22117204

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 2

Thursday 5 May 2011 (morning)

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

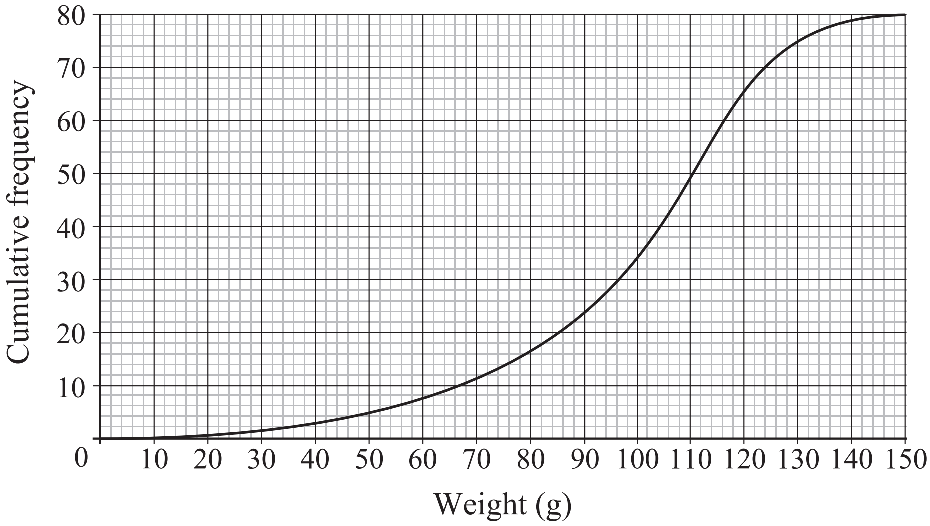
Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 4]*

The cumulative frequency graph below represents the weight in grams of 80 apples picked from a particular tree.



(a) Estimate the

(i) median weight of the apples;

(ii) 30^{th} percentile of the weight of the apples. *[2 marks]*

(b) Estimate the number of apples which weigh more than 110 grams. *[2 marks]*

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2. *[Maximum mark: 6]*

Consider the function $f(x) = x^3 - 3x^2 - 9x + 10$, $x \in \mathbb{R}$.

(a) Find the equation of the straight line passing through the maximum and minimum points of the graph $y = f(x)$. *[4 marks]*

(b) Show that the point of inflexion of the graph $y = f(x)$ lies on this straight line. *[2 marks]*

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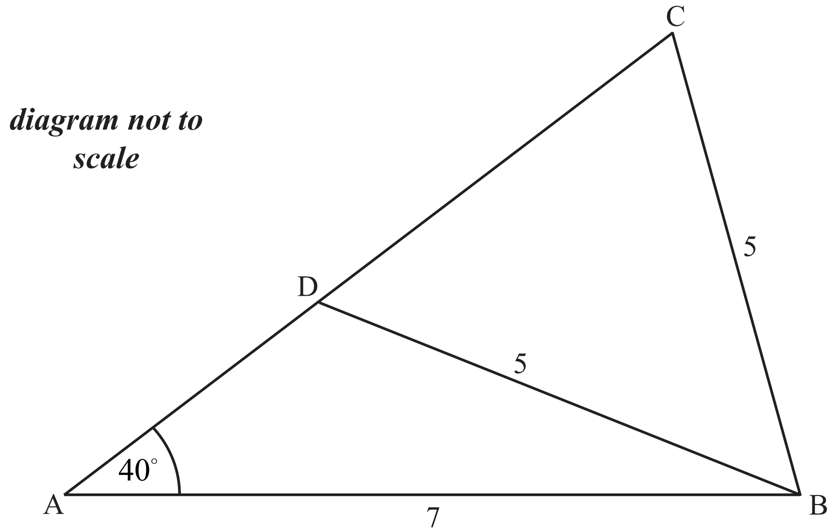
3. *[Maximum mark: 5]*

Given ΔABC, with lengths shown in the diagram below, find the length of the line segment [CD].



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4. *[Maximum mark: 5]*

The function $f(x) = 4x^3 + 2ax - 7a$, $a \in \mathbb{R}$, leaves a remainder of -10 when divided by $(x - a)$.

(a) Find the value of a. *[3 marks]*

(b) Show that for this value of a there is a unique real solution to the equation $f(x) = 0$. *[2 marks]*

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5. *[Maximum mark: 5]*

(a) Write down the quadratic expression $2x^2 + x - 3$ as the product of two linear factors. *[1 mark]*

(b) Hence, or otherwise, find the coefficient of x in the expansion of $(2x^2 + x - 3)^8$. *[4 marks]*

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6. *[Maximum mark: 7]*

The radius of the circle with centre C is 7 cm and the radius of the circle with centre D is 5 cm. If the length of the chord [AB] is 9 cm, find the area of the shaded region enclosed by the two arcs AB.

