplifications/inclusions	Exclusions
1.1	Arithmetic sequences and series; sum of finite arithmetic series; geometric sequences and series; sum of finite and infinite geometric series. Sigma notation.	Examples of applications: compound interest and population growth.	
1.2	Exponents and logarithms. Laws of exponents; laws of logarithms. Change of base.	Elementary treatment only is required. $\log_b a = \dfrac{\log_c a}{\log_c b}$.	
1.3	Counting principles, including permutations and combinations. The binomial theorem: expansion of $(a+b)^n$, $n \in \mathbb{N}$.	Simple applications only. The formula for $\binom{n}{r}$ also denoted by ${}^n\mathrm{C}_r$.	Formula for ${}^n\mathrm{P}_r$. Permutations where some objects are identical. **See SL guide**

Topic 1—Core: Algebra (continued)

	Content	Amplifications/inclusions	Exclusions
1.4	Proof by mathematical induction. Forming conjectures to be proved by mathematical induction.		Proof of binomial theorem.
1.5	Complex numbers: the number $\mathrm{i}=\sqrt{-1}$; the terms real part, imaginary part, conjugate, modulus and argument. Cartesian form $z=a+\mathrm{i}b$. Modulus–argument form $z=r\left(\cos\theta+\mathrm{i}\sin\theta\right)$. The complex plane.	Awareness that $z=r\left(\cos\theta+\mathrm{i}\sin\theta\right)$ can be written as $z=r\mathrm{e}^{\mathrm{i}\theta}$ and $z=r\,\mathrm{cis}\,\theta$. The complex plane is also known as the Argand diagram.	Loci in the complex plane.
1.6	Sums, products and quotients of complex numbers.		
1.7	De Moivre's theorem. Powers and roots of a complex number.	Proof by mathematical induction for $n\in\mathbb{Z}^{+}$.	
1.8	Conjugate roots of polynomial equations with real coefficients.		Equations with complex coefficients.

Topic 2—Core: Functions and equations 26 hrs

Aims

The aims of this section are to explore the notion of function as a unifying theme in mathematics, and to apply functional methods to a variety of mathematical situations. It is expected that extensive use will be made of a GDC in both the development and the application of this topic.

Details

	Content	Amplifications/inclusions	Exclusions
2.1	Concept of function $f: x \mapsto f(x)$: domain, range; image (value). Composite functions $f \circ g$; identity function. Inverse function f^{-1}.	On examination papers: if the domain is the set of real numbers then the statement "$x \in \mathbb{R}$" will be omitted. The composite function $(f \circ g)(x)$ is defined as $f(g(x))$. Distinction between one-to-one and many-to-one functions. Domain restriction.	The term "codomain". **See SL guide**
2.2	The graph of a function; its equation $y = f(x)$. Function graphing skills: use of a GDC to graph a variety of functions investigation of key features of graphs solutions of equations graphically.	On examination papers: questions may be set that require the graphing of functions that do not explicitly appear on the syllabus. Identification of asymptotes. May be referred to as roots of equations, or zeros of functions.	

	Content	Amplifications/inclusions	Exclusions
2.3	Transformations of graphs: translations; stretches; reflections in the axes.	Translations: $y = f(x) + b$; $y = f(x - a)$. Stretches: $y = pf(x)$; $y = f(x/q)$. Reflections (in both axes): $y = -f(x)$; $y = f(-x)$. Examples: $y = x^2$ used to obtain $y = 3x^2 + 2$ by a stretch of scale factor 3 in the y-direction followed by a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. $y = \sin x$ used to obtain $y = 3\sin 2x$ by a stretch of scale factor 3 in the y-direction and a stretch of scale factor $\frac{1}{2}$ in the x-direction.	
	The graph of $y = f^{-1}(x)$ as the reflection in the line $y = x$ of the graph of $y = f(x)$.		
	The graph of $y = \frac{1}{f(x)}$ from $y = f(x)$.		
	The graphs of the absolute value functions, $y = \lvert f(x) \rvert$ and $y = f(\lvert x \rvert)$.		
2.4	The reciprocal function $x \mapsto \frac{1}{x}$, $x \neq 0$: its graph; its self-inverse nature.		

	Content	Amplifications/inclusions	Exclusions
2.5	The quadratic function $x \mapsto ax^2 + bx + c$: its graph. Axis of symmetry $x = -\frac{b}{2a}$. The form $x \mapsto a(x-h)^2 + k$. The form $x \mapsto a(x-p)(x-q)$.	Real coefficients only.	
2.6	The solution of $ax^2 + bx + c = 0,\ a \neq 0$. The quadratic formula. Use of the discriminant $\Delta = b^2 - 4ac$.		On examination papers: questions requiring elaborate factorization techniques will not be set.
2.7	The function: $x \mapsto a^x,\ a > 0$. The inverse function $x \mapsto \log_a x,\ x > 0$. Graphs of $y = a^x$ and $y = \log_a x$. Solution of $a^x = b$ using logarithms.	$\log_a a^x = x$; $a^{\log_a x} = x,\ x > 0$.	

Topic 2—Core: Functions and equations (continued)

	Content	Amplifications/inclusions	Exclusions
2.8	The exponential function $x \mapsto e^x$. The logarithmic function $x \mapsto \ln x$, $x > 0$.	$a^x = e^{x \ln a}$. Examples of applications: compound interest, growth and decay.	
2.9	Inequalities in one variable, using their graphical representation. Solution of $g(x) \geq f(x)$, where *f*, *g* are linear or quadratic.	Use of the absolute value sign in inequalities. Analytical solution for simple cases.	On examination papers: questions requiring elaborate manipulation will not be set.
2.10	Polynomial functions. The factor and remainder theorems, with application to the solution of polynomial equations and inequalities.	The graphical significance of repeated roots.	

Topic 3—Core: Circular functions and trigonometry 22 hrs

Aims

The aims of this section are to explore the circular functions, to introduce some important trigonometric identities and to solve triangles using trigonometry.

Details

	Content	Amplifications/inclusions	Exclusions
3.1	The circle: radian measure of angles; length of an arc; area of a sector.	Radian measure may be expressed as multiples of π, or decimals.	
3.2	Definition of $\cos\theta$ and $\sin\theta$ in terms of the unit circle. Definition of $\tan\theta$ as $\frac{\sin\theta}{\cos\theta}$. Definition of $\sec\theta$, $\csc\theta$ and $\cot\theta$. Pythagorean identities: $\cos^2\theta+\sin^2\theta=1$; $1+\tan^2\theta=\sec^2\theta$; $1+\cot^2\theta=\csc^2\theta$.		**See SL guide**
3.3	Compound angle identities. Double angle identities.	Proof of compound angle identities: $\sin(A\pm B)$, $\cos(A\pm B)$, $\tan(A\pm B)$. Proof of double angle identities. Given $\sin\theta$, finding possible values of other ratios (for example $\sin 2\theta$) without finding θ.	**See SL guide**

Topic 3—Core: Circular functions and trigonometry (continued)

	Content	Amplifications/inclusions	Exclusions
3.4	The circular functions $\sin x$, $\cos x$ and $\tan x$; their domains and ranges; their periodic nature; their graphs.	On examination papers: radian measure should be assumed unless otherwise indicated, for example, by $x \mapsto \sin x°$.	
	Composite functions of the form $f(x) = a\sin(b(x+c)) + d$.	Example: $f(x) = 3\tan(4(x-2)) + 1$.	
	The inverse functions $x \mapsto \arcsin x$, $x \mapsto \arccos x$, $x \mapsto \arctan x$; their domains and ranges; their graphs.		On examination papers: questions requiring elaborate analytical treatment of inverse trigonometric functions will not be set.
		Examples of applications: height of tide; Ferris wheel.	
			See SL guide
3.5	Solution of trigonometric equations in a finite interval.	Examples: $2\sin x = 3\cos x$, $0 \le x \le 2\pi$. $2\sin 2x = 3\cos x$, $0° \le x \le 180°$. $2\sin x = \cos 2x$, $-\pi \le x \le \pi$.	The general solution of trigonometric equations.
	Use of trigonometric identities and factorization to transform equations.	Both analytical and graphical methods required.	

	Content	Amplifications/inclusions	Exclusions
3.6	Solution of triangles.		
	The cosine rule: $c^2 = a^2 + b^2 - 2ab\cos C$.		
	The sine rule: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.	The ambiguous case of the sine rule.	
	Area of a triangle as $\frac{1}{2}ab\sin C$.		
		Applications to real-life situations in two dimensions, and simple cases in three dimensions, for example, navigation.	

Topic 4—Core: Matrices 12 hrs

Aims

The aim of this section is to provide an elementary introduction to matrices, a fundamental concept of linear algebra.

