

MARKSCHEME

May 2003

MATHEMATICS

Higher Level

Paper 2

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Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

- (a) All marking must be done using a **red** pen.
- (b) Marks should be noted on candidates' scripts as in the markscheme:
 - show the breakdown of individual marks using the abbreviations (***M1***), (***A2***) etc.
 - write down each part mark total, indicated on the markscheme (for example, [***3 marks***]) – it is suggested that this be written at the end of each part, and underlined;
 - write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Errors made at any step of a solution can affect all working that follows. To limit the severity of the penalty, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

- (i) penalise an error when it **first occurs**;
- (ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent working;
- (iii) award **M** marks for a correct method, and **A(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
$\$ 600 \times 1.02$	<i>MI</i>	Amount earned = $\$ 600 \times 1.02$	✓	<i>MI</i>
= $\$ 612$	<i>AI</i>	= $\$602$	×	<i>A0</i>
$\$ (306 \times 1.02) + (306 \times 1.04)$	<i>MI</i>	Amount = $301 \times 1.02 + 301 \times 1.04$	✓	<i>MI</i>
= $\$ 630.36$	<i>AI</i>	= $\$ 620.06$	✓	<i>AI(ft)</i>

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

- (i) **fewer** marks should be awarded at the discretion of the Examiner;
- (ii) marks awarded should be followed by “(d)” (to indicate that these marks have been awarded at the **discretion** of the Examiner);
- (iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

- (a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

- (i) a mark should be awarded followed by “(d)” (to indicate that these marks have been awarded at the **discretion** of the Examiner);
- (ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc.* Other alternative solutions, including graphic display calculator alternative solutions are indicated by **OR**. For example:

$$\begin{aligned} \text{Mean} &= 7906/134 && \text{(M1)} \\ &= 59 && \text{(A1)} \end{aligned}$$

OR

$$\text{Mean} = 59 \qquad \qquad \qquad \text{(G2)}$$

- (b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin \theta}{\cos \theta}$ for $\tan \theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect ie, once the correct answer is seen, ignore further working.

- (c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7, 1·7, 1,7; different forms of vector notation such as \vec{u} , \bar{u} , \underline{u} ; $\tan^{-1} x$ for $\arctan x$.

5 Accuracy of Answers

There are two types of accuracy errors, incorrect level of accuracy, and rounding errors.

Unless the level of accuracy is specified in the question, candidates should be penalized **once only IN THE PAPER** for any accuracy error (**AP**). This could be an incorrect level of accuracy (**only applies to fewer than three significant figures**), or a rounding error. Hence, on the **first** occasion in the paper when a correct answer is given to the wrong degree of accuracy, or rounded incorrectly, maximum marks are **not** awarded, but on **all subsequent occasions** when accuracy errors occur, then maximum marks **are** awarded.

(a) Level of accuracy

- (i) In the case when the accuracy of the answer is **specified in the question** (for example: “find the size of angle A to the nearest degree”) the maximum mark is awarded **only if** the correct answer is given to the accuracy required.
- (ii) When the accuracy is **not** specified in the question, then the general rule applies:

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

However, if candidates give their answers to more than three significant figures, this is acceptable

(b) Rounding errors

Rounding errors should only be penalized at the **final answer** stage. This does **not** apply to intermediate answers, only those asked for as part of a question. Premature rounding which leads to incorrect answers should only be penalized at the answer stage.

Incorrect answers are wrong, and should not be considered under (a) or (b).

Examples

A question leads to the answer 4.6789....

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : 4.7 should be penalised the first time this type of error occurs, but 4.679 is **not** penalized, as it has more than three significant figures.
- 4.67 is incorrectly rounded – penalise on the first occurrence.
- 4.678 is incorrectly rounded, but has more than the required accuracy, do **not** penalize.

Note: All these “incorrect” answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Calculator penalties

Candidates are instructed to write the make and model of their calculator on the front cover. Please apply the following penalties where appropriate.

(i) Illegal calculators

If candidates note that they are using an illegal calculator, please report this on a PRF, and deduct 10 % of their overall mark. Note this on the front cover.

(ii) Calculator box not filled in.

