88097209

International Baccalaureate®
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**MATHEMATICS**
**HIGHER LEVEL**
**PAPER 3 – SETS, RELATIONS AND GROUPS**

Wednesday 18 November 2009 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.



3 pages

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*Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.*

**1.** *[Maximum mark: 13]*

The binary operation $*$ is defined on the set $S = \{0, 1, 2, 3\}$ by

$$a * b = a + 2b + ab \,(\text{mod}\, 4)\,.$$

(a) (i) Construct the Cayley table.

(ii) Write down, with a reason, whether or not your table is a Latin square. *[4 marks]*

(b) (i) Write down, with a reason, whether or not $*$ is commutative.

(ii) Determine whether or not $*$ is associative, justifying your answer. *[5 marks]*

(c) Find all solutions to the equation $x * 1 = 2 * x$, for $x \in S$. *[4 marks]*

**2.** *[Maximum mark: 10]*

The function $f : [0, \infty[ \to [0, \infty[$ is defined by $f(x) = 2\mathrm{e}^{x} + \mathrm{e}^{-x} - 3$.

(a) Find $f'(x)$. *[1 mark]*

(b) Show that $f$ is a bijection. *[3 marks]*

(c) Find an expression for $f^{-1}(x)$. *[6 marks]*

**3.** *[Maximum mark: 12]*

The relations $R$ and $S$ are defined on quadratic polynomials $P$ of the form

$$P(z) = z^2 + az + b, \text{ where } a, b \in \mathbb{R}, z \in \mathbb{C}.$$

(a) The relation $R$ is defined by $P_1 R P_2$ if and only if the sum of the two zeros of $P_1$ is equal to the sum of the two zeros of $P_2$.

(i) Show that $R$ is an equivalence relation.

(ii) Determine the equivalence class containing $z^2 - 4z + 5$. *[9 marks]*

(b) The relation $S$ is defined by $P_1 S P_2$ if and only if $P_1$ and $P_2$ have at least one zero in common. Determine whether or not $S$ is transitive. *[3 marks]*

**4.** *[Maximum mark: 16]*

(a) Show that the set of matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \text{ where } a, b \in \mathbb{R}^+$$

is a group $G$ under matrix multiplication.
(You may assume that matrix multiplication is associative.) *[7 marks]*

(b) Given that the set of matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & ab \end{pmatrix}, \text{ where } a, b \in \mathbb{R}^+$$

is a group $H$ under matrix multiplication, show that $G$ and $H$ are isomorphic. *[9 marks]*

**5.** *[Maximum mark: 9]*

Let $\{G, *\}$ be a finite group of order $n$ and let $H$ be a non-empty subset of $G$.

(a) Show that any element $h \in H$ has order smaller than or equal to $n$. *[3 marks]*

(b) If $H$ is closed under $*$, show that $\{H, *\}$ is a subgroup of $\{G, *\}$. *[6 marks]*