MATHEMATICS HL

Overall grade boundaries

Higher level

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 17	18 – 32	33 – 44	45 – 56	57 – 67	68 – 79	80 – 100

This subject report is written by the principal examiners. Each of the authors provides general comments on performance, taking into account the comments of the assistant examiners and team leaders. This report is the only means of communication between the senior examiners and the classroom teachers and should therefore be read by all teachers of Mathematics HL.

The grade award team studied the responses in the G2 forms, the assistant examiners' reports and the grade descriptors (a description of the criteria to be satisfied for each of the individual grade levels) before determining the grade boundaries.

Internal Assessment

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 4	5 – 6	7 – 8	9 – 11	12 – 13	14 – 16	17 – 20

Candidates generally fared well in their portfolio work this session, in spite of a few difficulties. Most of the observations of the moderators appear to be in three areas – the tasks, suggestions to teachers, and the candidates' performance against the criteria.

The tasks:

The majority of the tasks were taken from the teacher support material (TSM) for mathematics HL. Many samples contained a good selection of activities designed by teachers themselves than was the case a year ago.

That said, some tasks were incorrectly categorised as type I or III by the teacher. Some teacher-designed type I tasks were lacking in suitable components that directed students to generate and observe patterns, formulate a conjecture, then produce an inductive generalisation with proof to satisfy criterion E.

Thankfully, moderators did not report the misuse of tasks from the Mathematical Methods TSM this session.

Suggestions to teachers:

It is critical that teachers provide more feedback to students on their work. Very few samples contained teacher comments to students. (Some samples were entirely devoid of teacher marks!) Moderation was extremely difficult when it was not possible to determine the basis upon which the teacher awarded marks.

Only a few teachers explained the background to the portfolio tasks, which moderators need and appreciate.

If a teacher-designed task is submitted, the solution key must accompany the portfolios for moderators to justify the accuracy and appropriateness of the work.

The strengths and weaknesses of the candidates

Candidates generally performed well against criterion A (Use of Notation and Terminology). The use of computer notation, such as "x^3 – 5x +2" or "2.3E6", was not a problem this session.

Criterion B (Communication) was often assessed with little attention being paid to the expected care and detail of the presentation of the work. It does not seem that all candidates were given sufficient direction in meeting the expectations under this criterion. Some samples contained very poorly presented work. Students must be directed to acquire some skills in technical writing. Many of them have merely shown the steps to the solutions of problems and their work was found to be severely lacking in introduction, explanation, annotation, or justification. A perennial concern of moderators is that graphs are not always presented with titles and axes properly labelled.

Some candidates were generously rewarded by their teacher with the highest level of achievement under criterion C (Mathematical Content) or criterion D (Results and Conclusions) for adequate work, which, although complete, did not manifest much insight or sophistication.

Criterion E (Making Conjectures) was treated inconsistently by teachers and students. In two instances, candidates were awarded full marks for utilising trigonometric identities to "prove" their validity when applied to specific examples. Many candidates were not given the task of engaging in formulating a conjecture or in presenting an inductive generalisation with formal argument, as noted above.

Success in criterion F (Use of Technology) varied considerably. The full capabilities of a GDC were often not realised in the limited scope of the tasks set. Full marks were erroneously given for an *appropriate* but often not a *resourceful* use of technology.

Paper 1

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 20	21 – 40	41 – 53	54 – 67	68 – 81	82 – 95	96 – 120

Summary of the G2 forms

- **Comparison with last year's paper:**

Much easier	A little easier	Similar standard	A little more difficult	Much more difficult
0	1	10	10	2

- **Suitability of question paper:**

	Too easy	Appropriate	Too Difficult
Level of difficulty	0	28	4
	Poor	**Satisfactory**	**Good**
Syllabus coverage	1	17	14
Clarity of wording	0	13	17
'resentation of paper	0	9	21

Based on the responses of teachers it was felt that the way in which the questions in this paper were structured was different from previous years. Some stated that candidates found it to be more difficult than papers from the previous years. A few commented that question 1 was not an appropriate first question and others stated that the questions did not progress in order of difficulty with too many difficult questions early in the paper. Some seemed to feel that the paper was long compared to previous years. Others stated that there was a good spread in the difficulty level of questions but raised concern that often candidates would not be able to demonstrate their knowledge of a topic because the question did not allow them to progress without getting the "first step." The paper was also described as "very fair with some interesting and different questions."