Please apply a calculator penalty (**CP**) of 1 mark if this information is not provided. Note this on the front cover.

1. (a) (i) $f'(x) = \frac{2x \cdot 2^x - x^2 2^x \ln 2}{2^{2x}}$ (MI)(A1)
 $= \frac{2x - x^2 \ln 2}{2^x}$ (AG)

(ii) $f''(x) = \frac{2^x [2 - 2x \ln 2] - 2^x \ln 2 [2x - x^2 \ln 2]}{2^{2x}}$ (MI)(A1)
 $= \frac{x^2 (\ln 2)^2 - 4x \ln 2 + 2}{2^x}$ (A1)

Note: Award the second (A1) for some form of simplification,
 e.g. accept $\frac{x \ln 2 (x \ln 2 - 4) + 2}{2^x}$.

[5 marks]

(b) (i) $2x - x^2 \ln 2 = 0$ giving $x = \frac{2}{\ln 2}$ (MI)(A1)

Note: Award (MI)(A0) for $x = 2.89$.

(ii) With this value of x ,
 $f''(x) = \frac{4 - 8 + 2}{\text{+ve number}} < 0$ (MI)(A1)
 Therefore, a maximum. (AG)

[4 marks]

(c) Points of inflexion satisfy $f''(0) = 0$, i.e.
 $x^2 (\ln 2)^2 - 4x \ln 2 + 2 = 0$ (MI)
 $\Rightarrow x = \frac{4 \ln 2 \pm \sqrt{8 (\ln 2)^2}}{2 (\ln 2)^2}$ (A1)
 $= \frac{2 \pm \sqrt{2}}{\ln 2} (= 0.845, 4.93)$ (A1)

OR

$x = 0.845, 4.93$ (MI)(G1)(G1)

[3 marks]

Total [12 marks]

2. (i) (a) $T_1 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}; T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; T_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ (A1)(A1)(A1)

[3 marks]

(b) (i) $T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (M1)(A1)

(ii) This is a reflection in the x-axis. (A1)(A1)

Note: Award (M1)(A0)(A1)ft(A1)ft for correctly multiplying the wrong way round – this gives $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ which is a reflection in the y-axis.

[4 marks]

(ii) (a) (i) Using row reduction,
 $x + 2y + z = k$ (M1)
 $-3y + 2z = 6 - 2k$ (A1)
 $6y - 4z = k - 9$ (A1)

$$\begin{aligned} x + 2y + z &= k \\ -3y + 2z &= 6 - 2k \\ 0z &= 3 - 3k \end{aligned}$$

Not a unique solution because the coefficient of z in the third equation is zero.

(ii) In order for the system to have a solution,
 $3 - 3k = 0,$ (R1)
 consistent for $k = 1.$ (M1)
(A1)

OR

(i) Consider
 $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & -4 & 5 \end{vmatrix} = 1 \times 21 - 2 \times 6 + 1 \times -9$ (M1)
 $= 0$ (A1)

The zero value confirms that the equations do not have a unique solution. (R1)

(ii) Consider
 $\begin{vmatrix} 1 & 2 & k \\ 2 & 1 & 4 \\ 1 & -4 & 5 \end{vmatrix} = 9 - 9k$ (M1)(A1)

Consistent when this determinant is zero, i.e. $k = 1.$ (R1)

[6 marks]

(b) The general solution is $z = \lambda, y = \frac{(2\lambda - 4)}{3}, x = \frac{(11 - 7\lambda)}{3}.$ (M1)(A1)(A1)

[3 marks]

Total [16 marks]

3. (a) The result is true for $n = 1$, since
 LHS = $\cos \theta + i \sin \theta$
 and RHS = $\cos \theta + i \sin \theta$ (R1)
 Let the proposition be true for $n = k$. (M1)
 Consider $(\cos \theta + i \sin \theta)^{k+1} = (\cos k\theta + i \sin k\theta)(\cos \theta + i \sin \theta)$ (M1)
 $= \cos k\theta \cos \theta - \sin k\theta \sin \theta + i(\sin k\theta \cos \theta + \cos k\theta \sin \theta)$
 $= \cos(k+1)\theta + i \sin(k+1)\theta$ (A1)
 Therefore, true for $n = k \Rightarrow$ true for $n = k + 1$ and the proposition is proved by
 induction. (R1)

