INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

N03/510/H(2)M+

MARKSCHEME

November 2003

MATHEMATICS

Higher Level

Paper 2

30 pages

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Errors made at any step of a solution can affect all working that follows. To limit the severity of the penalty, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent working;

(iii) award ***M*** marks for a correct method, and ***A*****(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
\$ 600 × 1.02	***M1***	Amount earned = \$ 600 × 1.02	√	***M1***
= \$ 612	***A1***	= \$602	×	***A0***
\$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	√	***M1***
= \$ 630.36	***A1***	= \$ 620.06	√	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative solutions, including graphic display calculator alternative solutions are indicated by **OR**. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect ie, once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

5 Accuracy of Answers

There are two types of accuracy errors, incorrect level of accuracy, and rounding errors.

Unless the level of accuracy is specified in the question, candidates should be penalized **once only IN THE PAPER** for any accuracy error **(AP)**. This could be an incorrect level of accuracy **(only applies to fewer than three significant figures)**, or a rounding error. Hence, on the **first** occasion in the paper when a correct answer is given to the wrong degree of accuracy, or rounded incorrectly, maximum marks are **not** awarded, but on **all subsequent occasions** when accuracy errors occur, then maximum marks **are** awarded.

(a) **Level of accuracy**

(i) In the case when the accuracy of the answer is **specified in the question** (for example: "find the size of angle A to the nearest degree") the maximum mark is awarded **only if** the correct answer is given to the accuracy required.

(ii) When the accuracy is **not** specified in the question, then the general rule applies:

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

However, if candidates give their answers to more than three significant figures, this is acceptable

(b) **Rounding errors**

Rounding errors should only be penalized at the **final answer** stage. This does **not** apply to intermediate answers, only those asked for as part of a question. Premature rounding which leads to incorrect answers should only be penalized at the answer stage.

Incorrect answers are wrong, and should not be considered under (a) or (b).

Examples

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : 4.7 should be penalised the first time this type of error occurs, but 4.679 is **not** penalized, as it has more than three significant figures.
- 4.67 is incorrectly rounded – penalise on the first occurrence.
- 4.678 is incorrectly rounded, but has more than the required accuracy, do **not** penalize.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Calculator penalties

Candidates are instructed to write the make and model of their calculator on the front cover. Please apply the following penalties where appropriate.

(i) Illegal calculators

If candidates note that they are using an illegal calculator, please report this on a PRF, and deduct 10 % of their overall mark. Note this on the front cover.

(ii) Calculator box not filled in.

Please apply a calculator penalty (***CP***) of 1 mark if this information is not provided. Note this on the front cover.

1. (a) $\boldsymbol{n}=\begin{pmatrix}1\\1\\1\end{pmatrix}$, hence equation of L through $A(2,5,-1)$ is given by $\dfrac{x-2}{1}=\dfrac{y-5}{1}=\dfrac{z+1}{1}$. ***(M1)(A1)***

[2 marks]

(b) A general point on L is $(2+\lambda, 5+\lambda, -1+\lambda)$. ***(A1)***

At intersection of line L and the plane

$(2+\lambda)+(5+\lambda)+(-1+\lambda)-1=0$ ***(M1)***

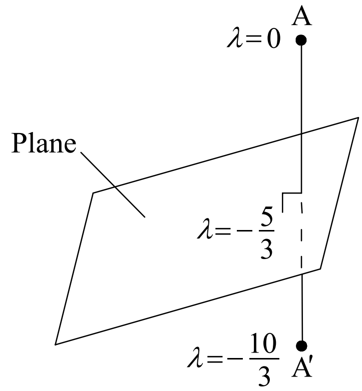
$$\Rightarrow 3\lambda=-5$$

$$\Rightarrow \lambda=-\frac{5}{3}$$ ***(A1)***

$$\Rightarrow \text{point of intersection}\left(\frac{1}{3},\frac{10}{3},-\frac{8}{3}\right)$$ ***(A1)***

[4 marks]

(c)



Let $A'(x,y,z)$ be the reflection of A.

Note:	Diagram **does not** have to be given.

EITHER

At A' $\lambda=-\dfrac{10}{3}$ ***(M1)***

$$\Rightarrow A'=\left(-\frac{4}{3},\frac{5}{3},-\frac{13}{3}\right)$$ ***(A1)***

OR

Since point of intersection of L and the plane is midpoint of AA'

$$\begin{pmatrix}2\\5\\-1\end{pmatrix}+\begin{pmatrix}x\\y\\z\end{pmatrix}=2\begin{pmatrix}\frac{1}{3}\\[4pt]\frac{10}{3}\\[4pt]-\frac{8}{3}\end{pmatrix}$$ ***(M1)***

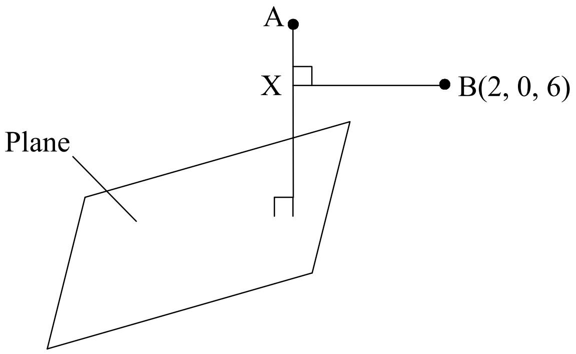
$$\Rightarrow A'=\left(-\frac{4}{3},\frac{5}{3},-\frac{13}{3}\right)$$ ***(A1)***

[2 marks]

continued…

Question 1 continued

(d)



Let X be foot of perpendicular from B to $L \Rightarrow d = \left|\overrightarrow{\text{BX}}\right|$

$$\overrightarrow{\text{BX}} = \overrightarrow{\text{OX}} - \overrightarrow{\text{OB}}$$

$$= \begin{pmatrix} 2+\lambda \\ 5+\lambda \\ -1+\lambda \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} = \begin{pmatrix} \lambda \\ 5+\lambda \\ -7+\lambda \end{pmatrix}$$ ***(A1)***

Now $\overrightarrow{\text{BX}} \cdot \boldsymbol{n} = 0$ ***(M1)***

$$\Rightarrow \lambda + (5+\lambda) + (-7+\lambda) = 0$$

$$\Rightarrow 3\lambda = 2$$

$$\Rightarrow \lambda = \frac{2}{3} \Rightarrow \overrightarrow{\text{BX}} = \begin{pmatrix} \frac{2}{3} \\ \frac{17}{3} \\ -\frac{19}{3} \end{pmatrix}$$ ***(A1)***

Hence $d = \sqrt{\frac{4}{9} + \frac{289}{9} + \frac{361}{9}}$

$$= \frac{\sqrt{654}}{3} = 8.52 \text{ (units)}$$ ***(A1)***

OR

$$d = \left|\frac{\overrightarrow{\text{AB}} \times \boldsymbol{n}}{|\boldsymbol{n}|}\right| \quad \text{where } \overrightarrow{\text{AB}} = \begin{pmatrix} 0 \\ -5 \\ 7 \end{pmatrix}$$ ***(M1)(A1)***

$$\overrightarrow{\text{AB}} \times \boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 0 & -5 & 7 \\ 1 & 1 & 1 \end{vmatrix} = -12\boldsymbol{i} + 7\boldsymbol{j} + 5\boldsymbol{k}$$ ***(A1)***

$$\Rightarrow d = \frac{\sqrt{144+49+25}}{\sqrt{3}} = 8.52 \text{ (units)}$$ ***(A1)***

