22097204

International Baccalaureate
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 2

Friday 8 May 2009 (morning)

2 hours

Candidate session number

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INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- A graphic display calculator is required for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Answer ***all*** *the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 6]*

Bob measured the heights of 63 students. After analysis, he conjectured that the height, H, of the students could be modelled by a normal distribution with mean 166.5 cm and standard deviation 5 cm.

(a) Based on this assumption, estimate the number of these students whose height is at least 170 cm. *[3 marks]*

Later Bob noticed that the tape he had used to measure the heights was faulty as it started at the 5 cm mark and not at the zero mark.

(b) What are the correct values of the mean and variance of the distribution of the heights of these students? *[3 marks]*

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2. *[Maximum mark: 6]*

(a) Show that the complex number i is a root of the equation

$$x^4 - 5x^3 + 7x^2 - 5x + 6 = 0 .$$

[2 marks]

(b) Find the other roots of this equation. *[4 marks]*

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3. *[Maximum mark: 7]*

Let $f(x)=\dfrac{1-x}{1+x}$ and $g(x)=\sqrt{x+1}$, $x>-1$.

Find the set of values of x for which $f'(x)\le f(x)\le g(x)$.

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4. *[Maximum mark: 7]*

Mr Lee is planning to go fishing this weekend. Assuming that the number of fish caught per hour follows a Poisson distribution with mean 0.6, find

(a) the probability that he catches at least one fish in the first hour; *[2 marks]*

(b) the probability that he catches exactly three fish if he fishes for four hours; *[2 marks]*

(c) the number of **complete** hours that Mr Lee needs to fish so that the probability of catching more than two fish exceeds 80 %. *[3 marks]*

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5. *[Maximum mark: 6]*

Find the angle between the lines $\frac{x-1}{2} = 1 - y = 2z$ and $x = y = 3z$.

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6. *[Maximum mark: 5]*

(a) Integrate $\int \frac{\sin\theta}{1-\cos\theta}\,\mathrm{d}\theta$. *[3 marks]*

(b) Given that $\int_{\frac{\pi}{2}}^{a} \frac{\sin\theta}{1-\cos\theta}\,\mathrm{d}\theta = \frac{1}{2}$ and $\frac{\pi}{2} < a < \pi$, find the value of a. *[2 marks]*

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7. *[Maximum mark: 8]*

Consider the planes defined by the equations $x + y + 2z = 2$, $2x - y + 3z = 2$ and $5x - y + az = 5$ where a is a real number.

(a) If $a = 4$ find the coordinates of the point of intersection of the three planes. *[2 marks]*

(b) (i) Find the value of a for which the planes do not meet at a unique point.

(ii) For this value of a show that the three planes do not have any common point. *[6 marks]*

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8. *[Maximum mark: 8]*

(a) Solve the differential equation $\frac{\cos^2 x}{e^y} - e^{e^y}\frac{dy}{dx} = 0$, given that $y = 0$ when $x = \pi$. *[7 marks]*

(b) Find the value of y when $x = \frac{\pi}{2}$. *[1 mark]*

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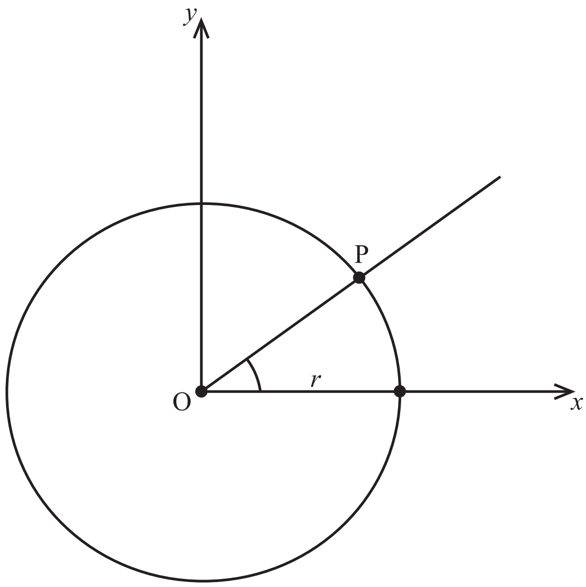
9. *[Maximum mark: 7]*

The diagram below shows a circle with centre at the origin O and radius $r > 0$.



A point $\text{P}(x,\ y),\ (x > 0,\ y > 0)$ is moving round the circumference of the circle.

Let $m = \tan\left(\arcsin \frac{y}{r}\right)$.

(a) Given that $\frac{\text{d}y}{\text{d}t} = 0.001r$, show that $\frac{\text{d}m}{\text{d}t} = \left(\frac{r}{10\sqrt{r^2 - y^2}}\right)^3$. *[6 marks]*

(b) State the geometrical meaning of $\frac{\text{d}m}{\text{d}t}$. *[1 mark]*

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SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

10. *[Maximum mark: 18]*

Let $\boldsymbol{A}=\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\end{pmatrix}$ and $\boldsymbol{B}=\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}$.

