MARKSCHEME

May 2006

MATHEMATICS

Higher Level

Paper 3

Instructions to Examiners

Abbreviations

M Marks awarded for attempting to use a correct **Method**; working must be seen.

(M) Marks awarded for **Method**; may be implied by **correct** subsequent working.

A Marks awarded for an **Answer** or for **Accuracy**: often dependent on preceding ***M*** marks.

(A) Marks awarded for an **Answer** or for **Accuracy**; may be implied by **correct** subsequent working.

R Marks awarded for clear **Reasoning**.

N Marks awarded for **correct** answers if **no** working shown (or working which gains no other marks).

AG Answer given in the question and so no marks are awarded.

Using the markscheme

1 General

Write the marks in red on candidates' scripts, in the right hand margin.

- Show the **breakdown** of individual marks awarded using the abbreviations ***M1***, ***A1***, *etc*.
- Write down the total for each **question** (at the end of the question) and **circle** it.

2 Method and Answer/Accuracy marks

- Do **not** automatically award full marks for a correct answer; all working **must** be checked, and marks awarded according to the markscheme.
- It is not possible to award ***M0*** followed by ***A1***, as ***A*** mark(s) depend on the preceding ***M*** mark(s), if any.
- Where ***M*** and ***A*** marks are noted on the same line, *e.g.* ***M1A1***, this usually means ***M1*** for an **attempt** to use an appropriate method (*e.g.* substitution into a formula) and ***A1*** for using the **correct** values.
- Where the markscheme specifies ***(M2)***, ***N3***, *etc.*, do **not** split the marks.
- Once a correct answer to a question or part-question is seen, ignore further working.

3 *N* marks

*Award **N** marks for **correct** answers where there is **no** working, (or working which gains no other marks).*

- Do **not** award a mixture of ***N*** and other marks.
- There may be fewer ***N*** marks available than the total of ***M***, ***A*** and ***R*** marks; this is deliberate as it penalizes candidates for not following the instruction to show their working.
- For consistency within the markscheme, ***N*** marks are noted for every part, even when these match the mark breakdown. In these cases, the marks may be recorded in either form *e.g.* ***A2*** or ***N2***.

4 Implied marks

Implied marks appear in ***brackets e.g. (M1)****, and can only be awarded if* ***correct*** *work is seen or if implied in subsequent working.*

- Normally the correct work is seen or implied in the next line.
- Marks **without** brackets can only be awarded for work that is **seen**.

5 Follow through marks

*Follow through (****FT****) marks are awarded where an incorrect answer from one* ***part*** *of a question is used correctly in* ***subsequent*** *part(s). To award* ***FT*** *marks,* ***there must be working present*** *and not just a final answer based on an incorrect answer to a previous part.*

- If the question becomes much simpler because of an error then use discretion to award fewer ***FT*** marks.
- If the error leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).
- Within a question part, once an error is made, no further **dependent** ***A*** marks can be awarded, but ***M*** marks may be awarded if appropriate.
- Exceptions to this rule will be explicitly noted on the markscheme.

6 Mis-read

*If a candidate incorrectly copies information from the question, this is a mis-read (****MR****). Apply a* ***MR*** *penalty of 1 mark to that question. Award the marks as usual and then write –1(****MR****) next to the total. Subtract 1 mark from the total for the question. A candidate should be penalised only once for a particular mis-read.*

- If the question becomes much simpler because of the ***MR***, then use discretion to award fewer marks.
- If the ***MR*** leads to an inappropriate value (*e.g.* $\sin\theta = 1.5$), do not award the mark(s) for the final answer(s).

7 Discretionary marks *(d)*

An examiner uses discretion to award a mark on the rare occasions when the markscheme does not cover the work seen. The mark should be labelled ***(d)*** *and a brief* ***note*** *written next to the mark explaining this decision.*

8 Alternative methods

Candidates will sometimes use methods other than those in the markscheme. Unless the question specifies a method, other correct methods should be marked in line with the markscheme. If in doubt, contact your team leader for advice.

- Alternative methods for complete questions are indicated by **METHOD 1**, **METHOD 2**, *etc*.
- Alternative solutions for part-questions are indicated by **EITHER** . . . **OR**.
- Where possible, alignment will also be used to assist examiners in identifying where these alternatives start and finish.