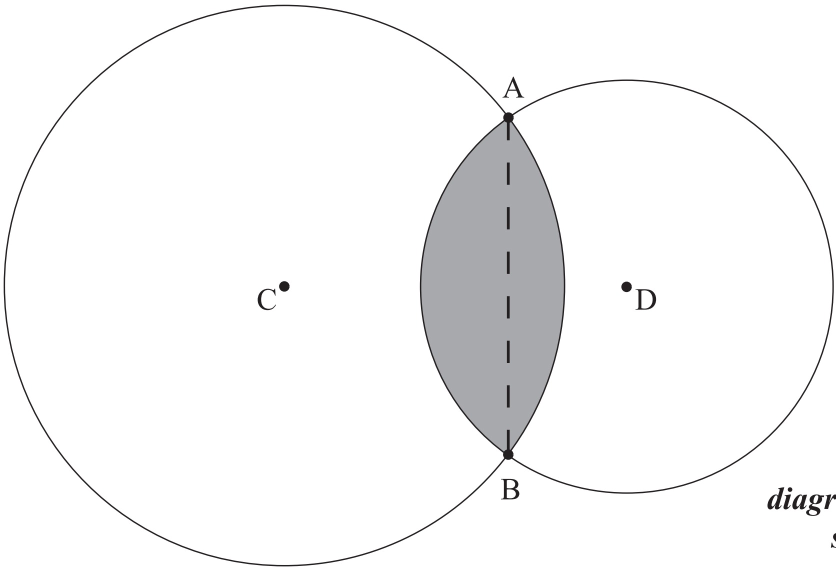

diagram not to scale

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7. *[Maximum mark: 7]*

A continuous random variable X has a probability density function given by the function $f(x)$, where

$$f(x) = \begin{cases} k(x+2)^2, & -2 \le x < 0 \\ k, & 0 \le x \le \frac{4}{3} \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of k. *[2 marks]*

(b) Hence find

(i) the mean of X;

(ii) the median of X. *[5 marks]*

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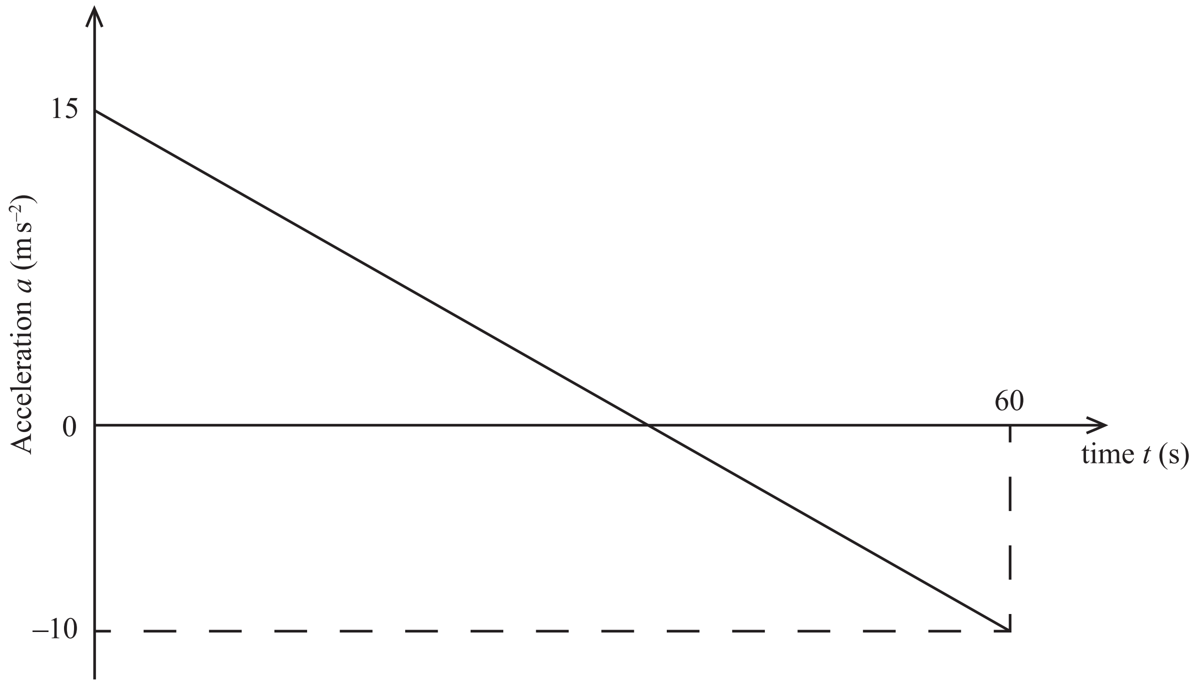
8. *[Maximum mark: 8]*

A jet plane travels horizontally along a straight path for one minute, starting at time $t = 0$, where t is measured in seconds. The acceleration, a, measured in m s^{-2}, of the jet plane is given by the straight line graph below.