[4 marks]

Total [12 marks]

2. (a) $(\cos\theta + \mathrm{i}\sin\theta)^n = \cos(n\theta) + \mathrm{i}\sin(n\theta),\ n \in \mathbb{Z}^+$

Let $n = 1 \Rightarrow \cos\theta + \mathrm{i}\sin\theta = \cos\theta + \mathrm{i}\sin\theta$ which is true. ***(A1)***

Assume true for $n = k \Rightarrow (\cos\theta + \mathrm{i}\sin\theta)^k = \cos(k\theta) + \mathrm{i}\sin(k\theta)$. ***(M1)***

Now show $n = k$ true implies $n = k + 1$ also true. ***(M1)***

$(\cos\theta + \mathrm{i}\sin\theta)^{k+1} = (\cos\theta + \mathrm{i}\sin\theta)^k(\cos\theta + \mathrm{i}\sin\theta)$ ***(M1)***

$= \left(\cos(k\theta) + \mathrm{i}\sin(k\theta)\right)(\cos\theta + \mathrm{i}\sin\theta)$

$= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + \mathrm{i}\left(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\right)$ ***(A1)***

$= \cos(k\theta + \theta) + \mathrm{i}\sin(k\theta + \theta)$

$= \cos(k+1)\theta + \mathrm{i}\sin(k+1)\theta \Rightarrow n = k + 1$ is true. ***(A1)***

Therefore by mathematical induction statement is true for $n \geq 1$. ***(R1)***

[7 marks]

(b) (i) $z_1 = 2\left(\cos\left(\frac{2\pi}{5}\right) + \mathrm{i}\sin\left(\frac{2\pi}{5}\right)\right)$

$\Rightarrow z_1^5 = 2^5(\cos 2\pi + \mathrm{i}\sin 2\pi)$ ***(M1)***

$= 32$

Therefore z_1 is a root of $z^5 - 32 = 0$. ***(AG)***

(ii) $z_1^2 = 4\left(\cos\frac{4\pi}{5} + \mathrm{i}\sin\frac{4\pi}{5}\right)$

$z_1^3 = 8\left(\cos\frac{6\pi}{5} + \mathrm{i}\sin\frac{6\pi}{5}\right)$

$z_1^4 = 16\left(\cos\frac{8\pi}{5} + \mathrm{i}\sin\frac{8\pi}{5}\right)$

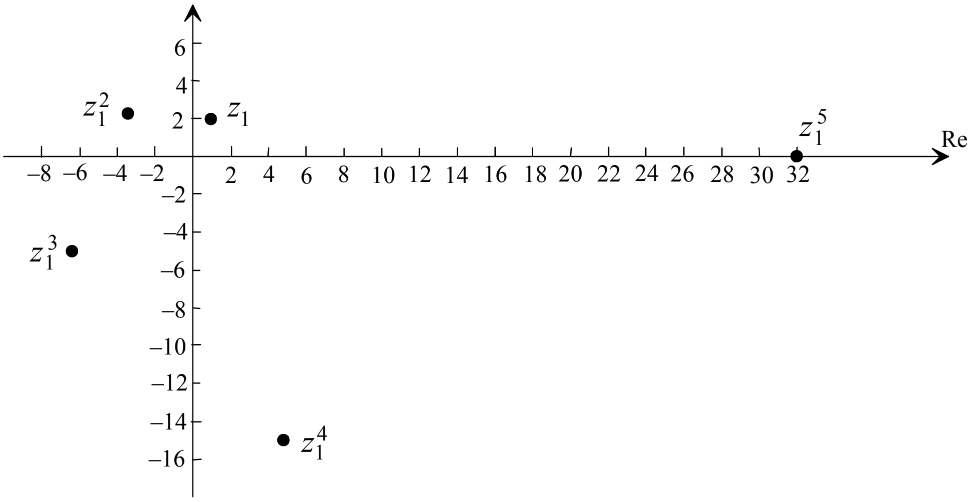
$z_1^5 = 32(\cos 2\pi + \mathrm{i}\sin 2\pi)\quad \left(= 32(\cos 0 + \mathrm{i}\sin 0) = 32\right)$ ***(A2)***

Note: Award ***(A2)*** for all 4 correct, ***(A1)*** for 3 correct, ***(A0)*** otherwise.

continued…

Question 2 (b) continued

(iii)



(A1)(A3)

Note:	Award ***(A1)*** for graph of reasonable size, scale, axes marked, ***(A3)*** for all 5 points correctly plotted, ***(A2)*** for 4 points correctly plotted. ***(A1)*** for 3 points correctly plotted.

(iv) Composite transformation is a combination of (in any order)

an enlargement scale factor 2, centre $(0, 0)$; ***(A1)***

a rotation (anti-clockwise) of $\frac{2\pi}{5}\,(72^\circ)$, centre $(0, 0)$ $\left(\text{or clockwise } \frac{8\pi}{5}\,(288^\circ)\right)$. ***(A1)***

Note:	Do not penalize if centre of enlargement or rotation not given.

[9 marks]

Total [16 marks]

3. (i) (a) $\sqrt{3}\cos\theta - \sin\theta = r\cos(\theta+\alpha)$

where $r = \sqrt{3+1} = 2$ ***(A1)***

and $\alpha = \arctan\dfrac{1}{\sqrt{3}} = \dfrac{\pi}{6}$ (or 30°) ***(M1)(A1)***

$\left(\Rightarrow \sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta+\dfrac{\pi}{6}\right)\right)$

[3 marks]

(b) Since $\sqrt{3}\cos\theta - \sin\theta = 2\cos\left(\theta+\dfrac{\pi}{6}\right)$ ***(M1)***

range will be $[-2, 2]$. ***(A1)***

[2 marks]

(c) $\sqrt{3}\cos\theta - \sin\theta = -1$

$\Rightarrow 2\cos\left(\theta+\dfrac{\pi}{6}\right) = -1$

$\Rightarrow \cos\left(\theta+\dfrac{\pi}{6}\right) = -\dfrac{1}{2}$ ***(M1)***

$\Rightarrow \theta+\dfrac{\pi}{6} = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ ***(A1)(A1)***

$\Rightarrow \theta = \dfrac{\pi}{2}, \dfrac{7\pi}{6}$ ***(A1)(A1)***

Note: Answers **must** be multiples of π.