9 Alternative forms

Unless the question specifies otherwise, ***accept*** *equivalent forms.*

- As this is an international examination, accept all alternative forms of **notation**.
- In the markscheme, equivalent **numerical** and **algebraic** forms will generally be written in brackets immediately following the answer.
- In the markscheme, **simplified** answers, (which candidates often do not write in examinations), will generally appear in brackets. Marks should be awarded for either the form preceding the bracket or the form in brackets (if it is seen).

  Example: for differentiating $f(x) = 2\sin(5x-3)$, the markscheme gives:

  $$f'(x) = \big(2\cos(5x-3)\big)5 \quad \big(=10\cos(5x-3)\big) \qquad \textbf{\textit{A1}}$$

  Award ***A1*** for $\big(2\cos(5x-3)\big)5$, even if $10\cos(5x-3)$ is not seen.

10 Accuracy of Answers

If the level of accuracy is specified in the question, a mark will be allocated for giving the answer to the required accuracy.

- **Rounding errors**: only applies to final answers not to intermediate steps.
- **Level of accuracy**: when this is not specified in the question the general rule applies: *unless otherwise stated in the question all numerical answers must be given exactly or correct to three significant figures*.

Candidates should be penalized ***once only IN THE PAPER*** *for an accuracy error* ***(AP)****. Award the marks as usual then write* ***(AP)*** *against the answer. On the* ***front*** *cover write –1****(AP).*** *Deduct 1 mark from the total for the paper, not the question*.

- If a final correct answer is incorrectly rounded, apply the ***AP***.
- If the level of accuracy is not specified in the question, apply the ***AP*** for correct answers not given to three significant figures.

If there is no working shown, and answers are given to the correct two significant figures, apply the ***AP***. However, do **not** accept answers to one significant figure without working.

11 Crossed out work

If a candidate has drawn a line through work on their examination script, or in some other way crossed out their work, do not award any marks for that work.

12 Examples

Exemplar material is available under examiner training on http://courses.triplealearning.co.uk. Please refer to this material before you start marking, and when you have any queries. Please also feel free to contact your Team Leader if you need further advice.

SECTION A

Statistics and probability

1. (a) $H_0 : p = 0.75 \quad H_1 : p < 0.75$ ***A1A1***

[2 marks]

(b) **EITHER**

Under H_0, number of patients (X) cured is $B(100, 0.75)$ ***(M1)(A1)***

p-value $= P(X \le 68) = 0.0693$ ***(M1)A1***

OR

Under H_0, the proportion cured is approximately $N\left(0.75, \frac{0.75 \times 0.25}{100}\right)$ ***(M1)(A1)***

p-value $= 0.0530$ ***(M1)A1***

[4 marks]

(c) (i) Accept H_1. ***A1***

(ii) Accept H_0. ***A1***

Note: Allow ***FT*** on incorrect p-value, but award ***A1A0*** if both conclusions are the same.

[2 marks]

Total [8 marks]

2. (a) $\bar{x} = \frac{224.4}{200} = 1.12(2)$ ***(M1)A1***

$s_{n-1}^2 = \frac{5.823}{199} = 0.0293$ ***(M1)A1***

Note: ***(M1)*** depends on correct use of 199.

[4 marks]

(b) 95 % confidence limits are

$1.12(2) \pm 1.96\sqrt{\frac{0.0292...}{200}} \quad \left(\text{or } 1.12(2) \pm 1.97\sqrt{\frac{0.0292...}{200}}\right)$ ***M1A1A1A1***

Note: Award ***M1*** for correct form, ***A1*** for 1.12(2), ***A1*** for 1.96 or 1.97, ***A1*** for correct SE.

leading to [1.10, 1.15] ***A1A1*** ***N6***

[6 marks]

(c) No ***A1***

The Central Limit Theorem ensures the (approximate) normality of $\bar{X}$. ***R1***

[2 marks]

Total [12 marks]

3. (a) $H_0 : \mu = 30$ $H_1 : \mu \neq 30$ ***A1A1***

[2 marks]

(b) **EITHER**

$t = 1.75$ ***(A2)***

p-value $= 0.114$ or critical value $= 2.26(2)$ ***A2***

Accept H_0 or equivalent. ***A1*** ***N0***

Note: Allow ***FT*** on incorrect *p*-value or critical value.