The results of the candidates did not seem to indicate that the examination was any more difficult than in the past. There was little evidence that the candidates did not have enough time to attempt all the questions. Certainly Question 20 proved to be the most difficult with very few correct solutions. The examiners noted that the candidates struggled with some of the easier questions at the beginning of the paper. Whilst Questions 2, 3 and 5 were considered very easy by the examiners the candidates in general did not perform well on these questions. However, they did handle Questions 8 to 13 with little difficulty. The printing of the cumulative frequency curve in Question 2 was poor and discretion was used in marking this question.

General comments about the strengths and weaknesses of the candidates

It was noted that many candidates were able to attempt almost all questions, giving evidence of their relatively sound knowledge of most topics in the programme, but also of their good time management skills during the examination. Examiners expressed concern about the poor algebra skills shown by many of the candidates. Examiners noted that the GDC in the integration problem was used well but felt that the candidates demonstrated little ability in working with given sketches of functions or in considering appropriate windows when comparing functions. Also there was concern about the candidates' inability to solve a simple matrix problem. Additionally, candidates who attempted

Question 20 seemed to have no appreciation for a realistic answer, giving answers such as "the speed of the airplane is 0.0004 kilometres per hour."

Question 1

Answer: Area = 21.9

Most candidates seemed to know what to do with this question but some made careless errors in the process.

Question 2

Answer: (a) 154

(b) $x = 44$ minutes

The poor quality of the graph caused problems in reading values from it although small variations from the markscheme solutions were permitted. In part (b), many candidates found the 20^{th} percentile instead of the 80^{th}.

Question 3

Answer: $\boldsymbol{X} = \boldsymbol{ACB}^{-1}$

There seemed to be a general lack of understanding of the importance of order in matrix multiplication.

Question 4

Answer: $m = \dfrac{\pi}{3}$

This question was correctly answered by a good number of candidates. A few started off by making the integral of $x \sin x$ equal to ½, a few gave the final answer to be ½, and a small number did not attempt the question.

Question 5

Answer: $p = -0.4$, $q = 1.2$

Although most candidates knew that they needed to derive a system of equations from the real and the imaginary part of both sides of the equation, many made algebraic errors in setting up the equations and then trying to solve them.

Question 6

There were relatively few correct answers to this question. Many did identify $x = 2$ as an asymptote. However, the shape of the curve was incorrect in a vast majority of answers; many reflected the curve in the x-axis and others seemed to think that the graph of the required function had discontinuities at $x = 1$ and $x = -1$.

Question 7

Answer: The angle is 48º

Some candidates had trouble finding the direction vector of the z -axis. Most obtained the angle 42° but many did not realise that the complementary angle was required.

Question 8

Answer: (a) $f'(t) = \dfrac{6\sin t}{\cos^3 t} + 5$

(b) $f(\pi) = 3 + 5\pi$
$f'(\pi) = 5$

Many candidates seemed to be uncomfortable with differentiating $\sec t$ directly. Those who changed it to $\dfrac{1}{\cos t}$ often made errors in the subsequent differentiation. A common mistake was the failure to give the exact values as required.

Question 9

Answer: $b = -3$
$a = 2$

This was correctly solved by many candidates although algebraic errors were not uncommon, especially with those candidates who introduced the common difference d as a third variable.

Question 10

Answer: $x = \pm 6$

This caused problems for many candidates who seemed unable to change the given form to an exponential form. Those who did manage this often failed to give the answer $x = -6$.

Question 11

Answer: Area = 0.201

This question was generally answered well by the majority of candidates. There was little evidence of candidates attempting this question without using a GDC. Many found the x-coordinates of the points of intersection of the curves successfully, but then quite a few wrote down and worked out an incorrect integral for the required area.

Question 12

Answer: $\mathrm{P}(X \le 3) = 0.91296...$ (0.913 to 3 s.f)

Many solved this problem successfully. The most common errors were to omit the $\mathrm{P}(X = 0)$ term, or worse, to omit the combinatorial terms.

Question 13

Answer: $-3 \le k \le 4.5$ (accept $-3 < k < 4.5$)

Practically every candidate realised they needed to find and work on the discriminant of the equation, but a variety of mistakes were made in the process. These included algebraic errors when working on the inequality, sign errors and the answer being given as the roots of the discriminant and not as a range of values.

Question 14

Answer: $x = 1.57$

Most candidates sketched the graph correctly but the required coordinates were not always correctly calculated. Quite a number lost the last mark by embarking on a complicated and involved analytical approach to the problem.

Question 15

Answer: (a) $y = -1$

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{4}{5}$

This question was well answered by quite a few candidates. Most were able to find the value of y in part (a) but some gave the positive rather than the negative value. Many found the derivative correctly although some included the value of the constant in the derivative.