[5 marks]

(ii) $$\frac{\sin 4\theta(1-\cos 2\theta)}{\cos 2\theta(1-\cos 4\theta)} \equiv \frac{2\sin 2\theta\cos 2\theta\left(1-(\cos^2\theta-\sin^2\theta)\right)}{\cos 2\theta\left(1-(\cos^2 2\theta-\sin^2 2\theta)\right)}$$ ***(M1)*** ***(M1)***

$$\equiv \frac{2\sin 2\theta(1-\cos^2\theta+\sin^2\theta)}{1-\cos^2 2\theta+\sin^2 2\theta}$$ ***(A1)***

$$\equiv \frac{2\sin 2\theta(2\sin^2\theta)}{2\sin^2 2\theta}$$ ***(A1)***

$$\equiv \frac{2\sin^2\theta}{\sin 2\theta}$$

$$\equiv \frac{2\sin^2\theta}{2\sin\theta\cos\theta}$$

$$\equiv \frac{\sin\theta}{\cos\theta}$$ ***(A1)***

$\equiv \tan\theta$ ***(AG)***

[5 marks]

Total [15 marks]

4. (i) $y = xv \Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x} = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$ ***(A1)***

Now $x^2 + y^2 + 2xy\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

$\Rightarrow x^2 + x^2v^2 + 2x^2v\left(v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}\right) = 0$ ***(A1)***

$\Rightarrow 1 + v^2 + 2v^2 + 2xv\dfrac{\mathrm{d}v}{\mathrm{d}x} = 0$ (since $x^2 > 0$)

$\Rightarrow 2xv\dfrac{\mathrm{d}v}{\mathrm{d}x} = -(1+3v^2)$

$\Rightarrow \int \dfrac{2v}{1+3v^2}\mathrm{d}v = \int -\dfrac{1}{x}\mathrm{d}x$ ***(M1)***

$\Rightarrow \dfrac{1}{3}\ln\left|1+3v^2\right| = -\ln x + \ln k$ ***(A1)(A1)***

$\Rightarrow \ln\left|1+3v^2\right| = -3\ln x + \ln k$

$\Rightarrow 1 + \dfrac{3y^2}{x^2} = \dfrac{k}{x^3}$ ***(A1)***

$\Rightarrow x^3 + 3xy^2 = k$ ***(AG)***

[6 marks]

(ii) $f(x) = \dfrac{a}{b + \mathrm{e}^{-cx}}, a \neq 0, b > 0, c > 0$

(a) $f'(x) = \dfrac{(b+\mathrm{e}^{-cx})(0) - (a)(-c\mathrm{e}^{-cx})}{(b+\mathrm{e}^{-cx})^2}$ ***(M1)***

$= \dfrac{ac\mathrm{e}^{-cx}}{(b+\mathrm{e}^{-cx})^2}$ ***(A1)***

$f''(x) = \dfrac{(b+\mathrm{e}^{-cx})^2(-ac^2\mathrm{e}^{-cx}) - (ac\mathrm{e}^{-cx})2(b+\mathrm{e}^{-cx})(-c\mathrm{e}^{-cx})}{(b+\mathrm{e}^{-cx})^4}$ ***(M1)***

$= \dfrac{-bac^2\mathrm{e}^{-cx} - ac^2(\mathrm{e}^{-cx})^2 + 2ac^2(\mathrm{e}^{-cx})^2}{(b+\mathrm{e}^{-cx})^3}$

$= \dfrac{ac^2(\mathrm{e}^{-cx})^2 - bac^2\mathrm{e}^{-cx}}{(b+\mathrm{e}^{-cx})^3}$ ***(A1)***

$= \dfrac{ac^2\mathrm{e}^{-cx}(\mathrm{e}^{-cx} - b)}{(b+\mathrm{e}^{-cx})^3}$ ***(AG)***

[4 marks]

(b) $f''(x) = 0 \Rightarrow \mathrm{e}^{-cx} = b$

$\Rightarrow -cx = \ln b$

$\Rightarrow x = -\dfrac{1}{c}\ln b \Rightarrow y = \dfrac{a}{b + \mathrm{e}^{\ln b}} = \dfrac{a}{2b}$

So coordinate $= \left(-\dfrac{1}{c}\ln b, \dfrac{a}{2b}\right)$ ***(A1)(A1)***

[2 marks]

continued…

Question 4 (ii) continued

(c) Now $e^{-cx} > b$ on one side of $x = -\frac{1}{c}\ln b$ and $e^{-cx} < b$ on the other side. ***(R1)***

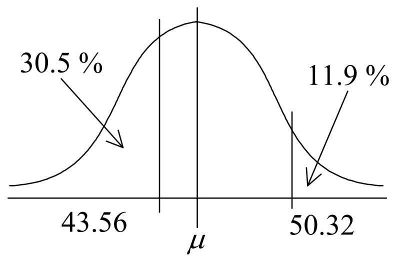
$\Rightarrow f''(x)$ changes sign at this point. ***(R1)***

$\Rightarrow$ It is a point of inflexion. ***(AG)***

[2 marks]

Total [14 marks]

5. (i) (a)



$\phi(z) = 0.305 \Rightarrow z = -0.51$ ***(A1)***

and $\phi(z) = 0.881 \Rightarrow z = 1.18$ ***(A1)***

$\dfrac{50.32-\mu}{\sigma} = 1.18$ and $\dfrac{43.56-\mu}{\sigma} = -0.51$ ***(M1)***

Solving simultaneously

$\Rightarrow 50.32 = \mu + 1.18\sigma$ and $43.56 = \mu - 0.51\sigma$

$\Rightarrow 1.69\sigma = 6.76$

$\Rightarrow \sigma = 4 \quad \Rightarrow \quad \mu = 45.6$ ***(A1)(A1)***

[5 marks]

(b) $\mathrm{P}\left(\left|X-\mu\right| < 5\right) = \mathrm{P}(40.6 < x < 50.6)$ ***(M1)***

$= 0.789$ ***(G1)***

[2 marks]

(ii) (a) $3x + y + z = 1$ ①

$2x + y - z = 4$ ②

$5x + y + bz = 1$ ③

Solving for z ③−② $\Rightarrow$ $3x + bz + z = -3$ ④ ***(M1)***

also ②−① $\Rightarrow$ $-x \quad -2z = 3$ ⑤ ***(M1)***

3×⑤+④ $\Rightarrow$ $bz - 5z = 6$ ***(A1)***

$\Rightarrow \quad z = \dfrac{6}{b-5}$ ***(A1)***

[4 marks]

(b) If $b = 5$, z is undefined. ***(R1)***

Hence equation has unique solution if $b \neq 5$. ***(A1)***

[2 marks]

Total [13 marks]

6. (i) (a) $\frac{e^{-\lambda}\times\lambda^4}{4!}=\frac{e^{-\lambda}\times\lambda^2}{2!}+\frac{e^{-\lambda}\times\lambda^3}{3!}$ ***(M1)***

$\lambda^2-4\lambda-12=0\Rightarrow\lambda=6$ ***(A1)(A1)***

[3 marks]

(b) (i) $P(X\geq 2)=1-e^{-3.2}-e^{-3.2}\times 3.2=0.829$ ***(M1)(A1)***

OR

$P(X\geq 2)=0.829$ ***(G2)***

(ii) $P\left(X\leq 3\,\middle|\,X\geq 2\right)=\frac{P(2\leq X\leq 3)}{P(X\geq 2)}$ ***(M1)***

$=\frac{\frac{e^{-3.2}\times 3.2^2}{2}+\frac{e^{-3.2}\times 3.2^3}{6}}{1-4.2e^{-3.2}}$ ***(A1)***

$=0.520$ ***(A1)***

OR

$P\left(X\leq 3\,\middle|\,X\geq 2\right)=0.520$ ***(G3)***

[5 marks]

(ii) $\hat{\mu}=\frac{55.5}{15}=3.7;\hat{\sigma}^2=\frac{215.8}{14}-\frac{55.5^2}{14\times 15}=0.746428\ldots$ ***(A1)(A1)***

$99\,\%$ confidence limits are $3.7\pm 2.977\sqrt{\frac{0.746428\ldots}{15}}$ ***(M1)(A1)***

giving [3.04, 4.36]. ***(A1)***

OR

$99\,\%$ C.I. is [3.04, 4.36]. ***(G3)***

[5 marks]

continued…

Question 6 continued

Note:	Candidates may obtain slightly different numerical answers depending on the calculator and approach used. Use discretion in marking.