OR

95 % confidence interval is $30.8(1) \pm 2.26(2) \times \frac{1.46...}{\sqrt{10}}$ ***(A1)(A1)(A1)***

$[29.8, 31.9]$ ***A1***

Accept H_0 or equivalent. ***A1*** ***N0***

Note: Allow ***FT*** on incorrect confidence interval.

[5 marks]

(c) (I used a *t*-test because)

The population is normal. ***R1***

The variance is unknown. ***R1***

[2 marks]

Total [9 marks]

4. (a) (i) $\mathrm{P}(T > t) = \int_t^{\infty} \frac{1}{10} \mathrm{e}^{-\frac{x}{10}} \, \mathrm{d}x$ ***M1A1***

$= -\left[\mathrm{e}^{-\frac{x}{10}} \right]_t^{\infty}$ ***A1***

$= \mathrm{e}^{-\frac{t}{10}}$ ***AG***

(ii) $\mathrm{P}\left(T \le t+s \mid T > t\right) = \frac{\mathrm{P}\left[(T \le t+s) \cap (T > t)\right]}{\mathrm{P}(T > t)}$ ***(M1)(A1)***

$= \frac{\mathrm{P}(t < T \le t+s)}{\mathrm{P}(T > t)}$ ***(A1)***

Numerator $= \int_t^{t+s} \frac{1}{10} \mathrm{e}^{-\frac{x}{10}} \, \mathrm{d}x$ ***A1***

$= \left[-\mathrm{e}^{-\frac{x}{10}} \right]_t^{t+s}$ ***A1***

$= \mathrm{e}^{-\frac{t}{10}} - \mathrm{e}^{-\frac{(t+s)}{10}}$ ***A1***

$\therefore \ \mathrm{P}\left(T \le t+s \mid T > t\right) = \frac{\mathrm{e}^{-\frac{t}{10}}\left(1 - \mathrm{e}^{-\frac{s}{10}}\right)}{\mathrm{e}^{-\frac{t}{10}}}$ ***A1***

$= 1 - \mathrm{e}^{-\frac{s}{10}}$ ***AG*** ***N0***

[10 marks]

(b) Here, $t = 5$ and $s = 10$ ***(A1)(A1)***

$\mathrm{P}\left(T \le 15 \mid T > 5\right) = 1 - \mathrm{e}^{-1}$ $(= 0.632)$ ***M1A1*** ***N2***

[4 marks]

Total [14 marks]

5. (a) (i) Mean $= \dfrac{1 \times 45 + ... + 5 \times 3}{100} = 2$ *A2*

> **Note:** The 5 or more row causes a problem in calculating the mean. The above calculation assumes that all 3 values are equal to 5; this is not necessarily the case. Allow candidates to assume any values greater than or equal to 5.

(ii) The distribution is geometric so *(M1)*

Estimated $p = \dfrac{1}{\text{mean}} = \dfrac{1}{2}$ *A1*

> **Note:** Award ***(M1)A1*** for writing $\dfrac{1}{\text{their mean}}$.

[4 marks]

(b) Expected frequencies are

x	f_o	f_e
1	45	50
2	26	25
3	16	12.5
4	10	6.25
5 or more	3	6.25

A1A1A1A1A1

$\chi^2 = \dfrac{5^2}{50} + \dfrac{1^2}{25} + \dfrac{3.5^2}{12.5} + \dfrac{3.75^2}{6.25} + \dfrac{3.25^2}{6.25}$ *(M1)(A1)*

$= 5.46$ *A1* *N3*

> **Note:** Allow ***FT*** from the values in the table.

DF = 3 *(A2)*

EITHER

Critical value $= 7.815$ *A1*

OR

$p = 0.141$ *A1*

THEN

We conclude, at the 5 % level, that the data fit the given distribution. ***R2***

> **Note:** Allow ***FT*** for ***R2***.