[5 marks]

- (b) (i) $\frac{1}{z} = \frac{1}{(\cos \theta + i \sin \theta)} \times \frac{(\cos \theta - i \sin \theta)}{(\cos \theta - i \sin \theta)}$ (M1)
 $= (\cos \theta - i \sin \theta)$ (A1)
 $= \cos(-\theta) + i \sin(-\theta)$ (AG)

OR

$$\frac{1}{z} = z^{-1} \quad \text{(M1)}$$

$$z^{-1} = \cos(-\theta) + i \sin(-\theta), \text{ by de Moivre's theorem (accept the cis notation).} \quad \text{(A1)(AG)}$$

Note: Award (M0)(A0) to candidates who use the result of part (a) with no consideration that in this part, $n < 0$.

- (ii) $z^{-n} = (z^{-1})^n = \cos(-n\theta) + i \sin(-n\theta)$ (A1)
 $z^n + z^{-n} = \cos n\theta + i \sin n\theta + \cos(-n\theta) + i \sin(-n\theta)$ (M1)
 $= \cos n\theta + i \sin n\theta + \cos(n\theta) - i \sin(n\theta)$ (A1)
 $= 2 \cos n\theta$ (AG)

[5 marks]

- (c) (i) $(z + z^{-1})^5 = z^5 + 5z^3 + 10z + 10z^{-1} + 5z^{-3} + z^{-5}$ (M1)(A1)
 (ii) $(2 \cos \theta)^5 = 2 \cos 5\theta + 10 \cos 3\theta + 20 \cos \theta$ (M1)(A1)
 giving $a = 1, b = 5$ and $c = 10$ (or $\cos^5 \theta = \frac{1}{16}(\cos 5\theta + 5 \cos 3\theta + 10 \cos \theta)$). (A1)

[5 marks]

Total [15 marks]

4. (a) (i) $\frac{1}{12} \int_0^2 x(8x - x^3) dx = E(X)$ (M1)(A1)

Note: Award (M1) for $\frac{1}{12} \int x(8x - x^3) dx$, and (A1) for the correct limits.

(ii)
$$E(X) = \frac{1}{12} \left[\frac{8}{3}x^3 - \frac{x^5}{5} \right]_0^2$$

$$= \frac{56}{45} = 1.24$$
 (A1)

OR

$E(X) = 1.24$ (G1)

[3 marks]

(b) (i) The median satisfies $\frac{1}{12} \int_0^m (8x - x^3) dx = \frac{1}{2}$ (M1)

$$\left[4x^2 - \frac{x^4}{4} \right]_0^m = 6$$
 (A1)

$$\left(4m^2 - \frac{m^4}{4} \right) - 0 = 6$$
 (A1)

$$m^4 - 16m^2 + 24 = 0$$
 (AG)

(ii) $m^2 = \frac{16 \pm \sqrt{256 - 96}}{2}$ (M1)

$$m = \sqrt{8 - \sqrt{40}} (=1.29)$$
 (A1)

OR

$m = 1.29$ (G2)

Note: Award (M1)(A0) or (G1) if other values in addition to 1.29 are given.

[5 marks]

(c) For the mode, we require the value of x which maximises $f(x)$. (M1)

$$f'(x) = 0 \Rightarrow 8 - 3x^2 = 0 \Rightarrow x = \sqrt{\frac{8}{3}} = 1.63$$
 (M1)(A1)

OR

$x = 1.63$ (G2)

[3 marks]

Total [11 marks]

5. (a) $\cos x + \sqrt{3} \sin x = R \cos \alpha \cos x + R \sin \alpha \sin x$ (M1)
 $\Rightarrow R \cos \alpha = 1, R \sin \alpha = \sqrt{3}$
 $\Rightarrow R = 2, \alpha = \frac{\pi}{3}$ (A1)(A1)

Note: Award (M1)(A1)(A0) if degrees used instead of radians.