(iii) **EITHER**

Let $H_0 : \mu_I = \mu_{II}$ and $H_1 : \mu_I < \mu_{II}$ ***(R1)***

Pooled variance $= \dfrac{150 \times 0.21^2 + 100 \times 0.23^2}{248} = 0.048004\ldots$ ***(M1)(A1)***

Test statistics $= \dfrac{3.56 - 3.51}{\sqrt{0.048004\ldots\left(\dfrac{1}{150} + \dfrac{1}{100}\right)}} = 1.77$ ***(M1)(A1)***

Critical value = 1.645 ***(A1)***

We conclude that the farmer's belief is supported by these results. ***(A1)***

OR

Let $H_0 : \mu_I = \mu_{II}$, $H_1 : \mu_I < \mu_{II}$ ***(R1)***

Using GDC, p-value = 0.0392. ***(G5)***

Note:	Award ***(G4)*** for $p = 0.0386$.

We conclude that the farmer's belief is supported by these results. ***(A1)***

There are several other methods of solution which, for various reasons, do not merit full credit. Possibilities are:

METHOD 1

Some candidates may use a GDC facility which does not require the assumption of equality of variance as required in the question. The results obtained (on a TI 83) are

$$t = 1.74,\ p\text{-value} = 0.0421$$

Note:	Award only ***(G3)*** (and not ***(G5)***) for this solution.

METHOD 2

Other candidates might think that, in view of the large samples involved, a 2-sample z-test can be used with variances replaced by their estimates. This is a valid approach but not in line with the syllabus which states that "if the population variance is unknown, the t-distribution should be used regardless of sample size". This method gives

$$\text{Test stat} = \frac{3.56 - 3.51}{\sqrt{\dfrac{0.21^2}{150} + \dfrac{0.23^2}{100}}} = 1.74$$

Critical value = 1.645 or using a GDC, p-value = 0.0407

Note:	Award only ***(G3)*** (and not ***(G5)***) for this solution.

[7 marks]

continued…

Question 6 continued

(iv) (a) (i) Mean $= \dfrac{1 \times 29 + \ldots + 6 \times 1}{100} = 1.98$ ***(A1)***

(ii) $\hat{p} = \dfrac{1.98}{6} = 0.33$ ***(A1)***

[2 marks]

(b) The calculated values are

f_o	f_e	$(f_o - f_e)^2$	
10	9.046	0.910	***(M1)***
29	26.732	5.14	***(A1)***
31	32.917	3.675	***(A1)***
18	21.617	13.083	***(A1)***
12	9.688	5.345	

Note: Award ***(M1)*** for the attempt to calculate expected values, ***(A1)*** for correct expected values, ***(A1)*** for correct $(f_0 - f_e)^2$ values, ***(A1)*** for combining cells.

$\chi^2 = \dfrac{0.910}{9.046} + \ldots + \dfrac{5.345}{9.688} = 1.56$ ***(A1)***

OR

$\chi^2 = 1.56$ ***(G5)***

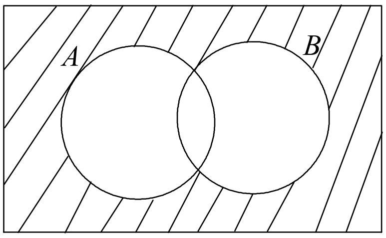
Degrees of freedom = 3; Critical value = 7.815 (or p-value = 0.668 (or 0.669)) ***(A1)(A1)***

We conclude that the binomial distribution does provide a good fit. ***(R1)***

[8 marks]

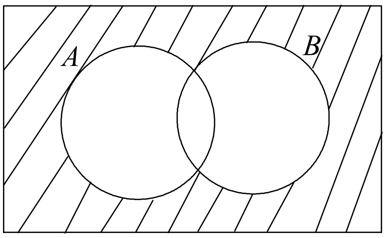
Total [30 marks]

7. (i) (a) $(A \cup B)'$ is given by



(A1)

$A' \cap B'$ is given by



(A1)

Hence $(A \cup B)' = A' \cap B'$. *(AG)*

[2 marks]

(b) $\left[(A' \cup B) \cap (A \cup B')\right]' = (A' \cup B)' \cup (A \cup B')'$ *(A1)*

$= (A \cap B') \cup (A' \cap B)$ *(A1)*

$= \left[(A \cap B') \cup A'\right] \cap \left[(A \cap B') \cup B\right]$ *(M1)*

$= \left[(A \cup A') \cap (B' \cup A'\right] \cap \left[(A \cup B) \cap (B' \cup B)\right]$

$= (B' \cup A') \cap (A \cup B)$ *(A1)*

$= (A \cap B)' \cap (A \cup B)$ *(AG)*

[4 marks]

(ii) (a)

$\circ$	f	g	h	j
f	f	g	h	j
g	g	f	j	h
h	h	j	f	g
j	j	h	g	f

(A3)

Note: Award *(A3)* for all correct, *(A2)* for 1 error, *(A1)* for 2 errors, *(A0)* otherwise.

[3 marks]

(b)

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\times_5$	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

To investigate which may be isomorphic we can consider the order of elements *(M1)*
for $+_4$, the identity is 0, 1 has order 4, 2 has order 2 and 3 has order 4, *(A1)*
for $\times_5$, the identity is 1, 2 has order 4, 3 has order 4 and 4 has order 2, *(A1)*
for $\circ$, the identity is *f*, and *g*, *h* and *j* all have order 2. *(A1)*
Hence $+_4$ is isomorphic with $\times_5$. *(A1)*
Corresponding elements are
$0 \leftrightarrow 1, 1 \leftrightarrow 2, 2 \leftrightarrow 4, 3 \leftrightarrow 3$, OR $0 \leftrightarrow 1, 1 \leftrightarrow 3, 2 \leftrightarrow 4, 3 \leftrightarrow 2$. *(A1)*

Note: Corresponding elements **must** be correct for final *(A1)*.