[13 marks]

Total [17 marks]

SECTION B

Sets, relations and groups

1. (a) Because $\sin x$ takes values in the interval $[-1, +1]$, ***(M1)***

$A = [e^{-1} - 1, e - 1]$. ***A1A1*** ***N3***

Note: Award ***A1A0*** for an open interval with the exact values, or for $[-0.632, 1.72]$.

[3 marks]

(b) (i) Using for example $f(0) = f(\pi) = 0$ or drawing a graph ***R1***

Note: Allow degrees instead of radians.

EITHER

f is not 1:1 ***R1***

OR

$f(x) = f(y)$ does not imply $x = y$ ***R1***

(ii) It is not a surjection since it can only take values in $[e^{-1} - 1, e - 1]$ (or equivalent reason). ***A1R1***

[4 marks]

(c) (i) The maximum value of k is $\frac{\pi}{2}$ (or 90°). ***A2***

(ii) $y = e^{\sin x} - 1$ ***M1***

$e^{\sin x} = 1 + y$ ***A1***

$\sin x = \ln(1 + y)$ ***A1***

$x = \arcsin \ln(1 + y)$ ***A1***

Note: Allow two of the three above ***A1*** marks to be implied.

so $g^{-1}(x) = \arcsin \ln(1 + x)$ ***A1***

(iii) Domain of g^{-1} is A or $[e^{-1} - 1, e - 1]$. ***A1***

[8 marks]

Total [15 marks]

2. (a) R is reflexive because aRa since $a^2 \equiv a^2 \pmod 6$ ***A1***

R is symmetric because $aRb \Rightarrow bRa$ since $a^2 \equiv b^2 \pmod 6 \Rightarrow b^2 \equiv a^2 \pmod 6$ ***A1***

Let aRb and bRc. It follows that $a^2 - b^2 = 6m$ and $b^2 - c^2 = 6n$ where *m, n* are integers. ***M1A1***

Then $a^2 - c^2 = 6(m+n)$ so aRc so transitive. ***M1A1***

[6 marks]

(b)

x	$x^2 \pmod 6$
2	4
4	4
6	0
8	4
10	4
12	0
14	4

M1A4

Note: Deduct 1 mark for each error.

Equivalence classes are
{2, 4, 8, 10, 14} ***(M1)A1***
and {6, 12} ***(M1)A1***

[9 marks]

Total [15 marks]

3. Closure – yes because for $a, b \in \mathbb{R}^+$, $\frac{a}{b} \in \mathbb{R}^+$. ***A1***

Associativity – consider $\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$ and $\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$ ***M1A1***

These are not equal so not associative ***A1***

Note: Accept a numerical counter example.

Identity – There is no identity because although $\frac{a}{1} = a$, $\frac{1}{a} \neq a$ in general. ***M1A1***

(Or equivalent argument)
Inverse – Without an identity there can be no inverse. ***A1***

[7 marks]

4. (a) Each row and each column contains each element exactly once. ***R1***

[1 mark]

(b) Use of a correct method ***(M1)***

e.g. pre and post multiply by the inverses of 2 and 7, namely 5 and 4 or trial and error or use of Abelian property.

$x = 5 * 4 * 4$ (or equivalent) ***A1***

$= 2 * 4$ (or equivalent) ***A1***

$= 8$ ***A1*** ***N2***

[4 marks]

(c) (i) The group is cyclic ***AG***

because the element 2 (or 5) has order 6. ***M1A2***

So 2 (or 5) is a generator. ***A1***

5 (or 2) is another generator. ***A1***

No other elements are of order 6. ***R1***

> **Note**: The two ***A1*** marks for finding the generators and the ***R1*** mark are not dependent upon the ***M1*** mark being awarded.

(ii) The proper subgroups are

$\{1, 8\}$ ***A2***

$\{1, 4, 7\}$ ***A2***

> **Note:** Ignore additional subgroups.

[10 marks]

Total [15 marks]

5. Associativity in $G \Rightarrow$ associativity in H. ***A1***

Taking $b = a \in H$, it follows that $a a^{-1} = e \in H$. ***M1A1***

Taking $a = e \in H$, it follows that for $b \in H$, $b^{-1} \in H$. ***M1A1***

For $a, b \in H$, we know that $b^{-1} \in H$ so that $a(b^{-1})^{-1} = ab \in H$. ***M1A1***

The group axioms are satisfied so H is a subgroup. ***R1***

> **Notes:** The ***R1*** mark can only be given if all three ***M1***s are awarded.
> The consideration of associativity is not necessary for ***R1***.