Details

	Content	Amplifications/inclusions	Exclusions
4.1	Definition of a matrix: the terms element, row, column and order.	Use of matrices to store data.	Use of matrices to represent transformations.
4.2	Algebra of matrices: equality; addition; subtraction; multiplication by a scalar. Multiplication of matrices. Identity and zero matrices.	Matrix operations to handle or process information.	
4.3	Determinant of a square matrix. Calculation of 2×2 and 3×3 determinants. Inverse of a matrix: conditions for its existence.	The terms singular and non-singular matrices. The result $\det \boldsymbol{AB} = \det \boldsymbol{A} \det \boldsymbol{B}$. Obtaining the inverse of a 3×3 matrix using a GDC.	 Cofactors and minors. Other methods for finding the inverse of a 3×3 matrix.
4.4	Solution of systems of linear equations (a maximum of three equations in three unknowns). Conditions for the existence of a unique solution, no solution and an infinity of solutions.	 These cases can be investigated using row reduction, including the use of augmented matrices. Unique solutions can also be found using inverse matrices.	

Topic 5—Core: Vectors

22 hrs

Aims

The aim of this section is to introduce the use of vectors in two and three dimensions, and to facilitate solving problems involving points, lines and planes.

Details

	Content	Amplifications/inclusions	Exclusions
5.1	Vectors as displacements in the plane and in three dimensions.	Distance between points in three dimensions.	
	Components of a vector; column representation $\boldsymbol{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$.	Components are with respect to the unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ (standard basis).	
	Algebraic and geometric approaches to the following topics:		
	the sum and difference of two vectors; the zero vector, the vector $-\boldsymbol{v}$;	The difference of $\boldsymbol{v}$ and $\boldsymbol{w}$ is $\boldsymbol{v} - \boldsymbol{w} = \boldsymbol{v} + (-\boldsymbol{w})$.	
	multiplication by a scalar, $k\boldsymbol{v}$;		
	magnitude of a vector, $\lvert\boldsymbol{v}\rvert$;		
	unit vectors; base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$;		
	position vectors $\overrightarrow{\text{OA}} = \boldsymbol{a}$.	$\overrightarrow{\text{AB}} = \overrightarrow{\text{OB}} - \overrightarrow{\text{OA}} = \boldsymbol{b} - \boldsymbol{a}$.	

	Content	Amplifications/inclusions	Exclusions
5.2	The scalar product of two vectors, $\boldsymbol{v}\cdot\boldsymbol{w}=\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert\cos\theta$; $\boldsymbol{v}\cdot\boldsymbol{w}=v_1w_1+v_2w_2+v_3w_3$. Algebraic properties of the scalar product. Perpendicular vectors; parallel vectors. The angle between two vectors.	The scalar product is also known as the "dot product" or "inner product". For non-zero perpendicular vectors $\boldsymbol{v}\cdot\boldsymbol{w}=0$; for non-zero parallel vectors $\boldsymbol{v}\cdot\boldsymbol{w}=\pm\lvert\boldsymbol{v}\rvert\lvert\boldsymbol{w}\rvert$.	Projections.
5.3	Vector equation of a line $\boldsymbol{r}=\boldsymbol{a}+\lambda\boldsymbol{b}$. The angle between two lines.	Lines in the plane and in three-dimensional space. Knowledge of the following forms for equations of lines. Parametric form: $x=x_0+\lambda l$, $y=y_0+\lambda m$, $z=z_0+\lambda n$. Cartesian form: $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$.	**See SL guide**
5.4	Coincident, parallel, intersecting and skew lines, distinguishing between these cases. Points of intersection.		

	Content	Amplifications/inclusions	Exclusions
5.5	The vector product of two vectors, $\boldsymbol{v} \times \boldsymbol{w}$. The determinant representation. Geometric interpretation of $\lvert \boldsymbol{v} \times \boldsymbol{w} \rvert$.	The vector product is also known as the cross product. Areas of triangles and parallelograms.	
5.6	Vector equation of a plane $\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b} + \mu \boldsymbol{c}$. Use of normal vector to obtain the form $\boldsymbol{r} \cdot \boldsymbol{n} = \boldsymbol{a} \cdot \boldsymbol{n}$. Cartesian equation of a plane $ax + by + cz = d$.		
5.7	Intersections of: a line with a plane; two planes; three planes. Angle between: a line and a plane; two planes.	Inverse matrix method and row reduction for finding the intersection of three planes. Awareness that three planes may intersect in a point, or in a line, or not at all.	

Topic 6—Core: Statistics and probability 40 hrs

Aims

The aim of this section is to introduce basic concepts. It may be considered as three parts: manipulation and presentation of statistical data (6.1–6.4), the laws of probability (6.5–6.8), and random variables and their probability distributions (6.9–6.11). It is expected that most of the calculations required will be done on a GDC. The emphasis is on understanding and interpreting the results obtained.

Details

	Content	Amplifications/inclusions	Exclusions
6.1	Concepts of population, sample, random sample and frequency distribution of discrete and continuous data.	Elementary treatment only.	
6.2	Presentation of data: frequency tables and diagrams, box and whisker plots. Grouped data: mid-interval values, interval width, upper and lower interval boundaries, frequency histograms.	Treatment of both continuous and discrete data. A frequency histogram uses equal class intervals.	 Histograms based on unequal class intervals.

	Content	Amplifications/inclusions	Exclusions
6.3	Mean, median, mode; quartiles, percentiles. Range; interquartile range; variance, standard deviation.	Awareness that the population mean, μ, is generally unknown, and that the sample mean, $\bar{x}$, serves as an unbiased estimate of this quantity. Awareness of the concept of dispersion and an understanding of the significance of the numerical value of the standard deviation. Obtain the standard deviation (and indirectly the variance) from a GDC and by other methods. Awareness that the population variance, σ^2, is generally unknown, and that $s_{n-1}^2 = \frac{n}{n-1} s_n^2$ serves as an unbiased estimate of σ^2.	Estimation of the mode from a histogram. Formal treatment of unbiased estimation. **See SL guide**
6.4	Cumulative frequency; cumulative frequency graphs; use to find median, quartiles, percentiles.		
6.5	Concepts of trial, outcome, equally likely outcomes, sample space (U) and event. The probability of an event A as $\mathrm{P}(A) = \frac{n(A)}{n(U)}$. The complementary events A and A' (not A); $\mathrm{P}(A) + \mathrm{P}(A') = 1$.	The calculation of $n(A)$ and $n(U)$ may involve counting principles.	

	Content	Amplifications/inclusions	Exclusions
6.6	Combined events, the formula: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. $P(A \cap B) = 0$ for mutually exclusive events.	Appreciation of the non-exclusivity of "or". Use of $P(A \cup B) = P(A) + P(B)$ for mutually exclusive events.	
6.7	Conditional probability; the definition: $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$. Independent events; the definition: $P(A \mid B) = P(A) = P(A \mid B')$. Use of Bayes' theorem for two events.	 The term "independent" is equivalent to "statistically independent". Use of $P(A \cap B) = P(A)P(B)$ for independent events. $P(B \mid A) = \frac{P(B)P(A \mid B)}{P(B)P(A \mid B) + P(B')P(A \mid B')}$.	
6.8	Use of Venn diagrams, tree diagrams and tables of outcomes to solve problems.		

	Content	Amplifications/inclusions	Exclusions
6.9	Concept of discrete and continuous random variables and their probability distributions. Definition and use of probability density functions. Expected value (mean), mode, median, variance and standard deviation.	Knowledge and use of the formulae for $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. Applications of expectations, for example, games of chance.	**See SL guide**
6.10	Binomial distribution, its mean and variance. Poisson distribution, its mean and variance.	Conditions under which random variables have these distributions.	Formal proof of means and variances. **See SL guide**
6.11	Normal distribution. Properties of the normal distribution. Standardization of normal variables.	Appreciation that the standardized value (z) gives the number of standard deviations from the mean. Use of calculator (or tables) to find normal probabilities; the reverse process.	Normal approximation to the binomial distribution.

Topic 7—Core: Calculus 48 hrs

Aims

The aim of this section is to introduce students to the basic concepts and techniques of differential and integral calculus and their application.

Details

	Content	Amplifications/inclusions	Exclusions
7.1	Informal ideas of limit and convergence.	Only an informal treatment of limit and convergence, including the result $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.	
	Definition of derivative as $f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$.	Use of this definition for differentiation of polynomials, and for justification of other derivatives. Familiarity with both forms of notation, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $f'(x)$, for the first derivative.	
	Derivative of x^n $(n \in \mathbb{Q})$, $\sin x$, $\cos x$, $\tan x$, e^x and $\ln x$.		On examination papers: students will not be required to prove these results.
	Derivative interpreted as a gradient function and as rate of change.	Finding equations of tangents and normals. Identifying increasing and decreasing functions.	
	Derivatives of reciprocal circular functions. Derivatives of a^x and $\log_a x$. Derivatives of $\arcsin x$, $\arccos x$, $\arctan x$.		**See SL guide**

	Content	Amplifications/inclusions	Exclusions
7.2	Differentiation of a sum and a real multiple of the functions in 7.1. The chain rule for composite functions. Application of chain rule to related rates of change. The product and quotient rules. The second derivative. Awareness of higher derivatives.	Familiarity with both forms of notation, $\frac{d^2y}{dx^2}$ and $f''(x)$, for the second derivative. Familiarity with the notations $\frac{d^ny}{dx^n}$, $f^{(n)}(x)$.	**See SL guide**
7.3	Local maximum and minimum points. Use of the first and second derivative in optimization problems.	Testing for the maximum or minimum using change of sign of the first derivative and using sign of second derivative. Examples of applications: profit, area, volume.	