[6 marks]

continued…

Question 7 continued

(iii) (a) $a\#b = a+b+1$

Now $b\#a = b+a+1$ ***(M1)***

Since + is commutative $a\#b = b\#a$ ***(A1)***

$\Rightarrow \#$ is also a commutative operation. ***(AG)***

$$(a\#b)\#c = (a+b+1)\#c$$
$$= a+b+1+c+1$$
$$= a+b+c+2 \quad \textbf{\textit{(A1)}}$$

$$a\#(b\#c) = a\#(b+c+1)$$
$$= a+b+c+1+1$$
$$= a+b+c+2 \quad \textbf{\textit{(A1)}}$$

$\Rightarrow \#$ is also associative operation. ***(AG)***

[4 marks]

(b) To show ($\mathbb{R}$, #) is a group we need to show closure, identity element exists, inverses exist and it is associative (already shown).

It is closed since $a+b+1 \in \mathbb{R}$ for $a, b \in \mathbb{R}$. ***(A1)***

There is a unique element $e(e \in \mathbb{R})$ such that $p\#e = e\#p = p$ where $p \in \mathbb{R}$

$\Rightarrow p+e+1 = e+p+1 = p$

$\Rightarrow e = -1$ as identity element ***(A1)***

There are unique inverse elements for each element in $\mathbb{R}$ such that

$p\#p^{-1} = p^{-1}\#p = -1$ ***(M1)***

$\Rightarrow p+p^{-1}+1 = p^{-1}+p+1 = -1$

$\Rightarrow p^{-1} = -p-2$ ***(A1)***

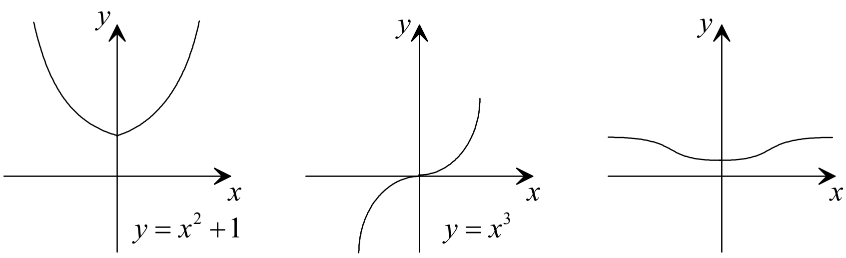
Hence ($\mathbb{R}$, #) forms a group. ***(AG)***

[4 marks]

continued…

Question 7 continued

(iv) (a) A bijection is both one-to-one and onto,
so by considering a sketch of each function



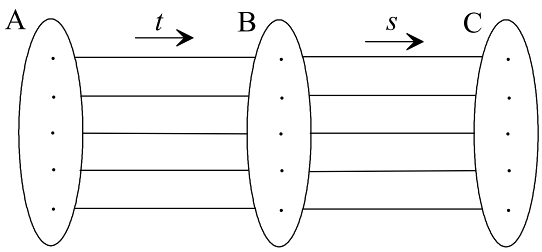
(A1)(A1)(A1)

we can see that for $\mathbb{R}$ to $\mathbb{R}$ only $y = x^3$ is one-to-one and onto. **(R1)**

[4 marks]

(b) t and s are bijections
so under t each element in set A has a unique image in set B and similarly under s each element in set B has a unique image in set C.

Hence



(M1)

and under the composite function $s \circ t$ each element of set A has a unique image in set C. **(R1)**
This is both one-to-one and onto **(R1)**
$\Rightarrow s \circ t$ is bijective. **(AG)**

[3 marks]

Total [30 marks]

8. (i) (a) $a \equiv b \pmod{c} \Rightarrow \dfrac{a-b}{c} = k, k \in \mathbb{Z}$

Now $a \equiv a \pmod{c}$ since $\dfrac{a-a}{c} = 0$

$\Rightarrow R$ is reflexive. ***(A1)***

If $a \equiv b \pmod{c}$ then $b \equiv a \pmod{c}$ is true,

since $\dfrac{b-a}{c} = -k \Rightarrow R$ is also symmetric. ***(A1)***

If $a \equiv b \pmod{c}$ and $b \equiv d \pmod{c}$ then $\dfrac{a-b}{c} = k_1$ and $\dfrac{b-d}{c} = k_2$

$\Rightarrow a - b = ck_1 \qquad b - d = ck_2$

$\Rightarrow a - b + b - d = ck_1 + ck_2$

$\Rightarrow a - d = (k_1 + k_2)c$ ***(M1)***

$\Rightarrow \dfrac{a-d}{c} = k_1 + k_2 \Rightarrow a \equiv d \pmod{c}$

$\Rightarrow R$ is transitive. ***(A1)***

Since R is reflexive, symmetric and transitive, ***(A1)***

R is an equivalence relation. ***(AG)***

[5 marks]

(b) (i) $5x \equiv 3 \pmod{8}$

$\Rightarrow \dfrac{5x-3}{8} = t \quad (t \in \mathbb{Z})$

$\Rightarrow x = \dfrac{8t+3}{5}$ ***(M1)***

By trial and error a solution is $t = 4$ and $x = 7$.

$\Rightarrow$ Required value $k = 7$. ***(A1)***

(ii) Other solutions for t and x are

$t = 9, \quad x = 15$

$t = 14, x = 23$ ***(M1)***

and in general t takes the form $5n + 4 \quad (n \in \mathbb{Z})$ ***(A1)***

$\Rightarrow x = \dfrac{8(5n+4)+3}{5}$

$= 8n + 7$. ***(A1)***

Hence all solutions are congruent to $k = 7$. ***(AG)***

OR

If k_1 and k_2 are two solutions

then $5k_1 \equiv 3$

$5k_2 \equiv 3$ ***(M1)***

$\Rightarrow 5(k_1 - k_2) \equiv 0 \pmod{8}$ ***(A1)***

As 5 is not a factor of 8, $k_1 - k_2$ must be a multiple of 8.

$\Rightarrow k_1 \equiv k_2 \pmod{8}$ ***(A1)***

Hence all solutions are congruent to $k = 7$. ***(AG)***

[5 marks]

continued…

Question 8 continued

(ii) $u_n = r^n \Rightarrow r^n = 3r^{n-1} + 4r^{n-2}$ ***(M1)(A1)***

$$\Rightarrow \frac{r^n}{r^{n-2}} = \frac{3r^{n-1}}{r^{n-2}} + \frac{4r^{n-2}}{r^{n-2}}$$

$$\Rightarrow r^2 = 3r + 4$$

$$\Rightarrow r^2 - 3r - 4 = 0$$ ***(A1)***

This is the Characteristic Equation of the Recurrence Relation.