[8 marks]

SECTION C

Series and differential equations

1. At (0, 1) $\frac{dy}{dx} = 3$ *(M1)(A1)*

Increment $= 3 \times 0.5 \quad (= 1.5)$ ***M1A1***

$y = 2.5$ (at $x = 0.5$) ***M1A1***

At $(0.5, 2.5)$, $\frac{dy}{dx} = 2$ *(A1)*

Increment $= 2 \times 0.5 \quad (= 1)$ ***A1***

Therefore $y \approx 3.5$ when $x = 1$. ***A1*** ***N0***

Note: Allow ***FT*** from their y value when $x = 0.5$.

[9 marks]

2. (a) $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\ln \cos x + C$ ***M1A1***

$= \ln \sec x + C$ ***AG*** ***N0***

Note: Accept a solution showing that the derivative of $\ln \sec x$ is $\tan x$.

[2 marks]

(b) Integrating factor $= e^{\int \tan x \, dx} (= e^{\ln \sec x + k})$ *(M1)*

$= (C) \sec x$ ***A1***

[2 marks]

(c) Multiply by integrating factor *(M1)*

$\sec x \frac{dy}{dx} + y \sec x \tan x = \sec^2 x$ *(A1)*

gives $y \sec x = \tan x + c$ ***A1A1A1***

Substitute (0, 2) $(2 = 0 + c)$ *(M1)*

So $c = 2$ ***A1***

$y \sec x = \tan x + 2$

$y = \frac{\tan x + 2}{\sec x} \quad (y = \sin x + 2\cos x)$ ***A1***

[8 marks]

Total [12 marks]

3. (a) $\lim\limits_{x\to 0}\dfrac{\sqrt{1+x}-1}{x}=\lim\limits_{x\to 0}\dfrac{0.5(1+x)^{-0.5}}{1}$ ***M1A1***

$=0.5$ ***A1*** ***N1***

[3 marks]

(b) $\lim\limits_{x\to 0}x\ln x=\lim\limits_{x\to 0}\dfrac{\ln x}{\frac{1}{x}}$ ***M1A1***

$=\lim\limits_{x\to 0}\dfrac{\frac{1}{x}}{-\frac{1}{x^2}}$ ***M1A1***

$=\lim\limits_{x\to 0}(-x)$ ***A1***

$=0$ ***A1*** ***N1***

[6 marks]

Total [9 marks]

4. (a) (i) For attempting to use the comparison test (could be in an example) ***M1***

If u_n is convergent, it follows that there exists N

such that for $n\ge N$, $u_n<1$. ***M1***

So, for $n\ge N$, ${u_n}^2<u_n$. ***A1***

It follows by the comparison test that $\sum {u_n}^2$ is convergent.

(ii) The converse is not true. ***A1***

A counterexample is

$\sum\dfrac{1}{n^2}$ is convergent but $\sum\dfrac{1}{n}$ is not. ***A1***

[5 marks]

(b) (i)(ii) Consider, for $k\ne 1$,

$\displaystyle\int_2^\infty\frac{\mathrm{d}x}{x(\ln x)^k}$ ***M1A1***

Recognizing the substitution $u=\ln x$ or attempting integration by parts ***(M2)***

$=\left[\dfrac{1}{(1-k)(\ln x)^{k-1}}\right]_2^\infty$ ***M1A1***

This integral, and therefore the series,

is convergent for $k>1$ and divergent for $k<1$. ***A1***

For $k=1$,

$\displaystyle\int_2^\infty\frac{\mathrm{d}x}{x\ln x}=\left[\ln(\ln x)\right]_2^\infty$ ***M1A1***

This integral, and therefore the series, is divergent for $k=1$. ***A1***

(The series is therefore convergent for $k>1$ and divergent for $k\le 1$.)