	Content	Amplifications/inclusions	Exclusions
7.4	Indefinite integration as anti-differentiation. Indefinite integral of x^n $(n \neq -1)$, $\sin x$, $\cos x$, e^x, $\frac{1}{x}$. The composites of any of these with the linear function $ax+b$.	Indefinite integral interpreted as a family of curves. $\int \frac{1}{x} dx = \ln\lvert x\rvert + C$. Examples: $f'(x) = \cos(2x+3) \Rightarrow f(x) = \frac{1}{2}\sin(2x+3) + C$.	**See SL guide**
7.5	Anti-differentiation with a boundary condition to determine the constant term. Definite integrals. Area between a curve and the *x*-axis or *y*-axis in a given interval, areas between curves. Volumes of revolution.	Example: if $\frac{dy}{dx} = 3x^2 + x$ and $y = 10$ when $x = 0$, then $y = x^3 + \frac{1}{2}x^2 + 10$. $\int_a^b y dx$ and $\int_a^b x dy$. Revolution about the *x*-axis or the *y*-axis. $V = \int_a^b \pi y^2 dx$, $V = \int_a^b \pi x^2 dy$.	**See SL guide**

	Content	Amplifications/inclusions	Exclusions
7.6	Kinematic problems involving displacement, s, velocity, v, and acceleration, a.	$v = \frac{ds}{dt}$, $a = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = v\frac{dv}{ds}$. Area under velocity–time graph represents distance.	**See SL guide**
	Graphical behaviour of functions: tangents and normals, behaviour for large $\lvert x \rvert$; asymptotes.	Included: both "global" and "local" behaviour. Oblique asymptotes.	
	The significance of the second derivative; distinction between maximum and minimum points.	Use of the terms "concave up" for $f''(x) > 0$, "concave down" for $f''(x) < 0$.	
	Points of inflexion with zero and non-zero gradients.	At a point of inflexion $f''(x) = 0$ and $f''(x)$	

	Content	Amplifications/inclusions	Exclusions
7.9	Further integration: integration by substitution integration by parts.	Limit changes in definite integrals. On examination papers: unusual substitutions may be given. Examples: $\int x \sin x \,\mathrm{d}x$ and $\int \ln x \,\mathrm{d}x$. Repeated integration by parts: examples: $\int x^2 \mathrm{e}^x \,\mathrm{d}x$ and $\int \mathrm{e}^x \sin x \,\mathrm{d}x$.	Integration using partial fractions. Reduction formulae.
7.10	Solution of first order differential equations by separation of variables.		

Option syllabus content

Topic 8—Option: Statistics and probability 40 hrs

Aims

The aims of this option are to allow students the opportunity to approach statistics in a practical way; to demonstrate a good level of statistical understanding; and to understand which situations apply and to interpret the given results. It is expected that GDCs will be used throughout this option and that the minimum requirement of a GDC will be to find pdf, cdf, inverse cdf, *p*-values and test statistics including calculations for the following distributions: binomial, Poisson, normal, *t* and chi-squared. Students are expected to set up the problem mathematically and then read the answers from the GDC, indicating this within their written answers. Calculator-specific or brand-specific language should not be used within these explanations.

Details

	Content	Amplifications/inclusions	Exclusions
8.1	Expectation algebra. Linear transformation of a single random variable. Mean and variance of linear combinations of two independent random variables. Extension to linear combinations of n independent random variables.	$\mathrm{E}(aX+b)=a\mathrm{E}(X)+b$; $\mathrm{Var}(aX+b)=a^2\mathrm{Var}(X)$. $\mathrm{E}(a_1X_1 \pm a_2X_2)=a_1\mathrm{E}(X_1)\pm a_2\mathrm{E}(X_2)$; $\mathrm{Var}(a_1X_1 \pm a_2X_2)=a_1{}^2\mathrm{Var}(X_1)+a_2{}^2\mathrm{Var}(X_2)$.	

	Content	Amplifications/inclusions	Exclusions
8.2	Cumulative distribution functions.		Formal treatment of proof of means and variances.
	Discrete distributions: uniform, Bernoulli, binomial, negative binomial, Poisson, geometric, hypergeometric.	Probability mass functions, means and variances.	
	Continuous distributions: uniform, exponential, normal.	Probability density functions, means and variances.	
8.3	Distribution of the sample mean.		Sampling without replacement.
	The distribution of linear combinations of independent normal random variables. In particular $X \sim \mathrm{N}\left(\mu, \sigma^2\right) \Rightarrow \bar{X} \sim \mathrm{N}\left(\mu, \frac{\sigma^2}{n}\right)$.	A linear combination of independent normally distributed random variables is also normally distributed.	
	The central limit theorem.		Proof of the central limit theorem.
	The approximate normality of the proportion of successes in a large sample.	The extension of these results for large samples to distributions that are not normal, using the central limit theorem.	Distributions that do not satisfy the central limit theorem.

	Content	Amplifications/inclusions	Exclusions
8.4	Finding confidence intervals for the mean of a population. Finding confidence intervals for the proportion of successes in a population.	Use of the normal distribution when σ is known and the *t*-distribution when σ is unknown (regardless of sample size). The case of paired samples (matched pairs) could be tested as an example of a single sample technique.	The difference of means and the difference of proportions.
8.5	Significance testing for a mean. Significance testing for a proportion. Null and alternative hypotheses H_0 and H_1. Type I and Type II errors. Significance levels; critical region, critical values, *p*-values; one-tailed and two-tailed tests.	Use of the normal distribution when σ is known and the *t*-distribution when σ is unknown. The case of paired samples (matched pairs) could be tested as an example of a single sample technique.	The difference of means and the difference of proportions.

	Content	Amplifications/inclusions	Exclusions
8.6	The chi-squared distribution: degrees of freedom, υ.		
	The χ^2 statistic, $\chi^2_{calc} = \sum \frac{(f_o - f_e)^2}{f_e}$.	Awareness of the fact that χ^2_{calc} is a measure of the discrepancy between observed and expected values.	
	The χ^2 goodness of fit test.	Test for goodness of fit for all of the above distributions; the requirement to combine classes with expected frequencies of less than 5.	
	Contingency tables: the χ^2 test for the independence of two variables.		Yates' continuity correction for $\upsilon = 1$.

Topic 9—Option: Sets, relations and groups 40 hrs

Aims

The aims of this option are to provide the opportunity to study some important mathematical concepts, and introduce the principles of proof through abstract algebra.

Details

	Content	Amplifications/inclusions	Exclusions
9.1	Finite and infinite sets. Subsets. Operations on sets: union; intersection; complement, set difference, symmetric difference.		
	De Morgan's laws; distributive, associative and commutative laws (for union and intersection).	Illustration of these laws using Venn diagrams.	Proofs of these laws.
9.2	Ordered pairs: the Cartesian product of two sets.		
	Relations; equivalence relations; equivalence classes.	An equivalence relation on a set induces a partition of the set.	
9.3	Functions: injections; surjections; bijections.	The term "codomain".	
	Composition of functions and inverse functions.	Knowledge that the function composition is not a commutative operation and that if f is a bijection from set A onto set B then f^{-1} exists and is a bijection from set B onto set A.	

Topic 9—Option: Sets, relations and groups (continued)

	Content	Amplifications/inclusions	Exclusions
9.4	Binary operations.	A binary operation $*$ on a non-empty set S is a rule for combining any two elements $a, b \in S$ to give a unique element c. That is, in this definition, a binary operation is not necessarily closed. On examination papers: candidates may be required to test whether a given operation satisfies the closure condition.	
	Operation tables (Cayley tables).	Operation tables with the Latin square property (every element appears once only in each row and each column).	
9.5	Binary operations with associative, distributive and commutative properties.	The arithmetic operations in $\mathbb{R}$ and $\mathbb{C}$; matrix operations.	
9.6	The identity element e.	Both the right-identity $a * e = a$ and left-identity $e * a = a$ must hold if e is an identity element.	
	The inverse a^{-1} of an element a.	Both $a * a^{-1} = e$ and $a^{-1} * a = e$ must hold.	
	Proof that left-cancellation and right-cancellation by an element a hold, provided that a has an inverse.		
	Proofs of the uniqueness of the identity and inverse elements.		