(a) Given that $\boldsymbol{X}=\boldsymbol{B}-\boldsymbol{A}^{-1}$ and $\boldsymbol{Y}=\boldsymbol{B}^{-1}-\boldsymbol{A}$,

(i) find $\boldsymbol{X}$ and $\boldsymbol{Y}$;

(ii) does $\boldsymbol{X}^{-1}+\boldsymbol{Y}^{-1}$ have an inverse? Justify your conclusion. *[5 marks]*

(b) Prove by induction that $\boldsymbol{A}^n=\begin{pmatrix}1&n&\frac{n(n+1)}{2}\\0&1&n\\0&0&1\end{pmatrix}$, for $n\in\mathbb{Z}^+$. *[7 marks]*

(c) Given that $(\boldsymbol{A}^n)^{-1}=\begin{pmatrix}1&x&y\\0&1&x\\0&0&1\end{pmatrix}$, for $n\in\mathbb{Z}^+$,

(i) find x and y in terms of n,

(ii) and hence find an expression for $\boldsymbol{A}^n+(\boldsymbol{A}^n)^{-1}$. *[6 marks]*

11. *[Maximum mark: 23]*

The position vector at time t of a point P is given by

$$\overrightarrow{\text{OP}} = (1+t)\boldsymbol{i} + (2-2t)\boldsymbol{j} + (3t-1)\boldsymbol{k},\ t \ge 0.$$

(a) Find the coordinates of P when $t = 0$. *[2 marks]*

(b) Show that P moves along the line L with Cartesian equations

$$x-1 = \frac{y-2}{-2} = \frac{z+1}{3}.$$ *[2 marks]*

(c) (i) Find the value of t when P lies on the plane with equation $2x + y + z = 6$.

(ii) State the coordinates of P at this time.

(iii) Hence find the total distance travelled by P before it meets the plane. *[6 marks]*

The position vector at time t of another point, Q, is given by

$$\overrightarrow{\text{OQ}} = \begin{pmatrix} t^2 \\ 1-t \\ 1-t^2 \end{pmatrix},\ t \ge 0.$$

(d) (i) Find the value of t for which the distance from Q to the origin is minimum.

(ii) Find the coordinates of Q at this time. *[6 marks]*

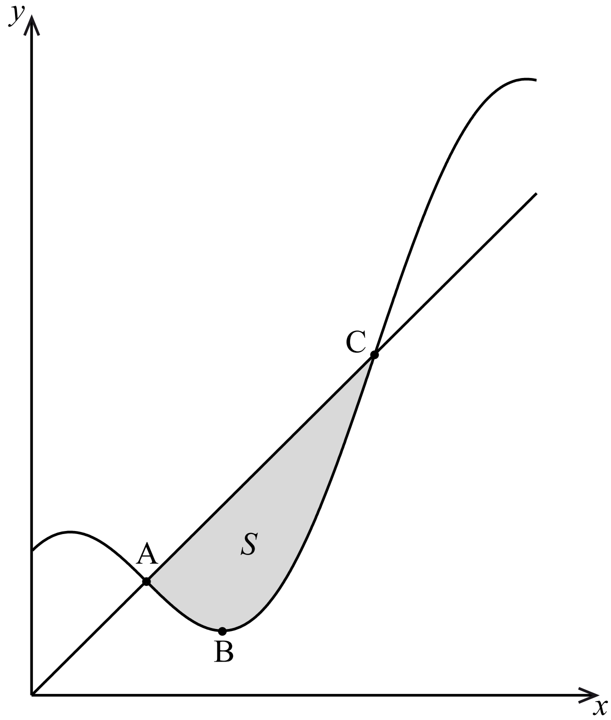
(e) Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be the position vectors of Q at times $t = 0$, $t = 1$ and $t = 2$ respectively.

(i) Show that the equation $\boldsymbol{a} - \boldsymbol{b} = k(\boldsymbol{b} - \boldsymbol{c})$ has no solution for k.

(ii) Hence show that the path of Q is not a straight line. *[7 marks]*

12. *[Maximum mark: 19]*

Let f be a function defined by $f(x) = x + 2\cos x$, $x \in [0, 2\pi]$. The diagram below shows a region S bound by the graph of f and the line $y = x$.



A and C are the points of intersection of the line $y = x$ and the graph of f, and B is the minimum point of f.

(a) If A, B and C have x-coordinates $a\frac{\pi}{2}$, $b\frac{\pi}{6}$ and $c\frac{\pi}{2}$, where $a, b, c \in \mathbb{N}$, find the values of a, b and c. *[4 marks]*

(b) Find the range of f. *[3 marks]*

(c) Find the equation of the normal to the graph of f at the point C, giving your answer in the form $y = px + q$. *[5 marks]*

(d) The region S is rotated through 2π about the x-axis to generate a solid.

(i) Write down an integral that represents the volume V of this solid.

(ii) Show that $V = 6\pi^2$. *[7 marks]*