(a) Find an expression for the acceleration of the jet plane during this time, in terms of t. *[1 mark]*

(b) Given that when $t = 0$ the jet plane is travelling at 125 m s^{-1}, find its maximum velocity in m s^{-1} during the minute that follows. *[4 marks]*

(c) Given that the jet plane breaks the sound barrier at 295 m s^{-1}, find out for how long the jet plane is travelling greater than this speed. *[3 marks]*

9. *[Maximum mark: 6]*

Solve the following system of equations.

$$\log_{x+1} y = 2$$
$$\log_{y+1} x = \frac{1}{4}$$

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10. *[Maximum mark: 7]*

Port A is defined to be the origin of a set of coordinate axes and port B is located at the point $(70, 30)$, where distances are measured in kilometres. A ship S_1 sails from port A at 10:00 in a straight line such that its position t hours after 10:00 is given by $\boldsymbol{r} = t\begin{pmatrix}10\\20\end{pmatrix}$.

A speedboat S_2 is capable of three times the speed of S_1 and is to meet S_1 by travelling the shortest possible distance. What is the latest time that S_2 can leave port B?

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*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 14]*

The equations of three planes, are given by

$$ax + 2y + z = 3$$
$$-x + (a+1)y + 3z = 1$$
$$-2x + y + (a+2)z = k$$

where $a \in \mathbb{R}$.

(a) Given that $a = 0$, show that the three planes intersect at a point. *[3 marks]*

(b) Find the value of a such that the three planes do **not** meet at a point. *[5 marks]*

(c) Given a such that the three planes do **not** meet at a point, find the value of k such that the planes meet in one line and find an equation of this line in the form

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + \lambda \begin{pmatrix} l \\ m \\ n \end{pmatrix}.$$ *[6 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

12. *[Maximum mark: 17]*

A student arrives at a school X minutes after 08:00, where X may be assumed to be normally distributed. On a particular day it is observed that 40 % of the students arrive before 08:30 and 90 % arrive before 08:55.

(a) Find the mean and standard deviation of X. *[5 marks]*

(b) The school has 1200 students and classes start at 09:00. Estimate the number of students who will be late on that day. *[3 marks]*

(c) Maelis had not arrived by 08:30. Find the probability that she arrived late. *[2 marks]*

At 15:00 it is the end of the school day and it is assumed that the departure of the students from school can be modelled by a Poisson distribution. On average 24 students leave the school every minute.

(d) Find the probability that at least 700 students leave school before 15:30. *[3 marks]*

(e) There are 200 days in a school year. Given that Y denotes the number of days in the year that at least 700 students leave before 15:30, find

(i) $\mathrm{E}(Y)$;

(ii) $\mathrm{P}(Y > 150)$. *[4 marks]*

*Do **NOT** write solutions on this page. Any working on this page will **NOT** be marked.*

13. *[Maximum mark: 13]*

(a) Given that $\boldsymbol{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, show that $\boldsymbol{A}^2 = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ -\sin 2\theta & \cos 2\theta \end{pmatrix}$. *[3 marks]*

(b) Prove by induction that

$$\boldsymbol{A}^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}, \text{ for all } n \in \mathbb{Z}^+.$$ *[7 marks]*

(c) Given that $\boldsymbol{A}^{-1}$ is the inverse of matrix $\boldsymbol{A}$, show that the result in part (b) is true where $n = -1$. *[3 marks]*

14. *[Maximum mark: 16]*

An open glass is created by rotating the curve $y = x^2$, defined in the domain $x \in [0, 10]$, 2π radians about the y-axis. Units on the coordinate axes are defined to be in centimetres.

(a) When the glass contains water to a height h cm, find the volume V of water in terms of h. *[3 marks]*

(b) If the water in the glass evaporates at the rate of 3 cm^3 per hour for each cm^2 of exposed surface area of the water, show that,

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -3\sqrt{2\pi V}, \text{ where } t \text{ is measured in hours.}$$ *[6 marks]*

(c) If the glass is filled completely, how long will it take for all the water to evaporate? *[7 marks]*