	Content	Amplifications/inclusions	Exclusions
9.7	The axioms of a group $\{G,*\}$.	For the set G under a given operation $*$: • G is closed under $*$ • $*$ is associative • G contains an identity element • each element in G has an inverse in G.	
	Abelian groups.	$a*b=b*a$, for all $a,b\in G$.	
9.8	The groups: • $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{C}$ under addition • matrices of the same order under addition • 2×2 invertible matrices under multiplication • integers under addition modulo n		
	• groups of transformations	The composition T_1T_2 denotes T_2 followed by T_1.	
	• symmetries of an equilateral triangle, rectangle and square • invertible functions under composition of functions		
	• permutations under composition of permutations.	On examination papers: the form $p=\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}$ will be used to represent the mapping $1\rightarrow 3$, $2\rightarrow 1$, $3\rightarrow 2$.	

Topic 9—Option: Sets, relations and groups (continued)

	Content	Amplifications/inclusions	Exclusions
9.9	Finite and infinite groups. The order of a group element and the order of a group.	Latin square property of a group table.	
9.10	Cyclic groups. Proof that all cyclic groups are Abelian.	Generators.	
9.11	Subgroups, proper subgroups. Use and proof of subgroup tests.	 Suppose G is a group and H is a non-empty subset of G. H is a subgroup of G if $ab^{-1} \in H$ whenever $a, b \in H$. Suppose G is a finite group and H is a non-empty subset of G. H is a subgroup of G if H is closed under the group operation.	
	Lagrange's theorem. Use and proof of the result that the order of a finite group is divisible by the order of any element. (Corollary to Lagrange's theorem.)		On examination papers: questions requiring the proof of Lagrange's theorem will not be set.

	Content	Amplifications/inclusions	Exclusions
9.12	Isomorphism of groups. Proof of isomorphism properties for identities and inverses.	Infinite groups as well as finite groups. Two groups $\{G,\circ\}$ and $\{H,\bullet\}$ are isomorphic if there exists a bijection $f:G\to H$ such that $f(a\circ b)=f(a)\bullet f(b)$ for all $a,b\in G$. The function $f:G\to H$ is an isomorphism. Identity: let e_1 and e_2 be the identity elements of G, H respectively, then $f(e_1)=e_2$. Inverse: $f(a^{-1})=\left(f(a)\right)^{-1}$ for all $a\in G$.	

Topic 10—Option: Series and differential equations 40 hrs

Aims

The aims of this option are to introduce limit theorems and convergence of series, and to use calculus results to solve differential equations.

Details

	Content	Amplifications/inclusions	Exclusions
10.1	Infinite sequences of real numbers.		
	Limit theorems as n approaches infinity.	Limit of sum, difference, product, quotient; squeeze theorem.	
	Limit of a sequence.	Formal definition: the sequence $\{u_n\}$ converges to the limit L, if for any $\varepsilon > 0$, there is a positive integer N such that $\lvert u_n - L \rvert < \varepsilon$, for all $n > N$.	
	Improper integrals of the type $\int_a^\infty f(x)\,dx$.		
	The integral as a limit of a sum; lower sum and upper sum.		

	Content	Amplifications/inclusions	Exclusions
10.2	Convergence of infinite series.	The sum of a series is the limit of the sequence of its partial sums.	
	Partial fractions and telescoping series (method of differences).	Simple linear non-repeated denominators.	
	Tests for convergence: comparison test; limit comparison test; ratio test; integral test.	Students should be aware that if $\lim_{x\to\infty} x_n = 0$ then the series is not necessarily convergent, but if $\lim_{x\to\infty} x_n \neq 0$, the series diverges.	
	The *p*-series, $\sum \frac{1}{n^p}$.	$\sum \frac{1}{n^p}$ is convergent for $p>1$ and divergent otherwise. When $p=1$, this is the harmonic series.	
	Use of integrals to estimate sums of series.		
10.3	Series that converge absolutely.		
	Series that converge conditionally.		
	Alternating series.	Conditions for convergence. The absolute value of the truncation error is less than the next term in the series.	
10.4	Power series: radius of convergence and interval of convergence. Determination of the radius of convergence by the ratio test.		

Topic 10—Option: Series and differential equations (continued)

	Content	Amplifications/inclusions	Exclusions
10.5	Taylor polynomials and series, including the error term.	Applications to the approximation of functions; formulae for the error term, both in terms of the value of the $(n+1)^{\text{th}}$ derivative at an intermediate point, and in terms of an integral of the $(n+1)^{\text{th}}$ derivative.	Proof of Taylor's theorem.
		Differentiation and integration of series (valid only on the interval of convergence of the initial series).	Use of products and quotients to obtain other series.
	Maclaurin series for e^x, $\sin x$, $\cos x$, $\arctan x$, $\ln(1+x)$, $(1+x)^p$. Use of substitution to obtain other series.	Intervals of convergence for these Maclaurin series. Example: e^{x^2}.	
	The evaluation of limits of the form $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ using l'Hôpital's Rule and/or the Taylor series.	Cases where the derivatives of $f(x)$ and $g(x)$ vanish for $x=a$.	Proof of l'Hôpital's Rule.

	Content	Amplifications/inclusions	Exclusions
10.6	First order differential equations: geometric interpretation using slope fields; numerical solution of $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(x, y\right)$ using Euler's method. Homogeneous differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = f\left(\frac{y}{x}\right)$ using the substitution $y = vx$. Solution of $y' + P(x)y = Q(x)$, using the integrating factor.	 $y_{n+1} = y_n + h \times f(x_n, y_n)$; $x_{n+1} = x_n + h$, where h is a constant.	

Topic 11—Option: Discrete mathematics 40 hrs

Aims

The aim of this option is to provide the opportunity for students to engage in logical reasoning, algorithmic thinkinglo

	Content	Amplifications/inclusions	Exclusions
11.6	Graphs, vertices, edges. Adjacent vertices, adjacent edges. Simple graphs; connected graphs; complete graphs; bipartite graphs; planar graphs, trees, weighted graphs. Subgraphs; complements of graphs. Graph isomorphism.	Two vertices are adjacent if they are joined by an edge. Two edges are adjacent if they have a common vertex. Euler's relation: $v-e+f=2$; theorems for planar graphs including $e \leq 3v-6$, $e \leq 2v-4$, κ_5 and $\kappa_{3,3}$ are not planar. Simple graphs only for isomorphism.	
11.7	Walks, trails, paths, circuits, cycles. Hamiltonian paths and cycles; Eulerian trails and circuits.	A connected graph contains a Eulerian circuit if and only if every vertex of the graph is of even degree.	Dirac's theorem for Hamiltonian cycles.
11.8	Adjacency matrix. Cost adjacency matrix.	Applications to isomorphism and of the powers of the adjacency matrix to number of walks.	
11.9	Graph algorithms: Prim's; Kruskal's; Dijkstra's.	These are examples of "greedy" algorithms.	

Topic 11—Option: Discrete mathematics (continued)

	Content	Amplifications/inclusions	Exclusions
11.10	"Chinese postman" problem ("route inspection").	To determine the shortest route around a weighted graph going along each edge at least once (route inspection algorithm).	Graphs with more than two vertices of odd degree.
	"Travelling salesman" problem.	To determine the Hamiltonian cycle of least weight in a weighted complete graph.	
	Algorithms for determining upper and lower bounds of the travelling salesman problem.		Graphs in which the triangle inequality is not satisfied.

Glossary of terminology for the discrete mathematics option

Introduction

Teachers and students should be aware that many different terminologies exist in graph theory and that different textbooks may employ different combinations of these. Examples of these are: vertex/node/junction/point; edge/route/arc; degree of a vertex/order; multiple edges/parallel edges; loop/self-loop.

In IBO examination questions, the terminology used will be as it appears in the syllabus. For clarity these terms are defined below.

