88107209

International Baccalaureate®
Baccalauréat International
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MATHEMATICS
HIGHER LEVEL
PAPER 3 – SETS, RELATIONS AND GROUPS

Monday 15 November 2010 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 8]*

Let R be a relation on the set $\mathbb{Z}$ such that $aRb \Leftrightarrow ab \geq 0$, for $a, b \in \mathbb{Z}$.

(a) Determine whether R is

(i) reflexive;

(ii) symmetric;

(iii) transitive. *[7 marks]*

(b) Write down with a reason whether or not R is an equivalence relation. *[1 mark]*

2. *[Maximum mark: 16]*

(a) Let $f : \mathbb{Z} \times \mathbb{R} \to \mathbb{R}$, $f(m, x) = (-1)^m x$. Determine whether f is

(i) surjective;

(ii) injective. *[4 marks]*

(b) P is the set of all polynomials such that $P = \left\{ \sum_{i=0}^{n} a_i x^i \;\middle|\; n \in \mathbb{N} \right\}$.
Let $g : P \to P$, $g(p) = xp$. Determine whether g is

(i) surjective;

(ii) injective. *[4 marks]*

(c) Let $h : \mathbb{Z} \to \mathbb{Z}^+$, $h(x) = \begin{cases} 2x, & x > 0 \\ 1 - 2x, & x \leq 0 \end{cases}$. Determine whether h is

(i) surjective;

(ii) injective. *[7 marks]*

(d) Write down which, if any, of the above functions are bijective. *[1 mark]*

3. *[Maximum mark: 8]*

Prove that for sets A and B

$$A \times (B \cap C) = (A \times B) \cap (A \times C).$$

4. *[Maximum mark: 20]*

Set $S = \{x_0, x_1, x_2, x_3, x_4, x_5\}$ and a binary operation $\circ$ on S is defined as $x_i \circ x_j = x_k$, where $i + j \equiv k \pmod 6$.

(a) (i) Construct the Cayley table for $\{S, \circ\}$ and hence show that it is a group.

(ii) Show that $\{S, \circ\}$ is cyclic. *[11 marks]*

(b) Let $\{G, *\}$ be an Abelian group of order 6. The element $a \in G$ has order 2 and the element $b \in G$ has order 3.

(i) Write down the six elements of $\{G, *\}$.

(ii) Find the order of $a * b$ and hence show that $\{G, *\}$ is isomorphic to $\{S, \circ\}$. *[9 marks]*

5. *[Maximum mark: 8]*

Let $\{G, *\}$ be a finite group that contains an element a (that is not the identity element) and $H = \{a^n \mid n \in \mathbb{Z}^+\}$, where $a^2 = a * a$, $a^3 = a * a * a$ *etc.*

Show that $\{H, *\}$ is a subgroup of $\{G, *\}$.