MATHEMATICS
HIGHER LEVEL
PAPER 2

Thursday 4 November 2004 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Indicate the make and model of your calculator in the appropriate box on your cover sheet.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

*Answer all **five** questions from this section.*

1. *[Maximum mark: 11]*

Consider the complex number $z = \cos\theta + \mathrm{i}\sin\theta$.

(a) Using De Moivre's theorem show that

$$z^n + \frac{1}{z^n} = 2\cos n\theta .$$ *[2 marks]*

(b) By expanding $\left(z + \frac{1}{z}\right)^4$ show that

$$\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3) .$$ *[4 marks]*

(c) Let $g(a) = \int_0^a \cos^4\theta \, \mathrm{d}\theta$.

(i) Find $g(a)$.

(ii) Solve $g(a) = 1$. *[5 marks]*

2. *[Maximum mark: 16]*

A line l_1 has equation $\frac{x+2}{3}=\frac{y}{1}=\frac{z-9}{-2}$.

(a) Let M be a point on l_1 with parameter μ. Express the coordinates of M in terms of μ. *[1 mark]*

(b) The line l_2 is parallel to l_1 and passes through $\mathrm{P}(4, 0, -3)$.

(i) Write down an equation for l_2.

(ii) Express $\overrightarrow{\mathrm{PM}}$ in terms of μ. *[4 marks]*

(c) The vector $\overrightarrow{\mathrm{PM}}$ is perpendicular to l_1.

(i) Find the value of μ.

(ii) Find the distance between l_1 and l_2. *[5 marks]*

(d) The plane π_1 contains l_1 and l_2. Find an equation for π_1, giving your answer in the form $Ax+By+Cz=D$. *[4 marks]*

(e) The plane π_2 has equation $x-5y-z=-11$. Verify that l_1 is the line of intersection of the planes π_1 and π_2. *[2 marks]*

3. *[Maximum mark: 15]*

(a) Let $\boldsymbol{T}_1$ be a rotation about the origin through angle A, and $\boldsymbol{T}_2$ be a rotation about the origin through angle B. By considering the matrices $\boldsymbol{T}_1$, $\boldsymbol{T}_2$, and the product $\boldsymbol{T}_1\boldsymbol{T}_2$, prove that

$$\sin(A+B) = \sin A\cos B + \cos A\sin B\,,$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B\,.$$

[4 marks]

(b) Hence prove that $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$. *[5 marks]*

(c) The matrix $\begin{pmatrix} \frac{5}{13} & \frac{-12}{13} \\ \frac{-12}{13} & \frac{-5}{13} \end{pmatrix}$ represents a reflection in a line through the origin. Using the result in **part (b)**, find the equation of the line. *[6 marks]*

4. *[Maximum mark: 17]*

Ian and Karl have been chosen to represent their countries in the Olympic discus throw. Assume that the distance thrown by each athlete is normally distributed. The mean distance thrown by Ian in the past year was 60.33 m with a standard deviation of 1.95 m.

(a) In the past year, 80 % of Ian's throws have been longer than x metres. Find x, correct to **two** decimal places. *[3 marks]*

(b) In the past year, 80 % of Karl's throws have been longer than 56.52 m. If the mean distance of his throws was 59.39 m, find the standard deviation of his throws, correct to **two** decimal places. *[3 marks]*

(c) This year, Karl's throws have a mean of 59.50 m and a standard deviation of 3.00 m. Ian's throws still have a mean of 60.33 m and standard deviation 1.95 m. In a competition an athlete must have at least one throw of 65 m or more in the first round to qualify for the final round. Each athlete is allowed three throws in the first round.

(i) Determine which of these two athletes is more likely to qualify for the final on their first throw.

(ii) Find the probability that **both** athletes qualify for the final. *[11 marks]*

5. *[Maximum mark: 11]*

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{y}{(\mathrm{e}^{2\theta}+1)}$.

(a) Use the substitution $x = \mathrm{e}^{\theta}$ to show that

$$\int \frac{\mathrm{d}y}{y} = \int \frac{\mathrm{d}x}{x(x^2+1)}.$$

[3 marks]

(b) Find $\int \frac{\mathrm{d}x}{x(x^2+1)}$. *[4 marks]*

(c) Hence find y in terms of θ, if $y = \sqrt{2}$ when $\theta = 0$. *[4 marks]*

SECTION B

*Answer **one** question from this section.*

Statistics

6. *[Maximum mark: 30]*

(i) Let X be a random variable with a Poisson distribution such that $\mathrm{Var}(X) = (\mathrm{E}(X))^2 - 6$.