Terminology

Graph	Consists of a set of vertices and a set of edges; an edge joins its endpoints (vertices)
Subgraph	A graph within a graph
Weighted graph	A graph in which each edge is allocated a number or weight
Loop	An edge whose endpoints are joined to the same vertex
Multiple edges	Occur if more than one edge joins the same pair of vertices
Walk	A sequence of linked edges
Trail	A walk in which no edge appears more than once
Path	A walk with no repeated vertices
Circuit	A walk that begins and ends at the same vertex, and has no repeated edges
Cycle	A walk that begins and ends at the same vertex, and has no other repeated vertices
Hamiltonian path	A path that contains all the vertices of the graph
Hamiltonian cycle	A cycle that contains all the vertices of the graph
Eulerian trail	A trail that contains every edge of a graph
Eulerian circuit	A circuit that contains every edge of a graph
Degree of a vertex	The number of edges joined to the vertex; a loop contributes two, one for each of its endpoints
Simple graph	A graph without loops or multiple edges
Complete graph	A simple graph where every vertex is joined to every other vertex
Connected graph	A graph that has a path joining every pair of vertices
Disconnected graph	A graph that has at least one pair of vertices not joined by a path

Tree	A connected graph that contains no cycles
Weighted tree	A tree in which each edge is allocated a number or weight
Spanning tree of a graph	A subgraph containing every vertex of the graph, which is also a tree
Minimum spanning tree	A spanning tree of a weighted graph that has the minimum total weight
Complement of a graph G	A graph with the same vertices as *G* but which has an edge between any two vertices if and only if *G* does not
Graph isomorphism between two simple graphs G and H	A one-to-one correspondence between vertices of *G* and *H* such that a pair of vertices in *G* is adjacent if and only if the corresponding pair in *H* is adjacent
Planar graph	A graph that can be drawn in the plane without any edge crossing another
Bipartite graph	A graph whose vertices can be divided into two sets and in which edges always join a vertex from one set to a vertex from the other set
Complete bipartite graph	A bipartite graph in which every vertex in one set is joined to every vertex in the other set
Adjacency matrix of G, denoted by $\boldsymbol{A}_G$	The adjacency matrix, $\boldsymbol{A}_G$, of a graph *G* with *n* vertices, is the $n\times n$ matrix in which the entry in row *i* and column *j* is the number of edges joining the vertices *i* and *j*. Hence, the adjacency matrix will be symmetric about the diagonal.
Cost adjacency matrix of G, denoted by $\boldsymbol{C}_G$	The cost adjacency matrix, $\boldsymbol{C}_G$, of a graph *G* with *n* vertices is the $n\times n$ matrix in which the entry in row *i* and column *j* is the weight of the edges joining the vertices *i* and *j*

ASSESSMENT OUTLINE

First examinations 2008

Mathematics HL

External assessment 5 hrs 80%

Written papers

Paper 1 2 hrs 30%

No calculator allowed

Section A 15%

Compulsory short-response questions based on the compulsory core of the syllabus

Section B 15%

Compulsory extended-response questions based on the compulsory core of the syllabus

Paper 2 2 hrs 30%

Graphic display calculator (GDC) required

Section A 15%

Compulsory short-response questions based on the compulsory core of the syllabus

Section B 15%

Compulsory extended-response questions based on the compulsory core of the syllabus

Paper 3 1 hr 20%

Graphic display calculator (GDC) required

Extended-response questions based mainly on the syllabus options

Internal assessment 20%

Portfolio

A collection of two pieces of work assigned by the teacher and completed by the student during the course. The pieces of work must be based on different areas of the syllabus and represent the two types of tasks:

- mathematical investigation
- mathematical modelling.

The portfolio is internally assessed by the teacher and externally moderated by the IBO. Procedures are provided in the *Vade Mecum*.

ASSESSMENT DETAILS

External assessment details 5 hrs 80%

General

Paper 1, paper 2 and paper 3

These papers are externally set and externally marked. Together they contribute 80% of the final mark for the course. These papers are designed to allow students to demonstrate what they know and what they can do.

Calculators

Paper 1

Students are not permitted access to any calculator. Questions will mainly involve analytic approaches to solutions, rather than requiring the use of a GDC. It is not intended to have complicated calculations, with the potential for careless errors. However, questions will include some arithmetical manipulations when they are essential to the development of the question.

Papers 2 and 3

Students must have access to a GDC at all times. However, not all questions will necessarily require the use of the GDC. Regulations covering the types of GDC allowed are provided in the *Vade Mecum*.

Mathematics HL information booklet

Each student must have access to a clean copy of the information booklet during the examination. One copy of this booklet is provided by the IBO as part of the examination papers mailing.

Awarding of marks

Marks may be awarded for method, accuracy, answers and reasoning, including interpretation.

In paper 1, paper 2 and paper 3, full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations (in the form of, for example, diagrams, graphs or calculations). Where an answer is incorrect, some marks may be given for correct method, provided this is shown by written working. All students should therefore be advised to show their working.

Paper 1 2 hrs 30%

This paper consists of section A, short-response questions, and section B, extended-response questions. Each section will be worth 15% of the total mark.

Syllabus coverage

- Knowledge of **all** topics in the core of the syllabus is required for this paper. However, not all topics are necessarily assessed in every examination session.

Mark allocation

- This paper is worth **120** marks, representing **30%** of the final mark.
- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

Section A

This section consists of compulsory short-response questions based on the core of the syllabus. It is worth 60 marks, representing 15% of the final mark.

- The intention of this section is to test students' knowledge across the breadth of the core. However, it should not be assumed that the separate topics from the core are given equal emphasis.

Question type

- A small number of steps is needed to solve each question.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

Section B

This section consists of compulsory extended-response questions based on the core of the syllabus. It is worth 60 marks, representing 15% of the final mark.

- Individual questions may require knowledge of more than one topic from the core.
- The intention of this section is to test students' knowledge of the core in depth. The range of syllabus topics tested in this paper may be narrower than that tested in section A.
- To provide appropriate syllabus coverage of each topic, some questions in this section are likely to contain two or more unconnected parts. Where this occurs, the unconnected parts will be clearly labelled as such.

Question type

- Questions require extended responses involving sustained reasoning.
- Individual questions may develop a single theme or be divided into unconnected parts.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is on problem solving.

Paper 2 2 hrs 30%

This paper consists of section A, short-response questions, and section B, extended-response questions. Each section will be worth 15% of the total mark.

Syllabus coverage

- Knowledge of **all** topics in the core of the syllabus is required for this paper. However, not all topics are necessarily assessed in every examination session.

Mark allocation

- This paper is worth **120** marks, representing **30%** of the final mark.
- Questions of varying levels of difficulty and length are set. Therefore, individual questions may not necessarily be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of the question.

Section A

This section consists of compulsory short-response questions based on the core of the syllabus. It is worth 60 marks, representing 15% of the final mark.

- The intention of this section is to test students' knowledge across the breadth of the core. However, it should not be assumed that the separate topics from the core are given equal emphasis.

Question type

- A small number of steps is needed to solve each question.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.

Section B

This section consists of compulsory extended-response questions based on the core of the syllabus. It is worth 60 marks, representing 15% of the final mark.

- Individual questions may require knowledge of more than one topic from the core.
- The intention of this section is to test students' knowledge of the core in depth. The range of syllabus topics tested in this paper may be narrower than that tested in section A.
- To provide appropriate syllabus coverage of each topic, some questions in this section are likely to contain two or more unconnected parts. Where this occurs, the unconnected parts will be clearly labelled as such.

Question type

- Questions require extended responses involving sustained reasoning.
- Individual questions may develop a single theme or be divided into unconnected parts.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is on problem solving.

Paper 3 — 1 hr — 20%

This paper consists of four sections, one on each of the options in the syllabus. Each section has a small number of extended-response questions based mainly on the option topic. Where possible, the first part of each question will be on core material leading to the option topic. When this is not readily achievable, as for example with the discrete mathematics option, the level of difficulty of the earlier part of a question will be comparable to that of the core questions.

Students must answer questions on one option topic only. Students must answer all the questions in the section chosen.

Syllabus coverage

- Students must answer all the questions based on the option they have studied.
- Knowledge of the entire content of the option studied is required for this paper, as well as the core material.

Question type

- Questions require extended responses involving sustained reasoning.
- Individual questions may develop a single theme or be divided into unconnected parts. Where this occurs, the unconnected parts will be clearly labelled as such.
- Questions may be presented in the form of words, symbols, diagrams or tables, or combinations of these.
- Normally, each question reflects an incline of difficulty, from relatively easy tasks at the start of a question to relatively difficult tasks at the end of a question. The emphasis is on problem solving.

Mark allocation

- This paper is worth **60** marks, representing **20%** of the final mark. Approximately **15** marks are allocated to core material (or work of a similar level).
- Questions in this section may be unequal in terms of length and level of difficulty. Therefore, individual questions may not necessarily be worth the same number of marks. The exact number of marks allocated to each question is indicated at the start of each question. Each section is worth **60** marks, and the overall level of difficulty of each section should be the same.

Guidelines

Notation

Of the various notations in use, the IBO has chosen to adopt a system of notation based on the recommendations of the International Organization for Standardization (ISO). This notation is used in the examination papers for this course without explanation. If forms of notation other than those listed in this guide are used on a particular examination paper, they are defined within the question in which they appear.

Because students are required to recognize, though not necessarily use, IBO notation in examinations, it is recommended that teachers introduce students to this notation at the earliest opportunity. Students are **not** allowed access to information about this notation in the examinations.

In a small number of cases, students may need to use alternative forms of notation in their written answers. This is because not all forms of IBO notation can be directly transferred into handwritten form. For vectors in particular the IBO notation uses a bold, italic typeface that cannot adequately be transferred into handwritten form. In this case, teachers should advise candidates to use alternative forms of notation in their written work (for example, $\vec{x}$, $\bar{x}$ or $\underline{x}$).

Students must always use correct mathematical notation, not calculator notation.

