MATHEMATICS
HIGHER LEVEL
PAPER 3

Friday 16 November 2007 (afternoon)

1 hour

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all the questions in one section only.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

Statistics and Probability

1. *[Maximum mark: 11]*

Juan plays a quiz game. The scores he achieves on the separate topics may be modelled by independent normal distributions.

(a) On the topic of sport, the scores have the distribution $\mathrm{N}(75, 12^2)$.

Find the probability that Juan scores less than 57 points on the topic of sport. *[2 marks]*

(b) On the topic of literature, Juan's scores have a mean of 45, and 30 % of his scores are greater than 50.

Find the standard deviation of his scores on the topic of literature. *[3 marks]*

(c) Juan claims that he scores better in current affairs than in sport. He achieves the following scores on current affairs in 10 separate quizzes.

91 84 75 92 88 71 83 90 85 78

Perform a hypothesis test at the 5 % significance level to decide whether there is evidence to support his claim. *[6 marks]*

2. *[Maximum mark: 20]*

(a) A horse breeder records the number of births for each of 100 horses during the past eight years. The results are summarized in the following table:

Number of births	0	1	2	3	4	5	6
Frequency	1	5	26	37	18	12	1

Stating null and alternative hypotheses carry out an appropriate test at the 5 % significance level to decide whether the results can be modelled by $\mathrm{B}(6,\ 0.5)$. *[10 marks]*

(b) Without doing any further calculations, explain briefly how you would carry out a test, at the 5 % significance level, to decide if the data can be modelled by $\mathrm{B}(6,\ p)$, where p is unspecified. *[2 marks]*

(c) A different horse breeder collected data on the time and outcome of births. The data are summarized in the following table:

Outcome	**Day**	**Night**
Live male	68	42
Live female	103	57
Dead male	8	18
Dead female	6	12

Carry out an appropriate test at the 5 % significance level to decide whether there is an association between time and outcome. *[8 marks]*

3. *[Maximum mark: 15]*

(a) The exponential distribution has the probability density function

$$f(x)=\begin{cases}\lambda e^{-\lambda x}, & \text{for } x \ge 0\\ 0, & \text{for } x<0.\end{cases}$$

Show that the mean is greater than the median. *[4 marks]*

(b) The time in seconds between arrivals of butterflies on a flowering bush can be modelled by an exponential distribution with parameter $\lambda = 0.1$.

A butterfly arrives on the bush.

(i) Calculate the probability that no other butterfly arrives within 20 seconds.

(ii) Given that no other butterfly has arrived within 20 seconds, calculate the probability that the next butterfly arrives within 50 seconds of the first. *[6 marks]*

(c) The number of goals scored by a soccer team in a period of duration t minutes follows a Poisson distribution with mean $\frac{t}{36}$.

(i) Write down the probability that no goals are scored during a period of duration t minutes.

(ii) The random variable T is defined as the length of time, in minutes, between successive goals. Show that T follows an exponential distribution. *[5 marks]*

4. *[Maximum mark: 14]*

(a) It was found that x people in a sample of 225 supported a smoking ban in public places. If the 95 % confidence interval for the proportion of people supporting the ban in the population from which the sample was taken is $[0.2297, 0.3481]$ calculate the value of x. *[4 marks]*

(b) A coin is thought to be biased.

To test the coin for bias, Amanda suggests that it should be tossed three times. If all three tosses are heads or all three tosses are tails, then we conclude that the coin is biased.

Roger suggests that it should be tossed eight times. If at least six tosses are heads or at least six tosses are tails, then we conclude that the coin is biased.

(i) Determine which of the two methods has the smaller probability of making a Type I error.

(ii) Determine the probability that Roger will make a Type II error when the probability of a head is actually 0.6. *[10 marks]*

SECTION B

Sets, relations and groups

1. *[Maximum mark: 11]*

For each $n \in \mathbb{Z}^+$, a subset of $\mathbb{Z}^+$ is defined by $S_n = \{x \in \mathbb{Z}^+ \mid n \text{ divides } x\}$.

(a) Express in simplest terms the membership of the following sets:

(i) S_1;

(ii) S_2';

(iii) $S_2 \cap S_3$;

(iv) $S_6 \setminus S_3$. *[7 marks]*

(b) Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$. *[4 marks]*

2. *[Maximum mark: 12]*

Two relations, *M* and *N*, are defined on $\mathbb{R}$ by:

$$xMy \text{ if and only if } |x| \le |y|;$$
$$xNy \text{ if and only if } x^2 - 2x = y^2 - 2y.$$

(a) Determine whether *M* is an equivalence relation. *[2 marks]*

(b) (i) Prove that *N* is an equivalence relation.

(ii) Determine the equivalence classes of *N*.

(iii) Find the equivalence class containing only one element. *[10 marks]*

3. *[Maximum mark: 15]*

Let $F(x) = x^2 - |x-2|$.

(a) The function f is defined by

$$f :]-\infty, 1] \rightarrow \mathbb{R}, \text{ where } f(x) = F(x).$$

Find the range of f and determine whether it is an injection. *[5 marks]*

(b) The function g is defined by

$$g : [1, \infty[\rightarrow [0, \infty[, \text{ where } g(x) = F(x).$$

Show that g has an inverse and find this inverse. *[10 marks]*

4. *[Maximum mark: 22]*

(a) S is the set of permutations $\{P, Q, R, T\}$ where

$$P = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix},\ Q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix},\ R = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix},\ T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Complete a Cayley table for S under the operation $*$ (composition of permutations) and show that $(S, *)$ is a cyclic group. *[9 marks]*

(b) A binary operation $\otimes$ is defined on $\mathbb{R}$ by

$$x \otimes y = x + y + sxy$$

where s is a fixed non-zero real number.