Question 16

Answer: (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3\mathrm{e}^{3x}\sin(\pi x) + \pi\mathrm{e}^{3x}\cos(\pi x)$

(b) $x = 0.7426\ldots(0.743$ to 3 s.f.)

A majority of candidates found the derivative correctly. However, relatively few were completely successful in finding the value required in the solution of $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Question 17

Answer: $x < -1$ or $4 < x \le 14$

There were few correct answers to this question. Many candidates simply cross-multiplied in the inequality, paying no attention to the signs of the denominators. Those who attempted it graphically usually sketched both curves instead of the difference between the curves. Most did not consider any possibility outside the standard window of the GDC.

Question 18

Answer: 55 ways

This question proved to be beyond the capabilities of most candidates. Candidates tended to write down combinatorial terms or even differences of combinatorial terms without any explanation. It was therefore difficult to know whether or not they were thinking along the right lines.

Question 19

Answer: $x = -1 + \log_5 3$

Very few candidates solved this correctly with only a few realising that it was essentially a quadratic equation in 5^x. Most candidates took logs too soon and then used incorrect properties of logs.

Question 20

Answer: The aeroplane is moving towards him at $240\ \text{km h}^{-1}$

Very few candidates seemed to recognise this as a related rate problem. Many tried to solve it without any reference to calculus. Some were confused by the term "moving towards him" and consequently used the sine function. Very few multiplied by 3600 to change to the appropriate units.

Paper 2

Component grade boundaries

Grade:	1	2	3	4	5	6	7
Mark range:	0 – 16	17 – 33	34 – 45	46 – 56	57 – 68	69 – 79	80 – 100

Summary of the G2 forms

- **Comparison with last year's paper:**

Much easier	A little easier	Similar standard	A little more difficult	Much more difficult
0	2	12	6	3

- **Suitability of question paper:**

	Too easy	Appropriate	Too Difficult
Level of difficulty	0	24	5
	Poor	**Satisfactory**	**Good**
Syllabus coverage	0	18	11
Clarity of wording	0	10	19
Presentation of paper	1	8	20

General comments

Teacher and assistant examiner response to this paper was mostly positive with comments generally along the lines that it was accessible and mostly straightforward with a few more challenging parts. With regard to section B, the option questions, the area where the comparable levels of difficulty have been the major issue of recent papers, these were considered to be very much in line with each other.

With regard to the advice that teachers give their students about examination technique, it is again necessary to remind teachers that candidates should take care of the following steps:

- Candidates should start each question on a fresh page.
- Questions do not need to be answered in the order asked in the examination. However, it is helpful if the pages are put in order at the end of the examination.
- Make sure any sub parts of each question are numbered clearly using the left hand margin.
- If a question requires a graph to be drawn (as opposed to sketched), this should be done on the graph paper provided. There was clear evidence in some batches of scripts that graph paper was not made available in some centres. If a sketch is asked for, this may be done on the answer sheets. In both cases, graphs should be neatly drawn and of an appropriate scale with both axes clearly marked. Candidates should also identify other suitable features, as required by the question. Candidates throw away marks when they fail to do this correctly

Comments on individual questions

Question 1 Three dimension vector geometry

Answer: (a) $\frac{x-2}{1}=\frac{y-5}{1}=\frac{z+1}{1}$

(b) $\left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)$

(c) $A'=\left(-\frac{4}{5},\frac{5}{3},-\frac{13}{3}\right)$

(d) $d=8.52$ (units)

A large number of candidates found this question to be straightforward and there were many completely correct solutions. However, there are still some problems in this topic:

(a) Some candidates still do not understand or appreciate the different forms the equation of the line can take and expressed the answer in a vector or parametric form rather than the required Cartesian form.

(b) Whatever form was used in part (a) many were still able to find the point of intersection of the line and plane.

(c) Only a few candidates recognized the most direct method of finding the point of reflection. Many candidates spent time using midpoint or equal distance methods to find A'. Teachers should advise their students that the mark allocation given within the question can be a useful guide in indicating how much work may be necessary in a particular part of a question.

(d) Some candidates simply calculated the distance between points A and B with no appreciation that the required distance would be perpendicular to the line *L*.