N	the set of positive integers and zero, $\{0, 1, 2, 3, ...\}$
Z	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, ...\}$
Z^+	the set of positive integers, $\{1, 2, 3, ...\}$
Q	the set of rational numbers
Q^+	the set of positive rational numbers, $\{x \mid x \in Q, x > 0\}$
R	the set of real numbers
R^+	the set of positive real numbers, $\{x \mid x \in R, x > 0\}$
$\mathbb{C}$	the set of complex numbers, $\{a + ib \mid a, b \in R\}$
i	$\sqrt{-1}$
z	a complex number
z^*	the complex conjugate of z
$\lvert z \rvert$	the modulus of z
$\arg z$	the argument of z
$\mathrm{Re}\, z$	the real part of z
$\mathrm{Im}\, z$	the imaginary part of z
$\{x_1, x_2, ...\}$	the set with elements $x_1, x_2, ...$
$n(A)$	the number of elements in the finite set A
$\{x \mid \quad \}$	the set of all x such that
$\in$	is an element of
$\notin$	is not an element of
$\varnothing$	the empty (null) set
U	the universal set
$\cup$	union
$\cap$	intersection
$\subset$	is a proper subset of

$\subseteq$	is a subset of
A'	the complement of the set A
$A \times B$	the Cartesian product of sets A and B (that is, $A \times B = \{(a, b) \mid a \in A,\ b \in B\}$)
$a \mid b$	a divides b
$a^{1/n}$, $\sqrt[n]{a}$	a to the power of $\frac{1}{n}$, n^{th} root of a (if $a \geq 0$ then $\sqrt[n]{a} \geq 0$)
$a^{1/2}$, $\sqrt{a}$	a to the power $\frac{1}{2}$, square root of a (if $a \geq 0$ then $\sqrt{a} \geq 0$)
$\lvert x \rvert$	the modulus or absolute value of x, that is $\begin{cases} x \text{ for } x \geq 0,\ x \in \mathbb{R} \\ -x \text{ for } x < 0,\ x \in \mathbb{R} \end{cases}$
$\equiv$	identity
$\approx$	is approximately equal to
$>$	is greater than
$\geq$	is greater than or equal to
$<$	is less than
$\leq$	is less than or equal to
$\ngtr$	is not greater than
$\nless$	is not less than
$[a, b]$	the closed interval $a \leq x \leq b$
$]a, b[$	the open interval $a < x < b$
u_n	the n^{th} term of a sequence or series
d	the common difference of an arithmetic sequence
r	the common ratio of a geometric sequence
S_n	the sum of the first n terms of a sequence, $u_1 + u_2 + ... + u_n$
S_∞	the sum to infinity of a sequence, $u_1 + u_2 + ...$

$\sum_{i=1}^{n} u_i$	$u_1 + u_2 + ... + u_n$
$\prod_{i=1}^{n} u_i$	$u_1 \times u_2 \times ... \times u_n$
$\binom{n}{r}$	$\frac{n!}{r!(n-r)!}$
$f : A \to B$	*f* is a function under which each element of set *A* has an image in set *B*
$f : x \mapsto y$	*f* is a function under which *x* is mapped to *y*
$f(x)$	the image of *x* under the function *f*
f^{-1}	the inverse function of the function *f*
$f \circ g$	the composite function of *f* and *g*
$\lim_{x \to a} f(x)$	the limit of $f(x)$ as *x* tends to *a*
$\frac{\mathrm{d}y}{\mathrm{d}x}$	the derivative of *y* with respect to *x*
$f'(x)$	the derivative of $f(x)$ with respect to *x*
$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}$	the second derivative of *y* with respect to *x*
$f''(x)$	the second derivative of $f(x)$ with respect to *x*
$\frac{\mathrm{d}^n y}{\mathrm{d}x^n}$	the n^{th} derivative of *y* with respect to *x*
$f^{(n)}(x)$	the n^{th} derivative of $f(x)$ with respect to *x*
$\int y \, \mathrm{d}x$	the indefinite integral of *y* with respect to *x*
$\int_a^b y \, \mathrm{d}x$	the definite integral of *y* with respect to *x* between the limits $x = a$ and $x = b$
e^x	exponential function of *x*
$\log_a x$	logarithm to the base *a* of *x*
$\ln x$	the natural logarithm of *x*, $\log_\mathrm{e} x$

sin, cos, tan	the circular functions
arcsin, arccos, arctan	the inverse circular functions
csc, sec, cot	the reciprocal circular functions
$\mathrm{A}(x, y)$	the point A in the plane with Cartesian coordinates x and y
$[\mathrm{AB}]$	the line segment with end points A and B
AB	the length of $[\mathrm{AB}]$
(AB)	the line containing points A and B
$\hat{A}$	the angle at A
$\mathrm{C\hat{A}B}$	the angle between $[\mathrm{CA}]$ and $[\mathrm{AB}]$
$\Delta\mathrm{ABC}$	the triangle whose vertices are A, B and C
$\boldsymbol{v}$	the vector $\boldsymbol{v}$
$\overrightarrow{\mathrm{AB}}$	the vector represented in magnitude and direction by the directed line segment from A to B
$\boldsymbol{a}$	the position vector $\overrightarrow{\mathrm{OA}}$
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	unit vectors in the directions of the Cartesian coordinate axes
$\lvert\boldsymbol{a}\rvert$	the magnitude of $\boldsymbol{a}$
$\lvert\overrightarrow{\mathrm{AB}}\rvert$	the magnitude of $\overrightarrow{\mathrm{AB}}$
$\boldsymbol{v} \cdot \boldsymbol{w}$	the scalar product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{v} \times \boldsymbol{w}$	the vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$
$\boldsymbol{A}^{-1}$	the inverse of the non-singular matrix $\boldsymbol{A}$
$\boldsymbol{A}^{\mathrm{T}}$	the transpose of the matrix $\boldsymbol{A}$
$\det \boldsymbol{A}$	the determinant of the square matrix $\boldsymbol{A}$
$\boldsymbol{I}$	the identity matrix
$\mathrm{P}(A)$	probability of event A

$\mathrm{P}(A')$	probability of the event "not A"
$\mathrm{P}(A \mid B)$	probability of the event A given B
$x_1, x_2, \ldots$	observations
$f_1, f_2, \ldots$	frequencies with which the observations $x_1, x_2, \ldots$ occur
P_x	probability distribution function $\mathrm{P}(X = x)$ of the discrete random variable X
$f(x)$	probability density function of the continuous random variable X
$F(x)$	cumulative distribution function of the continuous random variable X
$\mathrm{E}(X)$	the expected value of the random variable X
$\mathrm{Var}(X)$	the variance of the random variable X
μ	population mean
σ^2	population variance, $\sigma^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \mu)^2}{n}$, where $n = \sum_{i=1}^{k} f_i$
σ	population standard deviation
$\overline{x}$	sample mean
s_n^2	sample variance, $s_n^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n}$, where $n = \sum_{i=1}^{k} f_i$
s_n	standard deviation of the sample
s_{n-1}^2	unbiased estimate of the population variance, $s_{n-1}^2 = \dfrac{n}{n-1} s_n^2 = \dfrac{\sum_{i=1}^{k} f_i (x_i - \overline{x})^2}{n-1}$, where $n = \sum_{i=1}^{k} f_i$
$\mathrm{B}(n, p)$	binomial distribution with parameters n and p
$\mathrm{Po}(m)$	Poisson distribution with mean m
$\mathrm{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2

$X \sim \mathrm{B}(n, p)$	the random variable *X* has a binomial distribution with parameters *n* and *p*
$X \sim \mathrm{Po}(m)$	the random variable *X* has a Poisson distribution with mean *m*
$X \sim \mathrm{N}(\mu, \sigma^2)$	the random variable *X* has a normal distribution with mean

Glossary of command terms

The following command terms are used without explanation on examination papers. Teachers should familiarize themselves and their students with the terms and their meanings. This list is not exhaustive. Other command terms may be used, but it should be assumed that they have their usual meaning (for example, "explain" and "estimate"). The terms included here are those that sometimes have a meaning in mathematics that is different from the usual meaning.

Further clarification and examples can be found in the teacher support material.

Write down	Obtain the answer(s), usually by extracting information. Little or no calculation is required. Working does not need to be shown.
Calculate	Obtain the answer(s) showing all relevant working. "Find" and "determine" can also be used.
Find	Obtain the answer(s) showing all relevant working. "Calculate" and "determine" can also be used.
Determine	Obtain the answer(s) showing all relevant working. "Find" and "calculate" can also be used.
Differentiate	Obtain the derivative of a function.
Integrate	Obtain the integral of a function.
Solve	Obtain the solution(s) or root(s) of an equation.
Draw	Represent by means of a labelled, **accurate** diagram or graph, using a pencil. A ruler (straight edge) should be used for straight lines. Diagrams should be drawn to scale. Graphs should have points correctly plotted (if appropriate) and joined in a straight line or smooth curve.
Sketch	Represent by means of a diagram or graph, labelled if required. A sketch should give a general idea of the required shape of the diagram or graph. A sketch of a graph should include relevant features such as intercepts, maxima, minima, points of inflexion and asymptotes.
Plot	Mark the position of points on a diagram.
Compare	Describe the similarities and differences between two or more items.
Deduce	Show a result using known information.
Justify	Give a valid reason for an answer or conclusion.
Prove	Use a sequence of logical steps to obtain the required result in a formal way.
Show that	Obtain the required result (possibly using information given) without the formality of proof. "Show that" questions should not generally be "analysed" using a calculator.
Hence	Use the preceding work to obtain the required result.
Hence or otherwise	It is suggested that the preceding work is used, but other methods could also receive credit.