[10 marks]

Total [15 marks]

5. (a) (i) Consider

$$a_n = \frac{f^{(n)}(0)}{n!} \quad \text{or} \quad f^{(n)}(0) = n!a_n$$ ***(M1)(A1)***

> **Note:** Award ***M1A1*** if this statement, or its equivalent with at least 2 numerical values of *n*, is seen anywhere in the candidate's work.

Putting $x = 0$ in the given relationship ***M1***

$(n+2)!a_{n+2} - n^2 \times n!a_n = 0$ ***A1***

So $(n+1)(n+2)a_{n+2} = n^2 a_n ,\ n \ge 1$ ***AG***

(ii) We find that $a_3 = \frac{1^2}{2\times 3}$ ***(M1)***

$$a_5 = \frac{1^2}{2\times 3} \times \frac{3^2}{4\times 5}$$ ***(A1)***

and in general, for odd $n \ge 3$,

$$a_n = \frac{1^2 \times 3^2 ... (n-2)^2}{n!}$$ ***A1***

[7 marks]

(b) Using the ratio test. ***M1***

For odd *n*,

$$\frac{x^{n+2} \text{ term}}{x^n \text{ term}} = \frac{n^2}{(n+1)(n+2)} \times x^2$$ ***M1***

$\to x^2$ as $n \to \infty$ ***A1***

EITHER

The series is convergent for $|x| < 1$. ***A1***

OR

Radius of convergence is 1. ***A1***

[4 marks]

(c) $\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$ ***A1***

$$\approx \left(1\times\left(\frac{1}{2}\right) + \frac{1}{6}\times\left(\frac{1}{2}\right)^3 + \frac{3}{40}\times\left(\frac{1}{2}\right)^5\right)$$ ***M1A1***

$\pi \approx 3.139$ ***A1*** ***N0***

[4 marks]

Total [15 marks]

SECTION D

Discrete mathematics

1. (a) Using an appropriate method correctly *M1A1*

6 | 95 5
6 | 15 3
    2

The required base 6 number is 235. *A1* *N1*

[3 marks]

(b)

```
    235
    235
  51400      A1
  11530      A1
   2111      A1
 105 441     A1   N0
```

[4 marks]

(c) **EITHER**

$(105\,441)_6 = 6^5 + 5.6^3 + 4.6^2 + 4.6 + 1$ *M1*

$= 9025$ *A1* *N0*

OR

Using Horner's algorithm

	1	0	5	4	4	1
6	1	6	41	250	1504	9025

M1
A1 *N0*

[2 marks]

Total [9 marks]

2. (a) When $\lambda = 4$, $\gcd(2, 4)$ is not a divisor of the right hand side of the equation or equivalent *e.g.* the left hand side is even and the right hand side is odd. ***R2***

[2 marks]

(b) **EITHER**

When $\lambda = 3$, solving $3x - 2y = 1$ is the same as solving the linear congruence $3x \equiv 1 \pmod 2$ ***M1A1***

Solutions by inspection are 1, 3, 5, ... and the general solution is ***(A1)***

$x = 2n + 1$ for $n \in \mathbb{Z}$ ***A1***

Substituting,

$6n + 3 - 2y = 1$ ***M1***

giving $y = 3n + 1$ ***A1***

OR

Any particular solution *e.g.* $x = 1 \quad y = 1$ ***(M1)A1***

$\gcd(3, 2) = 1$ ***(M1)***

General solution given by $\left.\begin{aligned} x &= 1 + n \times \frac{-2}{1} \\ y &= 1 - n \times \frac{3}{1} \end{aligned}\right\}$ for $n \in \mathbb{Z}$ ***M1A1A1***

(Giving $x = 1 - 2n$, $y = 1 - 3n$)

[6 marks]

Total [8 marks]

3. (a) Let a graph contain e edges. ***M1***
Each edge contributes 2 towards the sum of the degrees of the vertices which is therefore $2e$. This is even. ***R2***

[3 marks]

(b) Consider a graph in which each of the 9 vertices represents a man. ***M1***
2 vertices are joined by an edge if and only if the corresponding men shake hands. ***M1***

Each vertex would be of order 5 and the sum of the orders of all the vertices would be 45. ***R1***
Since this is not even, the situation is impossible. ***R1***