Solving for $r \Rightarrow (r-4)(r+1) = 0$

$r = 4, r = -1$ ***(A1)***

general solution $u_n = A4^n + B(-1)^n$ ***(A1)***

$n = 1 \Rightarrow 1 = 4A - B$

$n = 2 \Rightarrow 2 = 16A + B$

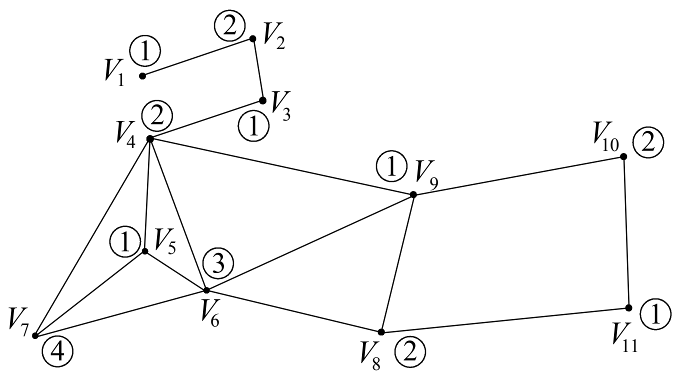
$\Rightarrow 3 = 20A$

$$\Rightarrow A = \frac{3}{20} \quad \Rightarrow B = -\frac{2}{5} \quad \left(\Rightarrow u_n = \frac{3}{20}(4)^n - \frac{2}{5}(-1)^n\right)$$ ***(A1)(A1)***

[7 marks]

(iii) (a) **EITHER**

Denoting different colours by the numbers ①,②,③… the diagram below shows the need to use 4 colours, otherwise two adjacent vertices would have equal numbers (same colour). ***(R1)***



(M1)(A1)

$\Rightarrow$ Chromatic number of graph $\chi(G) = 4$. ***(A1)***

OR

The chromatic number is ≤ 4 since G is planar and the chromatic number is ≥ 4, since G contains κ_4 as a sub graph (V_4, V_5, V_6, V_7). ***(A1)*** ***(A1)(A1)***

$\Rightarrow$ Chromatic number of graph $\chi(G) = 4$. ***(A1)***

[4 marks]

(b) Hamiltonian paths (all vertices appear once) between V_1 and V_{11}.

$V_1, V_2, V_3, V_4, V_5, V_7, V_6, V_8, V_9, V_{10}, V_{11}$ ***(A1)***

$V_1, V_2, V_3, V_4, V_7, V_5, V_6, V_8, V_9, V_{10}, V_{11}$ ***(A1)***

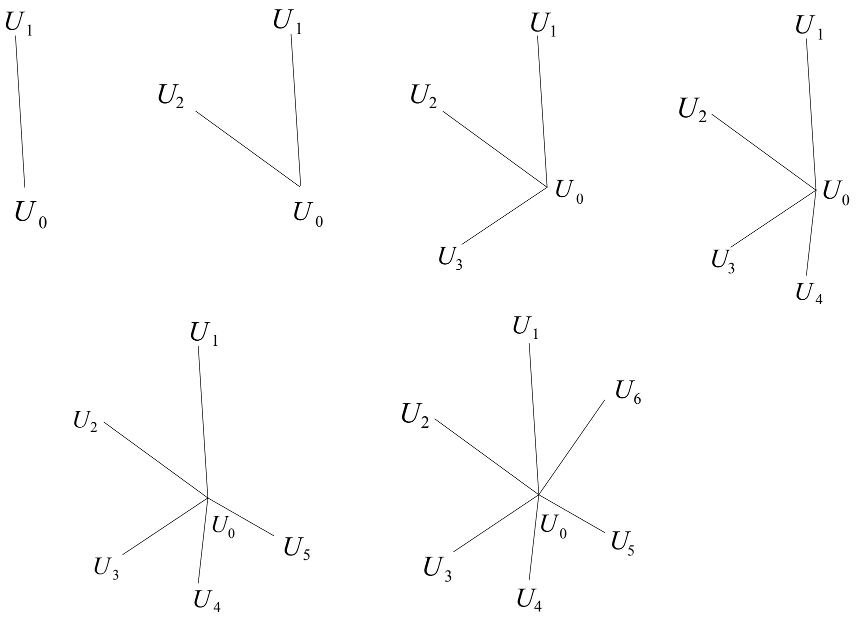
Hence there are two district Hamiltonian paths in *G*. ***(A1)***

[3 marks]

continued…

Question 8 continued

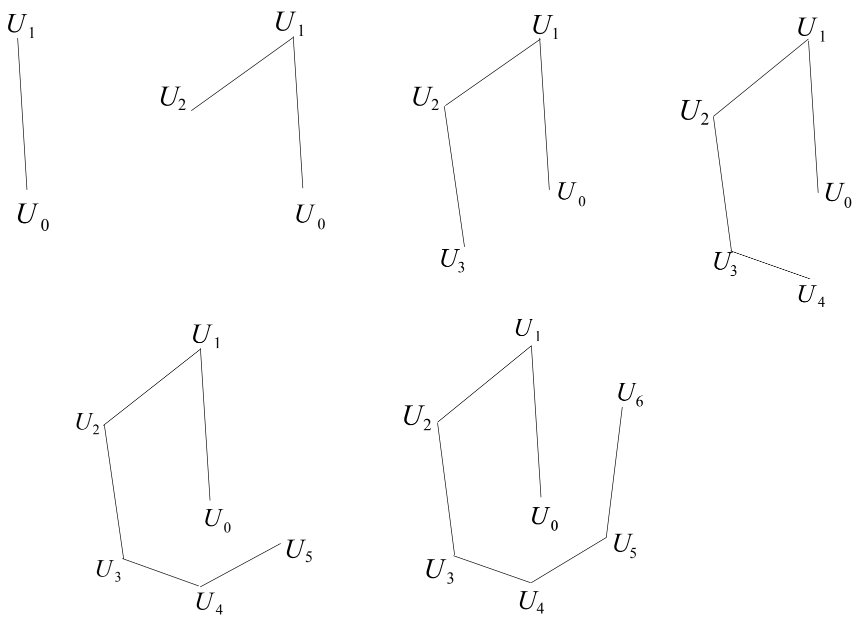
(iv) (a) For breadth-first search starting at U_0.



(M1)(A2)

Note:	Award ***(M1)*** for demonstration of breadth-first technique. Award ***(A2)*** for all correct, ***(A1)*** for 1 error, ***(A0)*** otherwise.

(b) For depth-first search starting at U_0.



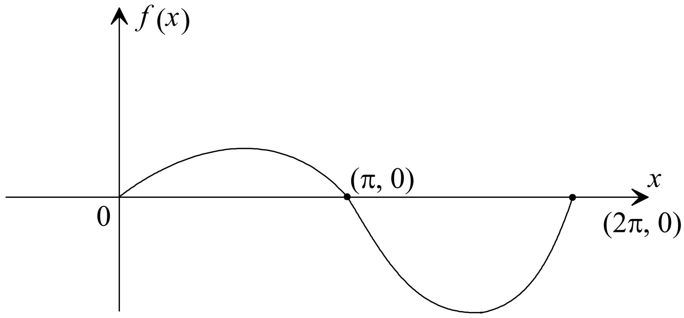
(M1)(A2)

Note:	Award ***(M1)*** for depth-first pattern. Award ***(A2)*** for all correct, ***(A1)*** for 1 error, ***(A0)*** otherwise.

[6 marks]

Total [30 marks]

9. (i) (a) (i)



(A1)(A1)

Note: Award *(A1)* for shape, *(A1)* for correct x-intercepts.

