Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – calculus

Monday 8 May 2017 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[50 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 7]

Use l'Hôpital's rule to determine the value of

$$\lim_{x \to 0} \frac{\sin^2 x}{x \ln(1+x)}.$$

2. [Maximum mark: 6]

Let the Maclaurin series for $\tan x$ be

$$\tan x = a_1 x + a_3 x^3 + a_5 x^5 + \ldots$$

where a_1, a_3 and a_5 are constants.

(a) Find series for $\sec^2 x$, in terms of a_1, a_3 and a_5, up to and including the x^4 term

(i) by differentiating the above series for $\tan x$;

(ii) by using the relationship $\sec^2 x = 1 + \tan^2 x$. [3]

(b) Hence, by comparing your two series, determine the values of a_1, a_3 and a_5. [3]

3. [Maximum mark: 9]

Use the integral test to determine whether the infinite series $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$ is convergent or divergent.

4. [Maximum mark: 13]

(a) Consider the differential equation

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), x > 0.$$

Use the substitution $y = vx$ to show that the general solution of this differential equation is

$$\int \frac{dv}{f(v) - v} = \ln x + \text{Constant}.$$

[3]

(b) Hence, or otherwise, solve the differential equation

$$\frac{dy}{dx} = \frac{x^2 + 3xy + y^2}{x^2}, \ x > 0,$$

given that $y = 1$ when $x = 1$. Give your answer in the form $y = g(x)$. [10]

5. [Maximum mark: 15]

Consider the curve $y = \frac{1}{x}, x > 0$.

(a) By drawing a diagram and considering the area of a suitable region under the curve, show that for $r > 0$,

$$\frac{1}{r+1} < \ln\left(\frac{r+1}{r}\right) < \frac{1}{r}.$$ [4]

(b) Hence, given that n is a positive integer greater than one, show that

(i) $\sum_{r=1}^{n} \frac{1}{r} > \ln(1+n)$;

(ii) $\sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n$. [6]

Let $U_n = \sum_{r=1}^{n} \frac{1}{r} - \ln n$.

(c) Hence, given that n is a positive integer greater than one, show that

(i) $U_n > 0$;

(ii) $U_{n+1} < U_n$. [4]

(d) Explain why these two results prove that $\{U_n\}$ is a convergent sequence. [1]