IB DIPLOMA PROGRAMME
PROGRAMME DU DIPLÔME DU BI
PROGRAMA DEL DIPLOMA DEL BI

N03/510/H(2)

MATHEMATICS
HIGHER LEVEL
PAPER 2

Wednesday 5 November 2003 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.
- Write the make and model of your calculator on the front cover of your answer booklets *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

Please start each question on a new page. You are advised to show all working, where possible. Where an answer is wrong, some marks may be given for correct method, provided this is shown by written working. Solutions found from a graphic display calculator should be supported by suitable working, e.g. if graphs are used to find a solution, you should sketch these as part of your answer.

SECTION A

Answer all ***five*** *questions from this section.*

1. *[Maximum mark: 12]*

The point $\mathrm{A}(2, 5, -1)$ is on the line L, which is perpendicular to the plane with equation $x + y + z - 1 = 0$.

(a) Find the Cartesian equation of the line L. *[2 marks]*

(b) Find the point of intersection of the line L and the plane. *[4 marks]*

(c) The point A is reflected in the plane. Find the coordinates of the image of A. *[2 marks]*

(d) Calculate the distance from the point $\mathrm{B}(2, 0, 6)$ to the line L. *[4 marks]*

2. *[Maximum mark: 16]*

(a) Use mathematical induction to prove De Moivre's theorem

$$(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta),\ n \in \mathbb{Z}^+.$$ *[7 marks]*

(b) Consider $z^5 - 32 = 0$.

(i) Show that $z_1 = 2\left(\cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right)\right)$ is one of the complex roots of this equation.

(ii) Find $z_1^{\ 2}, z_1^{\ 3}, z_1^{\ 4}, z_1^{\ 5}$, giving your answer in the modulus argument form.

(iii) Plot the points that represent $z_1, z_1^{\ 2}, z_1^{\ 3}, z_1^{\ 4}$ and $z_1^{\ 5}$, in the complex plane.

(iv) The point $z_1^{\ n}$ is mapped to $z_1^{\ n+1}$ by a composition of two linear transformations, where $n = 1, 2, 3, 4$. Give a full geometric description of the two transformations. *[9 marks]*

3. *[Maximum mark: 15]*

(i) (a) Express $\sqrt{3}\cos\theta - \sin\theta$ in the form $r\cos(\theta + \alpha)$, where $r > 0$ and $0 < \alpha < \frac{\pi}{2}$, giving r and α as **exact** values. *[3 marks]*

(b) Hence, or otherwise, for $0 \le \theta \le 2\pi$, find the range of values of $\sqrt{3}\cos\theta - \sin\theta$. *[2 marks]*

(c) Solve $\sqrt{3}\cos\theta - \sin\theta = -1$, for $0 \le \theta \le 2\pi$, giving your answers as **exact** values. *[5 marks]*

(ii) Prove that $\dfrac{\sin 4\theta(1 - \cos 2\theta)}{\cos 2\theta(1 - \cos 4\theta)} = \tan\theta$, for $0 < \theta < \frac{\pi}{2}$, and $\theta \ne \frac{\pi}{4}$. *[5 marks]*

4. *[Maximum mark: 14]*

(i) Use the substitution $y = xv$ to show that the general solution to the differential equation $(x^2 + y^2) + 2xy\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $x > 0$ is

$$x^3 + 3xy^2 = k, \text{ where } k \text{ is a constant.}$$

[6 marks]

(ii) A curve has equation $f(x) = \frac{a}{b + \mathrm{e}^{-cx}}, a \neq 0, b > 0, c > 0$.

(a) Show that $f''(x) = \frac{ac^2\mathrm{e}^{-cx}\left(\mathrm{e}^{-cx} - b\right)}{\left(b + \mathrm{e}^{-cx}\right)^3}$. *[4 marks]*

(b) Find the coordinates of the point on the curve where $f''(x) = 0$. *[2 marks]*

(c) Show that this is a point of inflexion. *[2 marks]*

5. *[Maximum mark: 13]*

(i) A random variable X is normally distributed with mean μ and standard deviation σ, such that $\mathrm{P}(X > 50.32) = 0.119$, and $\mathrm{P}(X < 43.56) = 0.305$.

(a) Find μ and σ. *[5 marks]*

(b) Hence find $\mathrm{P}\left(\left|X - \mu\right| < 5\right)$. *[2 marks]*

(ii) Consider the following system of equations where b is a constant.

$$3x + y + z = 1$$
$$2x + y - z = 4$$
$$5x + y + bz = 1$$

(a) Solve for z in terms of b. *[4 marks]*

(b) Hence write down, with a reason, the range of values of b for which this system of equations has a unique solution. *[2 marks]*

SECTION B

*Answer **one** question from this section.*

Statistics

6. *[Maximum mark: 30]*

(i) The random variable X has a Poisson distribution with mean λ.