(ii) Smallest positive root = π. *(A1)*

(iii) Newton-Raphson $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f(x) = x\sin x \Rightarrow f'(x) = \sin x + x\cos x$ *(A1)*

$\Rightarrow x_{n+1} = x_n - \dfrac{x_n \sin x_n}{\sin x_n + x_n \cos x_n}$ *(M1)*

$= \dfrac{x_n(\sin x_n + x_n \cos x_n) - x_n \sin x_n}{\sin x_n + x_n \cos x_n}$ *(A1)*

$= \dfrac{x_n^2 \cos x_n}{\sin x_n + x_n \cos x_n}$ *(AG)*

[6 marks]

(b) Area $= \int_0^{\pi} x\sin x \, dx = 3.141593$ (6 d.p.) *(G2)*

OR

$\int_0^{\pi} x\sin x \, dx = [-x\cos x + \sin x]_0^{\pi}$ *(M1)*

$= \pi$
$= 3.141593$ (6 d.p.) *(A1)*

[2 marks]

(c) Max error $= \dfrac{(b-a)^3}{12n^2} f''(c)$ where $f''(c)$ is max of $|f''|$ on $[a, b]$

$f''(x) = \cos x + \cos x + x(-\sin x) = 2\cos x - x\sin x$ *(M1)(A1)*

from sketch of $f''(x)$ use GDC max$|f''(c)| = 3.103$ *(M1)(A1)*

$\Rightarrow |\text{error}| \le \dfrac{(\pi - 0)^3}{12(10)^2}(3.103)$ *(M1)*

$= 0.0802$ (3 s.f.) *(A1)*

[6 marks]

continued…

Question 9 continued

(ii) (a) The integral test for $\sum a_n$.

Let $a_n = f(n)$ where $f(x)$ is a continuous and positive decreasing function for all $x \geq$ some positive integer, N. ***(R1)(R1)***

Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\mathrm{d}x$

both diverge or both converge. ***(R1)***

(*i.e.* if the integral is finite then $\sum a_n$ is finite, if the integral is infinite then $\sum a_n$ is infinite)

[3 marks]

(b) Let $f(x) = \frac{x}{\mathrm{e}^{x^2}}$ which satisfies the above conditions since from GDC we get for $x > 1$, $f(x) > 0$ and $f'(x) < 0$. ***(R1)***

Now $\int_1^{\infty} \frac{x}{\mathrm{e}^{x^2}}\mathrm{d}x = \lim_{t\to\infty}\int_1^t x\mathrm{e}^{-x^2}\mathrm{d}x$ ***(M1)***

$= \lim_{t\to\infty}\left[-\frac{1}{2}\mathrm{e}^{-x^2}\right]_1^t$ ***(A1)***

$= \lim_{t\to\infty}\left\{\left[-\frac{1}{2}\mathrm{e}^{-t^2}\right] - \left[-\frac{1}{2}\mathrm{e}^{-1}\right]\right\} = \frac{1}{2\mathrm{e}}$ ***(A1)***

Since the integral converges so the series $\sum_{n=1}^{\infty}\frac{n}{e^{n^2}}$ converges. ***(A1)***

[5 marks]

(iii) (a) (i) $f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \ldots$

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f'''(x) = -\cos x \qquad f'''(0) = -1$

$f^{iv}(x) = \sin x \qquad f^{iv}(0) = 0$

$f^{v}(x) = \cos x \qquad f^{v}(0) = 1$

$f^{vi}(x) = -\sin x \qquad f^{vi}(0) = 0$

$f^{vii}(x) = -\cos x \qquad f^{vii}(0) = -1$ ***(M1)***

Hence $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$ ***(A1)***

Note: Award ***(A2)*** if series is quoted as a standard result.

continued…

Question 9 (iii) (a) continued

(ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ (a standard result)

$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!}$ ***(M1)***

$= 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \dots$ ***(A1)***

[4 marks]

(b) $e^{x^2} \sin x = \left(1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!}\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)$ ***(M1)***

$= x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^3 - \frac{x^5}{3!} + \frac{x^5}{2!}$ (neglecting terms after x^5)

$= x + x^3 - \frac{x^3}{6} + \frac{x^5}{2} - \frac{x^5}{6} + \frac{x^5}{120}$

$= x + \frac{5}{6}x^3 + \frac{41}{120}x^5$ ***(A1)***

[2 marks]

(c) Using above result (part (iii) (b)) we get

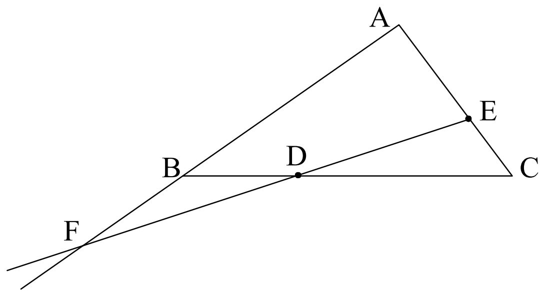
$\frac{e^{x^2} \sin x - x}{x^3} = \frac{5}{6} + \frac{41}{120}x^2$ ***(M1)***

Hence $\lim_{x \to 0}\left(\frac{e^{x^2} \sin x - x}{x^3}\right) = \frac{5}{6}$ ***(A1)***

[2 marks]

Total [30 marks]

10. (i)



By Menelaus' Theorem on ΔABC

$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=-1$ ***(M1)***

$\Rightarrow\frac{AF}{FB}\times\frac{1}{2}\times\frac{1}{2}=-1$ ***(A1)***

$\Rightarrow\frac{AF}{FB}=-4$ ***(A1)***

$\Rightarrow AF=-4FB$ and B divides AF in ratio 3 : 1 ***(R1)***

$\Rightarrow AB=3BF$ ***(AG)***

OR

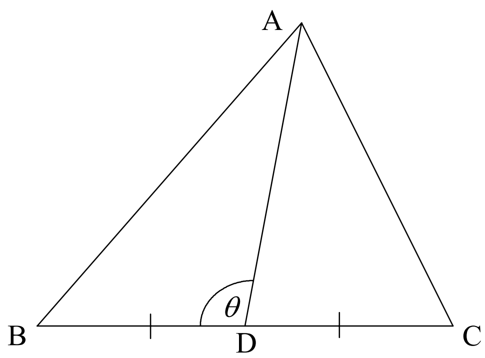
$AF=-4FB$

$\Rightarrow AB-FB=-4FB$ (Since $AB=AF+FB\Rightarrow AF=AB-FB$) ***(R1)***

$\Rightarrow AB=-3FB\Rightarrow AB=3BF$ ***(AG)***

[4 marks]

(ii) (a)



Prove $AB^2+AC^2=2(AD^2+BD^2)$

Using cosine rule we obtain

$AB^2=AD^2+BD^2-2AD\,BD\cos\theta$ ① ***(M1)(A1)***

and $AC^2=AD^2+DC^2-2AD\,DC\cos(180-\theta)$

$\Rightarrow AC^2=AD^2+DC^2+2AD\,DC\cos\theta$ ② ***(M1)(A1)***

Since BD = DC add ① and ② ***(R1)***

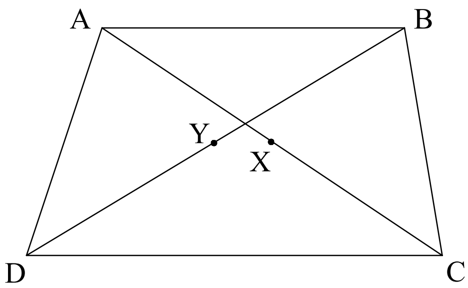
$\Rightarrow AB^2+AC^2=2(AD^2+BD^2)$ ***(AG)***