(a) Show that the mean of the distribution is 3. *[3 marks]*

(b) Find $\mathrm{P}(X \le 3)$. *[1 mark]*

Let Y be another random variable, independent of X, with a Poisson distribution such that $\mathrm{E}(Y) = 2$.

(c) Find $\mathrm{P}(X + Y < 4)$. *[2 marks]*

(d) Let $U = X + 2Y$.

(i) Find the mean and variance of U.

(ii) State with a reason whether or not U has a Poisson distribution. *[4 marks]*

(ii) A chicken farmer wishes to find a confidence interval for the mean weight of his chickens. He therefore randomly selects n chickens and weighs them. Based on his results, he obtains the following 95% confidence interval.

[2148 grams , 2188 grams]

The weights of the chickens are known to be normally distributed with a standard deviation of 100 grams.

(a) Find the value of n. *[5 marks]*

(b) Assuming that the same confidence interval had been obtained from weighing 166 chickens, what would be its level of confidence? *[3 marks]*

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(Question 6 continued)

(iii) The random variable Z has probability density function $f(z) = z - \dfrac{z^3}{4}$ for $z \in [0, 2]$ and 0 elsewhere. A physicist assumes that the lifetime of a certain particle can be modelled by this random variable. The interval [0, 2] is divided into the following equal intervals:

$I_1 = [0, 0.4[$
$I_2 = [0.4, 0.8[$
$I_3 = [0.8, 1.2[$
$I_4 = [1.2, 1.6[$
$I_5 = [1.6, 2]$

The physicist carried out 40 experiments and recorded the number of times the value of Z lay in each of the intervals I_k where $k = 1, 2, 3, 4, 5$ as shown in the following table:

I_1	I_2	I_3	I_4	I_5
2	12	9	8	9

(a) Assuming that the physicist's assumption is correct, for each value of k find $p_k = \mathrm{P}(Z \in I_k)$. *[4 marks]*

(b) At the $5\,\%$ significance level can his assumption be accepted? *[8 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) The relation R on $\mathbb{C}$ is defined as follows

$$z_1 R z_2 \Leftrightarrow |z_1| = |z_2| \text{ for } z_1, z_2 \in \mathbb{C}.$$

(a) Show that R is an equivalence relation on $\mathbb{C}$. *[3 marks]*

(b) Describe the equivalence classes under the relation R. *[2 marks]*

(ii) Consider the set $S = \{1, 3, 4, 9, 10, 12\}$ on which the operation $*$ is defined as multiplication modulo 13.

(a) Write down the operation table for S under $*$. *[4 marks]*

(b) Assuming multiplication modulo 13 is associative, show that $(S, *)$ is a commutative group. *[4 marks]*

(c) State the order of each element. *[3 marks]*

(d) Find all the subgroups of $(S, *)$. *[3 marks]*

(iii) Let $(H, \times)$ be a group and let a be one of its elements such that $a \times a = a$. Show that a must be the identity element of the group. *[4 marks]*

(iv) (a) Explain what is meant by a cyclic group. *[2 marks]*

Let $(G, \#)$ be a finite group such that its order p is a prime number.

(b) Show that $(G, \#)$ is cyclic. *[5 marks]*

Discrete Mathematics

8. *[Maximum mark: 30]*

(i) Let G be a weighted graph with 6 vertices L, M, N, P, Q, and R. The weight of the edges joining the vertices is given in the table below:

	L	M	N	P	Q	R
L	–	4	3	5	1	4
M	4	–	4	3	2	7
N	3	4	–	2	4	3
P	5	3	2	–	3	4
Q	1	2	4	3	–	5
R	4	7	3	4	5	–

For example the weight of the edge joining the vertices L and N is 3.

(a) Use Prim's algorithm to draw a minimum spanning tree starting at M. *[5 marks]*

(b) What is the total weight of the tree? *[1 mark]*

(ii) Let G_1 and G_2 be two graphs whose adjacency matrices are represented below.

G_1

	A	B	C	D	E	F
A	0	2	0	2	0	0
B	2	0	1	1	0	1
C	0	1	0	1	2	1
D	2	1	1	0	2	0
E	0	0	2	2	0	2
F	0	1	1	0	2	0

G_2

	a	b	c	d	e	f
a	0	1	3	0	1	2
b	1	0	1	3	2	0
c	3	1	0	2	1	3
d	0	3	2	0	2	0
e	1	2	1	2	0	1
f	2	0	3	0	1	0

(a) Which one of them does not have an Eulerian trail? Give a reason for your answer. *[2 marks]*

(b) Find an Eulerian trail for the other graph. *[4 marks]*

(This question continues on the following page)

(Question 8 continued)

(iii) Let $x_{n+2} = 2x_{n+1} - 2x_n$ be a difference equation for all positive integers n.