Question 2 Proof by induction and complex numbers

Answer: (b) (ii)

$$z_1^2 = 4\left(\cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5}\right)$$
$$z_1^3 = 8\left(\cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5}\right)$$
$$z_1^4 = 16\left(\cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}\right)$$
$$z_1^5 = 32\left(\cos 2\pi + \mathrm{i}\sin 2\pi\right)$$

(iv) an enlargement scale factor 2, centre (0, 0)
a rotation (anti-clockwise) of $\frac{2\pi}{5}$ (72º), centre (0, 0)

(a) It is good to report that the proof by induction has much improved. But candidates still write down the mathematics without the necessary explanations that are so important in this type of proof. Common errors are that candidates still do not clearly state that they are **assuming** the result to be true for $n = k$ and then show that this **leads** to it also being true for $n = k + 1$.

(b) (i) Well answered but often by unnecessarily finding all the roots of 32.

(ii) In calculating the powers of the root the modulus often remained at 2.

(iii) A large number of candidates found it quite difficult to set up an appropriate scale to plot the points correctly found in part (ii).

(iv) A disappointing number of correct answers to this part considering how guided the question had been up to this point. The need to find a transformation made many candidates search for a matrix representation of the answer.

Question 3 Trigonometry

Answer: (i) (a) $r = 2$
$\alpha = \frac{\pi}{6}$

(b) $[2, 2]$

(c) $\theta = \frac{\pi}{2}, \frac{7\pi}{6}$

(i) (a) Most candidates knew how to find r and θ, either as a standard result or from first principles. $\theta = \frac{\pi}{3}$ was a fairly common error.

(b) It was clear that many candidates used a GDC to find the range.

(c) Many candidates gave only one solution. Some again used a GDC (not the intention of the question) and then converted the solutions to the exact values required.

(ii) Some candidates were able to prove the identity quite quickly while others filled several pages getting nowhere. Teachers should advise their students that in situations like this that after a page of work consisting of expressions that are getting longer and more complex then something has probably gone wrong! They should check the first few steps and perhaps find a better strategy. Even with trigonometric identities available to candidates in their formula booklets there were many errors in substitutions being used.

Question 4 Calculus

Answer: (ii) (b) $\left(-\frac{1}{c}\ln b, \frac{a}{2b}\right)$

(i) Many candidates had no idea how to even start, giving the impression that some schools may not be covering this topic adequately. Those candidates who knew what to do found this problem fairly straightforward.

(ii) (a) The differentiation caused problems for many candidates. Many thought that the derivative of *a* was 1. Candidates that rearranged the expression and used the chain rule rather than the quotient rule did significantly better.

(b) Since the second derivative was given many candidates solved correctly for $f''(x) = 0$, but many did not find the *y* value of the point or did not simplify it as needed.

(c) It was clear from the scripts that the vast majority of candidates do not know the conditions for a point of inflexion. Many just stated that $f''(x) = 0$ guaranteed a point of inflexion.

Question 5 Normal distribution and solving a system of equations

Answer: (i) (a) $\sigma = 4$
$\mu = 45.6$

(b) $\mathrm{P}\left(\left|X - \mu\right| < 5\right) = 0.789$

(ii) (a) $z = \frac{6}{b-5}$

(b) $b \neq 5$

(i) (a) Setting up the required equations caused problems for many candidates. Some just used the given probabilities rather than the corresponding *z*-values, some failed to recognize that one of the *z*-values should be negative. Candidates who sketched a curve and illustrated the various areas under the curve usually avoided these errors. Many candidates went on to solve whatever system they had established and did not seem at all troubled by impossible answers such as negative solutions for the standard deviation.

(ii) (a) Good candidates solved for z with little difficulty but candidates with weak algebra skills often went around in circles getting nowhere.

(b) Some candidates used the determinant of the coefficient matrix to find the range for part (b) rather than the result found in part (a).

Question 6 Statistics

Answer: (i) (a) $\lambda = 6$

(b) (i) $\mathrm{P}(X \geq 2) = 0.829$

(ii) $\mathrm{P}(X \leq 3 \mid X \geq 2) = 0.520$

(ii) 99 % C.I. is $[3.04, 4.36]$

(iii) The farmer's belief is supported by these results

(iv) (a) (i) $\text{Mean} = 1.98$

(ii) $\hat{p} = 0.33$

(b) $\chi^2 = 1.56$
The binomial distribution provides a good fit

(i) (a) This was solved correctly by most candidates that attempted this option question. Some provided an extra unwanted solution to the equation.

(ii) (a) Also answered correctly by most candidates.

(b) (i) Many failed to notice that a conditional probability was required.

(ii) Solutions to finding the confidence limits were very weak. Some candidates took 55.5 to be the mean and 215.8 as the variance. Many candidates found a z-interval instead of the required t-interval.