(i) Show that $\otimes$ is commutative and associative on $\mathbb{R}$.

(ii) Find the identity, and hence find the inverse, of the element x. Show, however, that one particular value of x has no inverse. Call this value t.

(iii) Determine whether or not $\{\mathbb{R} \setminus t, \otimes\}$ is an Abelian group.
Find a subset of $\mathbb{R}$ which forms a group of order two. *[13 marks]*

SECTION C

Series and differential equations

1. *[Maximum mark: 19]*

(a) (i) State the domain and range of the function $f(x) = \arcsin(x)$.

(ii) Determine the first two non-zero terms in the Maclaurin series for $f(x)$. *[8 marks]*

(b) Use the small angle approximation

$$\cos(y) \approx 1 - \frac{y^2}{2} + \frac{y^4}{24}$$

to find a series for $\cos(\arcsin(x))$ up to and including the term in x^4. *[4 marks]*

(c) (i) Find the Maclaurin series for $(p + qx^2)^r$ up to and including the term in x^4 where $p, q, r \in \mathbb{R}$.

(ii) Find values of p, q, r such that your series in (c)(i) is identical to your answer to (b). Comment on this result. *[7 marks]*

2. *[Maximum mark: 10]*

(a) Determine $\lim\limits_{x \to 0}\left(\dfrac{\ln(a^2 + x^2)}{\ln(a - x^3)}\right)$, where a is a positive constant, not equal to 1. *[3 marks]*

(b) Calculate $\lim\limits_{x \to 0}\left(\dfrac{\ln(1 + x^2)}{\ln(1 - x^2)}\right)$. *[3 marks]*

(c) Show that $\lim\limits_{x \to 0}\left(\dfrac{2 + x^2 - 2\cos x}{e^x + e^{-x} - 2\cos x}\right) = 1$. *[4 marks]*

3. *[Maximum mark: 14]*

(a) A homogeneous differential equation has the form

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right).$$

Show that the substitution $v = \frac{y}{x}$ leads to a differential equation which can be solved by separation of variables. *[3 marks]*

(b) Show that the linear change of variables $X = x - 1, Y = y - 2$, transforms the equation

$$\frac{dy}{dx} = \frac{x + 3y - 7}{3x - y - 1} \text{ to a homogeneous form.}$$

Hence solve this equation. *[11 marks]*

4. *[Maximum mark: 17]*

(a) (i) By considering the graph of $y=\frac{1}{x}$ between $x=a$ and $x=a+1$, show that

$$\frac{1}{a+1}<\ln\left(\frac{a+1}{a}\right)<\frac{1}{2}\left(\frac{1}{a}+\frac{1}{a+1}\right), \text{ for all } a>0.$$

(ii) Deduce that

$$\frac{1}{2}<\ln 2<\frac{3}{4}.$$

(iii) Using (i) find $p, q \in \mathbb{Q}$ such that $p<\ln 3<q$. *[8 marks]*

(b) Given that H_n is the n^{th} partial sum of the harmonic series, show that

$$H_n-1<\ln n<H_n-\frac{1}{2}-\frac{1}{2n}, \text{ for } n>1.$$ *[5 marks]*

(c) If $\gamma_n=H_n-\ln n$, prove that the terms of the sequence $\{\gamma_n : n\geq 1\}$ decrease as n increases. *[4 marks]*

SECTION D

Discrete mathematics

1. *[Maximum mark: 15]*

(a) (i) Find the base 8 representation of the decimal number 1001.

(ii) Prove that a number is divisible by 7 if and only if the sum of the digits in its base 8 representation is also divisible by 7.

(iii) Use the results from (i) and (ii) to show that the decimal number 1001 is divisible by 7. *[9 marks]*

(b) Solve the system of simultaneous congruences:

$$x \equiv 4 \pmod 9$$
$$x \equiv 3 \pmod 5.$$ *[6 marks]*

2. *[Maximum mark: 9]*

The weights of the edges of a graph are given in the following table:

	A	B	C	D	E	F	G	H	I
A		5		2					
B	5		4	3	5	6			
C		4				3			
D	2	3			7		6	8	
E		5		7		1		3	
F		6	3		1			4	4
G				6				4	
H				8	3	4	4		2
I						4		2	

(a) Starting at A, use Prim's algorithm to find a minimum spanning tree for this graph. *[7 marks]*

(b) Draw this minimum spanning tree and state its weight. *[2 marks]*

3. *[Maximum mark: 19]*

A graph G has e edges and n vertices.

(a) Show that the sum of the degrees of the vertices is twice the number of edges. *[1 mark]*

(b) Deduce that G has an even number of vertices of odd degree. *[4 marks]*

(c) (i) Graph G is connected, planar and divides the plane into exactly four regions. If $(n-1)$ vertices have degree three and exactly one vertex has degree d, determine the possible values of (n, d).

(ii) For each possible (n, d), draw a graph which satisfies the conditions described in (c)(i). *[14 marks]*

4. *[Maximum mark: 17]*

(a) Show that $10^n + 3\times 4^{n+2} + 5$, $n \in \mathbb{Z}^+$ is a multiple of 9. *[8 marks]*

(b) (i) Use Fermat's Little Theorem to show that $x \equiv a^{p-2}b \pmod{p}$ is a solution to the linear congruence $ax \equiv b \pmod{p}$ where p is a prime number and $a, b \in \mathbb{Z}^+$.

Hence solve $4x \equiv 3 \pmod{7}$.

(ii) Find the last digit in the base 9 expansion of 5^{155}. *[9 marks]*