Weighting of objectives

Some objectives can be linked more easily to the different types of assessment. In particular, some will be assessed more appropriately in the internal assessment (as indicated in the following section) and only minimally in the examination papers.

Objective	**Percentage weighting**
Know and use mathematical concepts and principles.	15%
Read, interpret and solve a given problem using appropriate mathematical terms.	15%
Organize and present information and data in tabular, graphical and/or diagrammatic forms.	12%
Know and use appropriate notation and terminology (internal assessment).	5%
Formulate a mathematical argument and communicate it clearly.	10%
Select and use appropriate mathematical strategies and techniques.	15%
Demonstrate an understanding of both the significance and the reasonableness of results (internal assessment).	5%
Recognize patterns and structures in a variety of situations, and make generalizations (internal assessment).	3%
Recognize and demonstrate an understanding of the practical applications of mathematics (internal assessment).	3%
Use appropriate technological devices as mathematical tools (internal assessment).	15%
Demonstrate an understanding of and the appropriate use of mathematical modelling (internal assessment).	2%

Internal assessment details 20%

The purpose of the portfolio

The purpose of the portfolio is to provide students with opportunities to be rewarded for mathematics carried out under ordinary conditions, that is, without the time limitations and pressure associated with written examinations. Consequently, the emphasis should be on good mathematical writing and thoughtful reflection.

The portfolio is also intended to provide students with opportunities to increase their understanding of mathematical concepts and processes. It is hoped that, by doing portfolio work, students benefit from these mathematical activities and find them both stimulating and rewarding.

The specific purposes of portfolio work are to:

- develop students' personal insight into the nature of mathematics and to develop their ability to ask their own questions about mathematics
- provide opportunities for students to complete extended pieces of mathematical work without the time constraints of an examination
- enable students to develop individual skills and techniques, and to allow them to experience the satisfaction of applying mathematical processes on their own
- provide students with the opportunity to experience for themselves the beauty, power and usefulness of mathematics
- provide students with the opportunity to discover, use and appreciate the power of a calculator or computer as a tool for doing mathematics
- enable students to develop the qualities of patience and persistence, and to reflect on the significance of the results they obtain
- provide opportunities for students to show, with confidence, what they know and what they can do.

Objectives

The portfolio is internally assessed by the teacher and externally moderated by the IBO. Assessment criteria have been developed to relate to the mathematics objectives. In developing these criteria, particular attention has been given to the objectives listed here, since these cannot be easily addressed by means of timed, written examinations.

Where appropriate in the portfolio, students are expected to:

- know and use appropriate notation and terminology
- organize and present information and data in tabular, graphical and/or diagrammatic forms
- recognize patterns and structures in a variety of situations, and make generalizations
- demonstrate an understanding of and the appropriate use of mathematical modelling
- recognize and demonstrate an understanding of the practical applications of mathematics
- use appropriate technological devices as mathematical tools.

Requirements

The portfolio must consist of two pieces of work assigned by the teacher and completed by the student during the course.

Each piece of student work contained in the portfolio must be based on:

- an area of the syllabus
- one of the two types of tasks
  - type I—mathematical investigation
  - type II—mathematical modelling.

The level of sophistication of the students' mathematical work should be similar to that contained in the syllabus. It is not intended that additional topics are taught to students to enable them to complete a particular task.

Each portfolio must contain two pieces of student work, each of the two types of task: the portfolio must contain one type I and one type II piece of work.

Teaching considerations

These tasks should be completed at intervals throughout the course and should not be left until towards the end. Teachers are encouraged to allow students the opportunity to explore various aspects of as many different topics as possible.

Portfolio work should be integrated into the course of study so that it enhances student learning by introducing a topic, reinforcing mathematical meaning or taking the place of a revision exercise. Therefore, each task needs to correspond to the course of study devised by the individual teacher in terms of the knowledge and skills that the students have been taught.

Use of technology

The need for proper **mathematical** notation and terminology, as opposed to **calculator** or **computer** notation must be stressed and reinforced, as well as adequate documentation of technology usage. Students will therefore be required to reflect on the mathematical processes and algorithms the technology is performing, and communicate them clearly and succinctly.

Type I—mathematical investigation

While many teachers incorporate a problem-solving approach into their classroom practice, students also should be given the opportunity formally to carry out investigative work. The mathematical investigation is intended to highlight that:

- the idea of investigation is fundamental to the study of mathematics
- investigation work often leads to an appreciation of how mathematics can be applied to solve problems in a broad range of fields
- the discovery aspect of investigation work deepens understanding and provides intrinsic motivation
- during the process of investigation, students acquire mathematical knowledge, problem-solving techniques, a knowledge of fundamental concepts and an increase in self-confidence.

All investigations develop from an initial problem, the starting point. The problem must be clearly stated and contain no ambiguity. In addition, the problem should:

- provide a challenge and the opportunity for creativity
- contain multi-solution paths, that is, contain the potential for students to choose different courses of action from a range of options.

Essential skills to be assessed

- Producing a strategy
- Generating data
- Recognizing patterns or structures
- Searching for further cases
- Forming a general statement
- Testing a general statement
- Justifying a general statement
- Appropriate use of technology

Type II—Mathematical modelling

Problem solving usually elicits a process-oriented approach, whereas mathematical modelling requires an experimental approach. By considering different alternatives, students can use modelling to arrive at a specific conclusion, from which the problem can be solved. To focus on the actual process of modelling, the assessment should concentrate on the appropriateness of the model selected in relation to the given situation, and on a critical interpretation of the results of the model in the real-world situation chosen.

Mathematical modelling involves the following skills.

- Translating the real-world problem into mathematics
- Constructing a model
- Solving the problem
- Interpreting the solution in the real-world situation (that is, by the modification or amplification of the problem)
- Recognizing that different models may be used to solve the same problem
- Comparing different models
- Identifying ranges of validity of the models
- Identifying the possible limits of technology
- Manipulating data

Essential skills to be assessed

- Identifying the problem variables
- Constructing relationships between these variables
- Manipulating data relevant to the problem
- Estimating the values of parameters within the model that cannot be measured or calculated from the data
- Evaluating the usefulness of the model
- Communicating the entire process
- Appropriate use of technology

Follow-up and feedback

Teachers should ensure that students are aware of the significance of the results/conclusions that may be the outcome of a particular task. This is particularly important in the case when investigative work is used to introduce a topic on the syllabus. Teachers should allow class time for follow-up work when developing the course of study.

Students should also receive feedback on their own work so that they are aware of alternative strategies for developing their mathematical thinking and are provided with guidance for improving their skills in writing mathematics.

Management of the portfolio

Time allocation

The *Vade Mecum* states that a higher level course requires 240 teaching hours. In mathematics HL, 10 of these hours should be allocated to work connected with the portfolio. This allows time for teachers to explain to students the requirements of the portfolio and allows class time for students to work individually.

During the course students should have time to complete more than two pieces of work. This means they can then choose the best two for inclusion in their portfolios.

Setting of tasks

Teachers must set suitable tasks that comply with requirements for the portfolio.

There is no requirement to provide identical tasks for all students, nor to provide each student with a different task. The tasks set by teachers depend on the needs of their students.

Teachers can design their own tasks, use those contained in published teacher support material and the online curriculum centre (OCC), or modify tasks from other sources.

Submission of work

The finished piece of work should be submitted to the teacher for assessment 3–10 days after it has been set. Students should not be given the opportunity to resubmit a piece of work after it has been assessed.

There is no requirement for work to be word-processed. However, if the work is not word-processed, it must be presented in ink.

Please note that when sending sample work for moderation, original work with teachers' marks and comments on it must be sent. Photocopies are not acceptable.

Guidance and authenticity

Requirements

Students should be familiar with the requirements of the portfolio and the means by which it is assessed: time in the classroom can be used to allow students to assess work from previous years against the criteria.

Discussion in class

Time in the classroom can also be used for discussion of a particular task. This discussion can be between the teacher and the students (or an individual student), or between two or more students. If students ask specific questions, teachers should, where appropriate, guide them into productive routes of inquiry rather than provide a direct answer.

Authenticity

Students need to be aware that the written work they submit must be entirely their own. Teachers should try to encourage students to take responsibility for their learning, so that they accept ownership of the work and take pride in it. When completing a piece of work outside the classroom, students must work independently. Although group work can be educationally desirable in some situations, it is not appropriate for the portfolio.

If there is doubt about the authenticity of a piece of work, teachers can use one or more of the following methods to verify that the work is the student's own.