(iii) Many candidates used a z-test because the sample sizes were large and part marks were awarded for this solution. However, the syllabus states that in a case were the population variance is unknown and is estimated from the sample then a t-distribution should be used whatever the sample size.

(iv) (a) (i) Most candidates were able to estimate the mean correctly.

(ii) Finding the probability that an egg was brown was difficult for some and a common error was to simply state $p = 0.5$.

(b) Many candidates were able to calculate expected frequencies with whatever value of p found in part (a) (ii). But few realized that some of the cells needed to be combined.

Question 7 Sets, relations and groups

Answer: (ii) (b) $0 \leftrightarrow 1, 1 \leftrightarrow 2, 2 \leftrightarrow 4, 3 \leftrightarrow 3,$ OR $0 \leftrightarrow 1, 1 \leftrightarrow 3, 2 \leftrightarrow 4, 3 \leftrightarrow 2$

(iv) (a) $y = x^3$ is one-to-one and onto

(i) (a) These were simple Venn diagrams and most candidates attempting this option were able to answer this correctly.

(b) Some candidates seemed to manage the set algebra required with ease but many found it too much.

(ii) (a) A very large number of candidates were able to correctly construct the operation table without difficulty.

(b) Most candidates appreciated that a study of the order of the elements might be helpful in establishing the correct isomorphism but not all reached the correct conclusion or provided enough detail in their argument for which groups were isomorphic.

(iii) (a) Surprisingly there were quite a number of candidates who confused the terms commutative and associative.

(b) Proving that (□ ,#) formed a group was well done.

(iv) (a) Also well done but not many candidates provided sketches of the functions which would have helped.

(b) Very few correct answers were seen.

Question 8 Discrete mathematics

Answer: (i) (b) (i) $k = 7$

(ii) $u_n = \frac{3}{20}(4)^n - \frac{2}{5}(-1)^n$

(iii) (a) Chromatic number of graph $\chi(G) = 4$

(b) There are two district Hamiltonian paths in G

Judging by the number of candidates answering this question it is clear that this option is not widely taught.

(i) (a) All candidates knew the required properties but proving the equivalence relation with all the necessary steps was done poorly.

(b) (i) and (ii) There were many problems with the understanding of congruence.

(ii) Candidates seemed much more comfortable solving a standard recurrence relation and there were many completely correct solutions.

(iii) Most candidates who answered this option seemed well prepared in graph theory and again there were many correct solutions for parts (a) and (b).

Question 9 Analysis and approximation

Answer: (i) (a) (ii) Smallest positive root $=\pi$

(b) Area $=3.141593$ (6 d.p.)

(c) 0.0802 (3 s.f.)

(iii) (a) (i) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

(ii) $e^{x^2} = 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}$

(b) $e^{x^2}\sin x = x + \frac{5}{6}x^3 + \frac{41}{120}x^5$

(c) $\lim_{x\to 0}\left(\frac{e^{x^2}\sin x - x}{x^3}\right) = \frac{5}{6}$

(i) (a) Nearly all candidates drew the graph correctly but some went on to give the smallest positive root as 0. Many candidates were also able to derive the correct Newton-Raphson formula.

(b) and (c) Most candidates used a GDC to correctly find the area under the curve but very few were able to find an upper bound for the error. The main difficulty was finding the maximum value of $f''(x)$ over the interval $[0, \pi]$.

(ii) Few candidates were able to completely describe the necessary conditions required for the integral test and using it for part (b) was even less successful. Some candidates tried the ratio test with mixed results.

(iii) Nearly all candidates were able to find the correct series for sin*x* but the series for e^{x^2} was too much for those trying to find all the derivatives needed for four non-zero terms. Very few recognized that e^{x^2} could be found by substituting x^2 into the series for e^x.

(b) There were very few correct solutions for the series for $e^{x^2}\sin x$, and some candidates attempted to find the series by adding the individual series.

(c) Problems with the previous parts meant few candidates were able to attempt finding a limit.

Question 10 Euclidian geometry and conic sections

Answer: (iii) (b) $\mathrm{M} = \left(\dfrac{a^2}{x_0}, 0\right)$

(c) $\mathrm{PF}_1 = a - ex_0$

$\mathrm{PF}_2 = a + ex_0$

$\mathrm{MF}_1 = \dfrac{a}{x_0}(a - ex_0)$

$\mathrm{MF}_2 = \dfrac{a}{x_0}(a + ex_0)$

As with Question 8 there were very few attempts at this option question. However, of the few who attempted it there were several excellent solutions.