(a) Show that $1+i$ and $1-i$ are the roots of the characteristic equation. *[2 marks]*

(b) Hence write down the general solution to the difference equation. *[1 mark]*

(c) If $x_1 = 1$ and $x_2 = 2$, show that $x_n = \left(\sqrt{2}\right)^n \sin\frac{n\pi}{4}$. *[8 marks]*

(iv) Let p, q and $r \in \mathbb{Z}^+$ with p and q relatively prime.

Show that $r \equiv p \pmod q$ **and** $r \equiv q \pmod p$ if and only if $r \equiv p + q \pmod{pq}$. *[7 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

(i) Find the range of values of x for which the following series is convergent.

$$\sum_{n=0}^{\infty} \frac{x^n}{n+1}$$ *[7 marks]*

(ii) Let f be the function defined on the interval $\left[1, \frac{4}{3}\right]$ by $f(x) = \frac{(2+3x-x^3)}{3}$.

(a) Show that the range of f is a subset of its domain. *[5 marks]*

(b) Find the exact value of a such that $f(a) = a$. *[1 mark]*

The sequence $\{x_n\}, n = 0, 1, 2, \ldots$ is defined as follows

$$x_0 = 1, x_{n+1} = f(x_n).$$

(c) Using the mean value theorem or otherwise, show that $|x_{n+1} - a| \le \frac{7|x_n - a|}{9}$ for all values of n. *[5 marks]*

(d) Hence or otherwise show that the sequence $\{x_n\}$ converges to a. *[4 marks]*

(iii) (a) Show that the series $\sum_{n=1}^{\infty} \sin\frac{2\pi}{n^2}$ is convergent. *[3 marks]*

Let $S = \sum_{n=1}^{\infty} \sin\frac{2\pi}{n^2}$.

(b) Show that for positive integers $n \ge 2$,

$$\frac{1}{n^2} < \frac{1}{n-1} - \frac{1}{n}.$$ *[1 mark]*

(c) Hence or otherwise show that $1 \le S < 2\pi$. *[4 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) A parabola has equation $4py = x^2$, where $p > 0$.

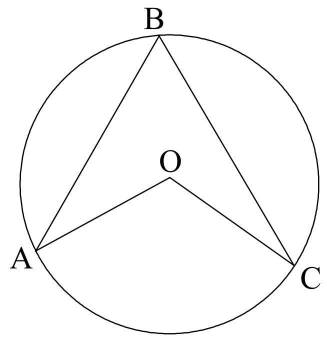
(a) Find the real numbers a and b such that for every point P on the parabola the distance from P to the point $A(0, a)$ is equal to the distance from P to the line $y = b$. *[6 marks]*

(b) Show that at every point P on the parabola the tangent to the parabola bisects the angle between (AP) and the line through P parallel to the y-axis. *[6 marks]*

(ii) In the triangle ABC let D be the point of intersection of the line (AC) and the bisector of the angle ABC.

Prove the bisector theorem, that is $\frac{\text{AD}}{\text{DC}} = \frac{\text{AB}}{\text{BC}}$. *[4 marks]*

(iii) Let A, B and C be three points on a circle of centre O such that B is on the longest arc joining A to C as shown below.



(a) Show that the angle AOC is twice the angle ABC. *[4 marks]*

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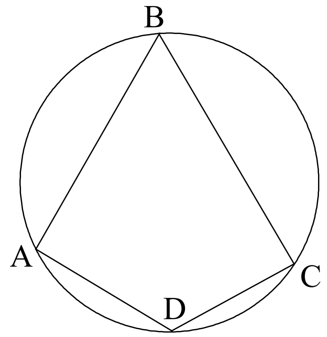
(Question 10 (iii) continued)

Let D be another point on the arc CA as shown below.



(b) Using part (a) or otherwise show that the sum of any two opposite angles of the quadrilateral is equal to $180°$. *[3 marks]*

(iv) Let M and N be any two points on a circle of centre O and let R be another point on the circle, equidistant from M and N.

Show that (NR) bisects the angle between (MN) and the tangent to the circle at N. *[7 marks]*