[4 marks]

(c) Start with

M1A1

For this graph, $v = 2$, $e = 1$ and $f = 1$ so Euler's relation is satisfied. ***A1***

Now add a vertex, which will require an extra edge – this will increase v and e by 1, f will remain unaffected. The relation will still be satisfied. ***M1R1***

Now add an edge between 2 existing vertices. This will increase e and f by 1, v will remain unaffected. The relation will still be satisfied. ***M1R1***

Thus as the graph is constructed, Euler's relation will remain satisfied. ***R2***

> **Note:** Accept starting with one vertex but award ***M1A0A0M1R1M1R1R2*** for starting with more than two vertices.

[9 marks]

Total [16 marks]

4. (a) $A_G = \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 1 \\ 2 & 1 & 0 \end{pmatrix}$ ***(M1)A1***

Note:	This assumes that A, B and C are represented by rows 1, 2 and 3 respectively. Accept any other representation.

$A_G^{\ 2} = \begin{pmatrix} 8 & 2 & 2 \\ 2 & 5 & 4 \\ 2 & 4 & 5 \end{pmatrix}$ ***A2***

Note:	Deduct 1 mark for each incorrect element.

There are 5 such paths (because the (3, 3) element of $A_G^{\ 2}$ is 5). ***A2*** ***N0***

Note:	***FT*** from the candidate's $A_G^{\ 2}$.

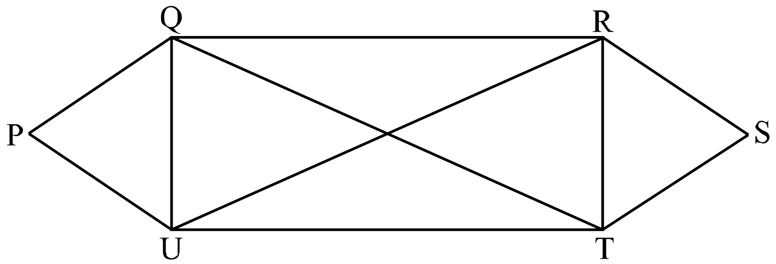
[6 marks]

(b) For an attempt to "label" the vertices or identify two disjoint sets of vertices. ***M1***
Disjoint sets of vertices are {H, J, L} and {I, K, M, N}. This can be inferred from a graph. ***A1A1***
The graph is bipartite. ***R1***

Note:	Award ***R1*** only if the previous three marks are awarded.

[4 marks]

(c) Graph has vertices of odd degree so it cannot have an Eulerian circuit. ***R2***



A2

Circuit is PQRSTRUQTUP. ***R2***

Note:	There are other possible Eulerian circuits.

[6 marks]

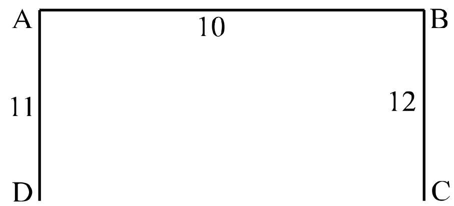
Total [16 marks]

5. (a) One upper bound is the length of any cycle, *e.g.* ABCDEA gives 73.
Other valid methods include:
double the weight of the minimum spanning tree which gives $2 \times 46 = 92$;
$5 \times$ the maximum weight of the edges $5 \times 19 = 95$;
twice the answer to (b)(ii). ***M1A1*** ***N0***

Note: The other possible cycles are:
ABCEDA – 64
ABDECA – 75
ABDCEA – 77
ABECDA – 66
ABEDCA – 73
ACBDEA – 79
ACBEDA – 68
ACDBEA – 81
ACEBDA – 72
ADBCEA – 72
ADCBEA – 70

[2 marks]

(b) (i) Using Kruskal's algorithm, the edges are introduced in the order AB, AD, BC. ***A1A1A1***



A1 ***N0***

(ii) Total weight of minimum spanning tree $= 33$. ***A1***
We rejoin E to the graph via the shortest edges, *i.e.* EB and EC. ***M1A1***
Therefore lower bound $= 33 + 13 + 14 = 60$. ***M1A1***

[9 marks]

Total [11 marks]