(a) Given that $\mathrm{P}(X=4)=\mathrm{P}(X=2)+\mathrm{P}(X=3)$, find the value of λ. *[3 marks]*

(b) Given that $\lambda = 3.2$, find the value of

(i) $\mathrm{P}(X \geq 2)$;

(ii) $\mathrm{P}\left(X \leq 3 \mid X \geq 2\right)$. *[5 marks]*

(ii) A medical statistician is studying the weights, x kg, of new-born babies in a hospital. She finds that, in one month, 15 babies were born.
For these babies, $\sum x = 55.5$ and $\sum x^2 = 215.8$.
Assuming that weights of babies are normally distributed, calculate a 99 % confidence interval for the mean weight of babies born in this hospital. *[5 marks]*

(iii) A farmer grows tomatoes using plants of two varieties, I and II. He believes that the mean yield from Variety II plants exceeds the mean yield from Variety I plants. He keeps a record of the yield from each plant and he obtains the following results.

Variety	I	II
Number of plants	150	100
Mean yield (kg)	3.51	3.56
Standard deviation of yield (kg)	0.21	0.23

Assume that the two samples are drawn from normal populations with equal variance. Using a 5 % significance level, determine whether or not these results support the farmer’s belief. *[7 marks]*

(This question continues on the following page)

(Question 6 continued)

(iv) Eggs at a farm are sold in boxes of six. Each egg is either brown or white. The owner believes that the number of brown eggs in a box can be modelled by a binomial distribution. He examines 100 boxes and obtains the following data.

Number of brown eggs in a box	Frequency
0	10
1	29
2	31
3	18
4	8
5	3
6	1

(a) (i) Calculate the mean number of brown eggs in a box.

(ii) Hence estimate p, the probability that a randomly chosen egg is brown. *[2 marks]*

(b) By calculating an appropriate χ^2 statistic, test, at the 5 % significance level, whether or not the binomial distribution gives a good fit to these data. *[8 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) (a) Use a Venn diagram to show that $(A \cup B)' = A' \cap B'$. *[2 marks]*

(b) Prove that $\left[(A' \cup B) \cap (A \cup B')\right]' = (A \cap B)' \cap (A \cup B)$. *[4 marks]*

(ii) Let $S = \{f, g, h, j\}$ be the set of functions defined by

$$f(x) = x,\ g(x) = -x,\ h(x) = \frac{1}{x},\ j(x) = -\frac{1}{x},\ \text{where } x \neq 0.$$

(a) Construct the operation table for the group $\{S, \circ\}$, where $\circ$ is the composition of functions. *[3 marks]*

(b) The following are the operation tables for the groups $\{0, 1, 2, 3\}$ under addition modulo 4, and $\{1, 2, 3, 4\}$ under multiplication modulo 5.

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

×	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

By comparing the elements in the two tables given plus the table constructed in part (a), find which groups are isomorphic. Give reasons for your answers. State clearly the corresponding elements. *[6 marks]*

(iii) (a) The binary operation # is defined on the set of real numbers by

$$a \# b = a + b + 1.$$

Show that the binary operation # is both commutative and associative. *[4 marks]*

(b) Show that the set of real numbers forms a group under the operation #. *[4 marks]*

(iv) (a) Determine with reasons which of the following functions is a bijection from $\mathbb{R}$ to $\mathbb{R}$.

$$p(x) = x^2 + 1,\ q(x) = x^3,\ r(x) = \frac{x^2 + 1}{x^2 + 2}$$ *[4 marks]*

(b) Let t be a function from set A to set B, and s be a function from set B to set C. Show that if both s and t are bijective then $s \circ t$ is also bijective. *[3 marks]*

Discrete Mathematics

8. *[Maximum mark: 30]*

(i) (a) The relation R is defined on $\mathbb{Z}$ by aRb if $a \equiv b \pmod{m}$, where $m \in \mathbb{Z}^+$. Show that R is an equivalence relation. *[5 marks]*

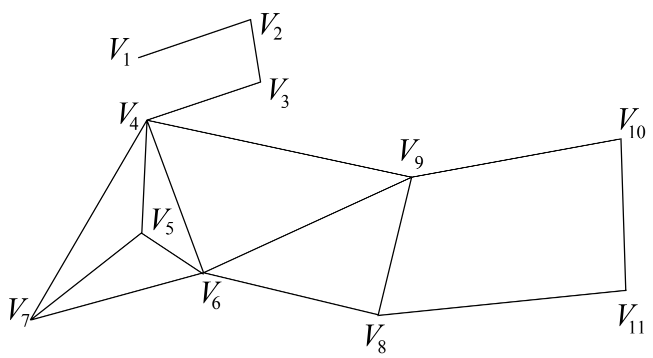
(b) (i) If k, $(0 \le k < 8)$ is a solution of the congruence $5x \equiv 3 \pmod 8$ find the value of k.