- Discussing the work with the student
- Asking the student to explain the methods used and to summarize the results
- Asking the student to repeat the task using a different set of data
- Asking the student to produce a list of resources

It is also appropriate for teachers to ask students to sign each task before submitting it to indicate that it is their own work.

All external sources quoted or used must be fully referenced with a bibliography and footnotes.

Student work should include definitions of terminology not previously studied in class.

Record keeping

Teachers must keep careful records to ensure that all students can complete portfolios that comply with the requirements.

For each piece of work, the following information must be recorded.

- Exact details of the task given to the student(s)
- The areas of the syllabus on which the task is based
- The date the task was given to the student(s) and the date of submission
- The type of task (type I or type II)
- The background to the task, in relation to the skills and concepts from the syllabus that had, or had not, been taught to the student at the time the task was set

Please refer to the teacher support materials for sample forms that could be used.

Internal assessment criteria

Form of the assessment criteria

Each piece of work is assessed against all six criteria. Criteria A, B, E and F are identical for both types of task. Criteria C and D are different for the two types of task.

Assessment criteria for type I—mathematical investigation

Type I tasks must be assessed against the following criteria.

Criterion A	Use of notation and terminology
Criterion B	Communication
Criterion C	Mathematical process—searching for patterns
Criterion D	Results—generalization
Criterion E	Use of technology
Criterion F	Quality of work

Assessment criteria for type II—mathematical modelling

Type II tasks must be assessed against the following criteria.

Criterion A	Use of notation and terminology
Criterion B	Communication
Criterion C	Mathematical process—developing a model
Criterion D	Results—interpretation
Criterion E	Use of technology
Criterion F	Quality of work

Applying the assessment criteria

The method of assessment used is criterion referenced, not norm referenced. That is, the method of assessing each portfolio judges students by their performance in relation to identified assessment criteria and not in relation to the work of other students.

The aim is to find, for each criterion, the level descriptor that conveys most adequately the achievement levels attained by the student.

Read the description of each achievement level, starting with level 0, until one is reached that describes a level of achievement that has not been reached. The level of achievement gained by the student is therefore the preceding one and it is this that should be recorded.

For example, if, when considering successive achievement levels for a particular criterion, the description for level 3 does not apply then level 2 should be recorded.

For each criterion, whole numbers only may be recorded; fractions and decimals are not acceptable.

The highest achievement levels do not imply faultless performance and teachers should not hesitate to use the extremes, including zero, if they are appropriate descriptions of the work to be assessed.

The whole range of achievement levels should be awarded as appropriate. For a particular piece of work, a student who attains a high achievement level in relation to one criterion may not necessarily attain high achievement levels in relation to other criteria.

A student who attains a particular level of achievement in relation to one criterion does not necessarily attain similar levels of achievement in relation to the others. Teachers should not assume that the overall assessment of the students produces any particular distribution of scores.

It is recommended that the assessment criteria be available to students at all times.

The final mark

Each portfolio must contain two pieces of work (if more than two pieces have been completed the best two should be selected for submission).

To calculate the final mark:

- add all the achievement levels for both pieces of work together to give a total out of 40.

For example:

Criterion/tasks	A	B	C	D	E	F	Final mark
Type I	1	3	3	2	3	2	
Type II	2	3	4	2	2	2	
	3 +	6 +	7 +	4 +	5 +	4	**= 29**

The final mark is 29.

Incomplete portfolios

If only one piece of work is submitted, award zero for each criterion for the missing work.

Non-compliant portfolios

If two pieces of work are submitted, but they do not represent a type I and a type II task (for example, they are both type I or both type II tasks), mark both tasks. Apply a penalty of 10 marks to the final mark.

Level of tasks

Teachers should set tasks that are appropriate to the level of the course. In particular, tasks appropriate to a higher level course, rather than to a standard level course, should be set.

Achievement levels

Criterion A: use of notation and terminology

Achievement level	
0	The student does not use appropriate notation and terminology.
1	The student uses some appropriate notation and/or terminology.
2	The student uses appropriate notation and terminology in a consistent manner and does so throughout the work.

Tasks will probably be set before students are aware of the notation and/or terminology to be used. Therefore the key idea behind this criterion is to assess how well the students' use of terminology describes the context. Teachers should provide an appropriate level of background knowledge in the form of notes given to students at the time the task is set.

Correct mathematical notation is required, but it can be accompanied by calculator notation, particularly when students are substantiating their use of technology.

This criterion addresses appropriate use of mathematical symbols (for example, use of "≈" instead of "=", and proper vector notation).

Word processing a document does not increase the level of achievement for this criterion or for criterion B.

Students should take care to write in appropriate mathematical symbols if the word processing software does not supply them. For example, using *x*^2 instead of x^2 would be considered a lack of proper usage and the candidate would not achieve a level 2.

Criterion B: communication

Achievement level	
0	The student neither provides explanations nor uses appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).
1	The student attempts to provide explanations or uses some appropriate forms of representation (for example, symbols, tables, graphs and/or diagrams).
2	The student provides adequate explanations or arguments, and communicates them using appropriate forms of representation (for example, symbols, tables, graphs, and/or diagrams).
3	The student provides complete, coherent explanations or arguments, and communicates them clearly using appropriate forms of representation (for example, symbols, tables, graphs, and/or diagrams).

This criterion also assesses how coherent the work is. The work can achieve a good mark if the reader does not need to refer to the wording used to set the task. In other words, the task can be marked independently.

Level 2 cannot be achieved if the student only writes down mathematical computations without explanation.

Graphs, tables and diagrams should accompany the work in the appropriate place and not be attached to the end of the document. Graphs must be correctly labelled and must be neatly drawn on graph paper. Graphs generated by a computer program or a calculator "screen dump" are acceptable providing that all items are correctly labelled, even if the labels are written in by hand. Colour keying the graphs can increase clarity of communication.

Criterion C: mathematical process

Type I—mathematical investigation: searching for patterns

Achievement level	
0	The student does not attempt to use a mathematical strategy.
1	The student uses a mathematical strategy to produce data.
2	The student organizes the data generated.
3	The student attempts to analyse data to enable the formulation of a general statement.
4	The student successfully analyses the correct data to enable the formulation of a general statement.
5	The student tests the validity of the general statement by considering further examples.

Students can only achieve a level 3 if the amount of data generated is sufficient to warrant an analysis.

Type II—mathematical modelling: developing a model

Achievement level	
0	The student does not define variables, parameters or constraints of the task.
1	The student defines some variables, parameters or constraints of the task.
2	The student defines variables, parameters and constraints of the task and attempts to create a mathematical model.
3	The student correctly analyses variables, parameters and constraints of the task to enable the formulation of a mathematical model that is relevant to the task and consistent with the level of the course.
4	The student considers how well the model fits the data.
5	The student applies the model to other situations.

At achievement level 5, applying the model to other situations could include, for example, a change of parameter or more data.

Criterion D: results

Type I—mathematical investigation: generalization

Achievement level	
0	The student does not produce any general statement consistent with the patterns and/or structures generated.
1	The student attempts to produce a general statement that is consistent with the patterns and/or structures generated.
2	The student correctly produces a general statement that is consistent with the patterns and/or structures generated.
3	The student expresses the correct general statement in appropriate mathematical terminology.
4	The student correctly states the scope or limitations of the general statement.
5	The student gives a correct, formal proof of the general statement.

A student who gives a correct formal proof of the general statement that does not take into account scope or limitations would achieve level 4.

Type II—mathematical modelling: interpretation

Achievement level	
0	The student has not arrived at any results.
1	The student has arrived at some results.
2	The student has not interpreted the reasonableness of the results of the model in the context of the task.
3	The student has attempted to interpret the reasonableness of the results of the model in the context of the task, to the appropriate degree of accuracy.
4	The student has correctly interpreted the reasonableness of the results of the model in the context of the task, to the appropriate degree of accuracy.
5	The student has correctly and critically interpreted the reasonableness of the results of the model in the context of the task, to include possible limitations and modifications of the results, to the appropriate degree of accuracy.

Criterion E: use of technology

Achievement level	
0	The student uses a calculator or computer for only routine calculations.
1	The student attempts to use a calculator or computer in a manner that could enhance the development of the task.
2	The student makes limited use of a calculator or computer in a manner that enhances the development of the task.
3	The student makes full and resourceful use of a calculator or computer in a manner that significantly enhances the development of the task.

The level of calculator or computer technology varies from school to school. Therefore, teachers should state the level of the technology that is available to their students.

Using a computer and/or a GDC to generate graphs or tables may not significantly contribute to the development of the task.

Criterion F: quality of work

Achievement level	
0	The student has shown a poor quality of work.
1	The student has shown a satisfactory quality of work.
2	The student has shown an outstanding quality of work.

Students who satisfy all the requirements correctly achieve level 1. For a student to achieve level 2, work must show precision, insight and a sophisticated level of mathematical understanding.