[5 marks]

continued…

Question 10 (ii) continued

(b)



In ΔABC $AB^2 + BC^2 = 2(BX^2 + AX^2)$ from part (a) ***(M1)***

In ΔADC $AD^2 + DC^2 = 2(DX^2 + AX^2)$ from part (a) ***(M1)***

Addition $\Rightarrow AB^2 + BC^2 + CD^2 + DA^2 = 4AX^2 + 2BX^2 + 2DX^2$ ***(M1)***

Now in ΔBXD, $BX^2 + DX^2 = 2(XY^2 + BY^2)$ ***(M1)***

$\Rightarrow AB^2 + BC^2 + CD^2 + DA^2 = 4AX^2 + 4XY^2 + 4BY^2$ ***(A1)***

Also $2AX = AC$ and $2BY = BD$ ***(M1)***

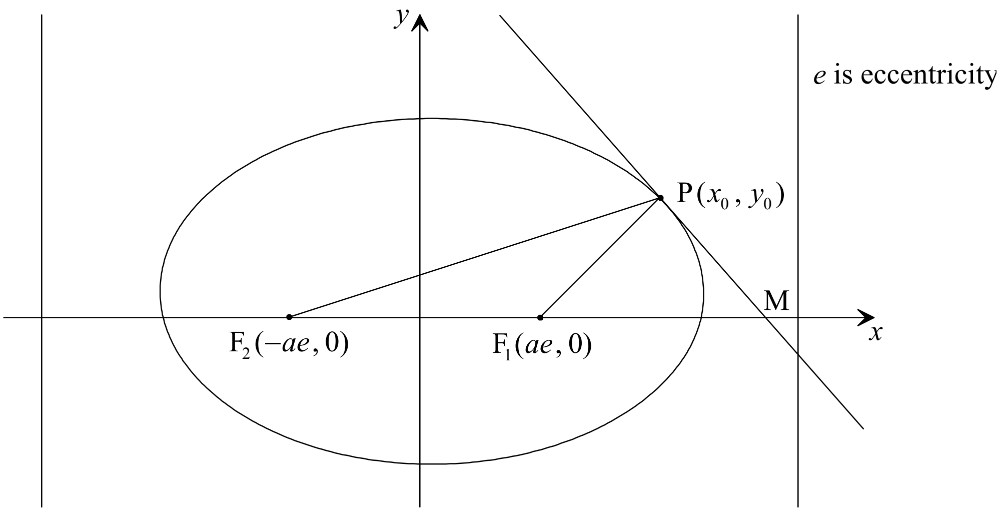
Hence $AB^2 + BC^2 + CD^2 + DA^2 = AC^2 + BD^2 + 4XY^2$ ***(AG)***

[6 marks]

continued…

Question 10 continued

(iii) $x = -\dfrac{a}{e}$ $x = \dfrac{a}{e}$



(a) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

Differentiate with respect to $x \Rightarrow \dfrac{2x}{a^2} + \dfrac{2yy^1}{b^2} = 0$ ***(M1)***

$\Rightarrow y^1 = \dfrac{-b^2 x}{a^2 y}$ ***(A1)***

$\Rightarrow$ at $P(x_0, y_0)$ Slope $= \dfrac{-b^2 x_0}{a^2 y_0} \Rightarrow$ equation of tangent is $y - y_0 = \dfrac{-b^2 x_0}{a^2 y_0}(x - x_0)$ ***(M1)(A1)***

$\Rightarrow a^2 yy_0 - a^2 y_0^2 = -b^2 x_0 x + b^2 x_0^2$

$\Rightarrow xx_0 b^2 + yy_0^2 = a^2 y_0^2 + b^2 x_0^2$

$\Rightarrow \dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = \dfrac{x_0^2}{a^2} + \dfrac{y_0^2}{b^2}$ ***(A1)***

$\Rightarrow \dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$ ***(AG)***

[5 marks]

(b) At M $y = 0 \Rightarrow x = \dfrac{a^2}{x_0}$ ***(M1)***

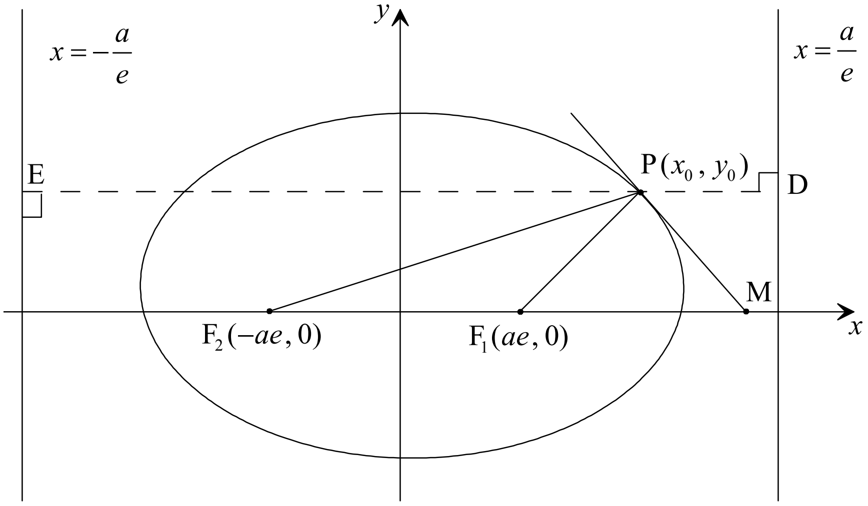
$\Rightarrow$ Coordinate of $M = \left(\dfrac{a^2}{x_0}, 0\right)$ ***(A1)***

[2 marks]

continued…

Question 10 (iii) continued

(c)



$$PF_1 = e(PD) = e\left(\frac{a}{e} - x_0\right) = a - ex_0$$ ***(A1)***

$$PF_2 = e(PE) = e\left(\frac{a}{e} + x_0\right) = a + ex_0$$ ***(A1)***

$$MF_1 = \frac{a^2}{x_0} - ae = \frac{a}{x_0}(a - ex_0)$$ ***(A1)***

$$MF_2 = \frac{a^2}{x_0} + ae = \frac{a}{x_0}(a + ex_0)$$ ***(A1)***

Note: Award ***(A1)*** marks for either form.

[4 marks]

(d) Using the converse of the angle bisector theorem we need to show $\frac{F_2M}{F_1M} = \frac{F_2P}{F_1P}$. ***(M1)***

Now from part (c) $\frac{PF_2}{PF_1} = \frac{a + ex_0}{a - ex_0}$ ***(A1)***

and $$\frac{MF_2}{MF_1} = \frac{\frac{a}{x_0}(a + ex_0)}{\frac{a}{x_0}(a - ex_0)} = \frac{a + ex_0}{a - ex_0}$$ ***(A1)***

$$\Rightarrow \frac{F_2M}{F_1M} = \frac{F_2P}{F_1P}$$ ***(A1)***

So PM is the external bisector of $F_2\hat{P}F_1$. ***(AG)***

[4 marks]

Total [30 marks]