(ii) Show that all solutions of $5x \equiv 3 \pmod 8$ are congruent to k. *[5 marks]*

(ii) Solve the recurrence relation

$$u_n = 3u_{n-1} + 4u_{n-2}\ ;\ u_1 = 1, u_2 = 2.$$ *[7 marks]*

(iii) Consider the graph G with vertices $V_1, V_2, V_3, \ldots\ldots V_{11}$ as shown in the diagram given below.



(a) Find the chromatic number for G, justifying your answer. *[4 marks]*

(b) Find the number of distinct Hamiltonian paths between V_1 and V_{11} and give their vertex sequence. *[3 marks]*

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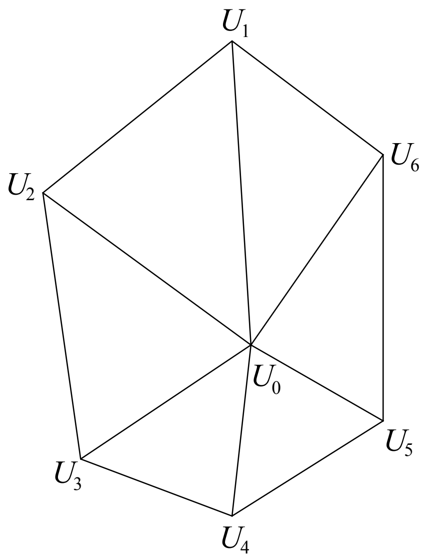
(Question 8 continued)

(iv) Consider the graph K, with vertices $U_0, U_1, U_2, ...U_6$.



(a) Starting at U_0, carry out a breadth-first search of graph K, to obtain a spanning tree. Draw the current tree T_r for $r = 1$ to 6, at each stage of search.

(b) Starting at U_0, carry out a depth-first search of graph K, to obtain a spanning tree. Draw the current tree T_r, for $r = 1$ to 6, at each stage of search. *[6 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

(i) (a) Consider the curve $f(x) = x\sin x$, for $0 \le x \le 2\pi$.

(i) Sketch the graph of $y = f(x)$.

(ii) Find the smallest positive root for $f(x) = 0$, and give your answer as an **exact** value.

(iii) Using the Newton-Raphson method for finding the root of $f(x) = 0$, show that $x_{n+1} = \dfrac{x_n^2 \cos x_n}{x_n \cos x_n + \sin x_n}$. *[6 marks]*

(b) Find the area enclosed between the curve $y = x\sin x$, and the line $y = 0$, for $0 \le x \le \pi$. Give your answer to **six** decimal places. *[2 marks]*

(c) If the area of part (b) were to be calculated using the trapezium rule with 10 intervals, find an upper bound for the error in the estimate of the area. (Do not calculate the area using the trapezium rule, just find the upper bound for the error involved in the calculation). *[6 marks]*

(ii) (a) Describe how the integral test is used to show that a series is convergent. Clearly state all the necessary conditions. *[3 marks]*

(b) Test the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{e^{n^2}}$ for convergence. *[5 marks]*

(iii) (a) Find the first four non-zero terms of the Maclaurin series for

(i) $\sin x$;

(ii) e^{x^2}. *[4 marks]*

(b) Hence find the Maclaurin series for $e^{x^2}\sin x$, up to the term containing x^5. *[2 marks]*

(c) Use the result of part (b) to find $\displaystyle\lim_{x \to 0}\left(\frac{e^{x^2}\sin x - x}{x^3}\right)$. *[2 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

(i) In a ΔABC, D and E are points on the sides [BC] and [CA] such that DC = 2BD and EA = 2CE. If the lines (DE) and (AB) intersect at the point F, then prove that AB = 3BF. *[4 marks]*

(ii) (a) In a ΔABC, let D be the midpoint of the segment [BC] and [AD] be a median. Prove Apollonius' theorem, that

$$\mathrm{AB}^2 + \mathrm{AC}^2 = 2(\mathrm{AD}^2 + \mathrm{BD}^2).$$ *[5 marks]*

(b) In a quadrilateral ABCD, X and Y are the midpoints of [AC] and [BD], respectively. Prove that

$$\mathrm{AB}^2 + \mathrm{BC}^2 + \mathrm{CD}^2 + \mathrm{DA}^2 = \mathrm{AC}^2 + \mathrm{BD}^2 + 4\mathrm{XY}^2.$$ *[6 marks]*

(iii) Let F_1 and F_2 be the foci of the ellipse whose equation is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b.$$

Let P be the point on the ellipse with coordinates (x_0, y_0) with $x_0, y_0 > 0$.

(a) Show that the equation of the tangent to the ellipse at the point P is given by

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$ *[5 marks]*

(b) The tangent at P meets the x-axis at the point M. Find the coordinates of point M. *[2 marks]*

(c) Find the lengths $\mathrm{PF}_1, \mathrm{PF}_2, \mathrm{MF}_1, \mathrm{MF}_2$. *[4 marks]*

(d) Use the converse of the angle bisector theorem to prove that [PM] is the external bisector of $\mathrm{F}_2\hat{\mathrm{P}}\mathrm{F}_1$. *[4 marks]*