Diploma Programme
Programme du diplôme
Programa del Diploma

Mathematics
Higher level
Paper 3 – sets, relations and groups

Thursday 21 May 2015 (afternoon)

1 hour

Instructions to candidates

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the **mathematics HL and further mathematics HL formula booklet** is required for this paper.
- The maximum mark for this examination paper is **[60 marks]**.

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Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. [Maximum mark: 13]

Consider the set $S_3 = \{p, q, r, s, t, u\}$ of permutations of the elements of the set $\{1, 2, 3\}$, defined by

$$p = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ q = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ r = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix},$$

$$s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ t = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ u = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

Let $\circ$ denote composition of permutations, so $a \circ b$ means b followed by a. You may assume that $(S_3, \circ)$ forms a group.

(a) Complete the following Cayley table

$\circ$	***p***	***q***	***r***	***s***	***t***	***u***
p						
q			t			s
r		u		t	s	q
s		t	u			r
t		s	q	r		
u		r	s	q		

[5]

(b) (i) State the inverse of each element.

(ii) Determine the order of each element. [6]

(c) Write down the subgroups containing

(i) r,

(ii) u. [2]

2. [Maximum mark: 10]

The binary operation $*$ is defined for $x, y \in S = \{0, 1, 2, 3, 4, 5, 6\}$ by

$$x * y = \left(x^3 y - xy\right) \bmod 7 .$$

(a) Find the element e such that $e * y = y$, for all $y \in S$. [2]

(b) (i) Find the least solution of $x * x = e$.

(ii) Deduce that $(S, *)$ is not a group. [5]

(c) Determine whether or not e is an identity element. [3]

3. [Maximum mark: 12]

The relation R is defined on $\mathbb{Z}$ by xRy if and only if $x^2 y \equiv y \bmod 6$.

(a) Show that the product of three consecutive integers is divisible by 6. [2]

(b) Hence prove that R is reflexive. [3]

(c) Find the set of all y for which $5Ry$. [3]

(d) Find the set of all y for which $3Ry$. [2]

(e) Using your answers for (c) and (d) show that R is not symmetric. [2]

4. [Maximum mark: 9]

Let X and Y be sets. The functions $f : X \to Y$ and $g : Y \to X$ are such that $g \circ f$ is the identity function on X.

(a) Prove that

(i) f is an injection,

(ii) g is a surjection. [6]

(b) Given that $X = \mathbb{R}^+ \cup \{0\}$ and $Y = \mathbb{R}$, choose a suitable pair of functions f and g to show that g is not necessarily a bijection. [3]

5. [Maximum mark: 16]

Consider the sets

$$G = \left\{ \frac{n}{6^i} \mid n \in \mathbb{Z}, i \in \mathbb{N} \right\}, \; H = \left\{ \frac{m}{3^j} \mid m \in \mathbb{Z}, j \in \mathbb{N} \right\}.$$

(a) Show that $(G, +)$ forms a group where $+$ denotes addition on $\mathbb{Q}$. Associativity may be assumed. [5]

(b) Assuming that $(H, +)$ forms a group, show that it is a proper subgroup of $(G, +)$. [4]

The mapping $\phi : G \to G$ is given by $\phi(g) = g + g$, for $g \in G$.

(c) Prove that ϕ is an isomorphism. [7]