[3 marks]

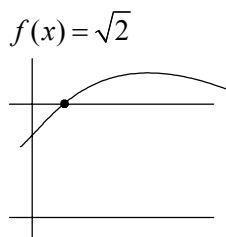
- (b) (i) Since $f(x) = 2 \cos\left(x - \frac{\pi}{3}\right)$,
 $f_{\max} = 2$ (when $x = \frac{\pi}{3}$); $f_{\min} = 1$ (when $x = 0$) (A1)(A1)
 Range is $[1, 2]$ (A1)
- (ii) Inverse does not exist because f is not 1:1. (R2)

Note: Award (R2) for a correct answer with a valid reason.
 Award (R1) for a correct answer with an attempt at a valid reason,
 e.g. horizontal line test.
 Award (R0) for just saying inverse does not exist, without any reason.

[5 marks]

- (c) $f(x) = \sqrt{2} \Rightarrow \cos\left(x - \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$ (M1)
 $x - \frac{\pi}{3} = \pm \frac{\pi}{4}$ (A1)
 $x = \frac{\pi}{12}$ (A1)

OR



$x = 0.262$

$\Rightarrow x = \frac{\pi}{12}$

(M1)

(G1)

(A1)

[3 marks]

Question 5 continued

$$(d) \quad I = \frac{1}{2} \int_0^{\frac{\pi}{2}} \sec\left(x - \frac{\pi}{3}\right) dx \quad (M1)$$

$$= \frac{1}{2} \left[\ln\left(\sec\left(x - \frac{\pi}{3}\right) + \tan\left(x - \frac{\pi}{3}\right)\right) \right]_0^{\frac{\pi}{2}} \quad (A1)$$

$$= \frac{1}{2} \ln\left(\frac{\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{2 - \sqrt{3}}\right) \quad (A1)(A1)$$

$$= \frac{1}{2} \ln\left(\frac{\sqrt{3}(2 + \sqrt{3})}{(2 - \sqrt{3})(2 + \sqrt{3})}\right) = \frac{1}{2} \ln(3 + 2\sqrt{3}). \quad (M1)(AG)$$

Note: Award zero marks for any work using GDC.

[5 marks]

Total [16 marks]

6. **Note:** Do not penalize candidates who give answers to more than the required accuracy.

(i) (a) (i)
$$\bar{v} = \frac{1}{1000}(45 \times 9 + 55 \times 35 + 65 \times 93 + 75 \times 139 + 85 \times 261 + 95 \times 295 + 105 \times 131 + 115 \times 26 + 125 \times 11)$$

$$= 87.13 \text{ (2 d.p.)}$$
 (M1)
(A1)

Note: Award (M1)(A0) for candidates dividing by 999 to get = 87.22.

OR

$$\bar{v} = 87.13$$
 (G2)

(ii)
$$s^2 = \frac{1}{999}[(87.13 - 45)^2 \times 9 + (87.13 - 55)^2 \times 35 + (87.13 - 65)^2 \times 93 + (87.13 - 75)^2 \times 139 + (87.13 - 85)^2 \times 261 + (87.13 - 95)^2 \times 295 + (87.13 - 105)^2 \times 131 + (87.13 - 115)^2 \times 26 + (87.13 - 125)^2 \times 11]$$

$$= 215.58 \text{ (2 d.p.)}$$
 (M1)
(A1)

OR

$$s^2 = 215.58$$
 (G2)

Note: Award (M1)(A0) for finding the standard deviation 14.68 instead of the variance.

[4 marks]

(b) **Note:** Do not penalize candidates who use z-values rather than t-values.

(i) The confidence interval is given by

$$\bar{v} - \frac{t_{0.025, 999} \times s}{\sqrt{n}}, \bar{v} + \frac{t_{0.025, 999} \times s}{\sqrt{n}} \left(\text{or } \bar{v} \pm \frac{z_{0.025} \times s}{\sqrt{n}} \right).$$
 (M1)
 Hence the confidence interval is [86.22, 88.04]. (A1)

OR

Confidence interval is [86.22, 88.04]. (G2)

(ii) The confidence interval is given by

$$\bar{v} - t_{0.05, 999} \times \frac{s}{\sqrt{n}}, \bar{v} + t_{0.05, 999} \times \frac{s}{\sqrt{n}} \left(\text{or } \bar{v} \pm \frac{z_{0.05} \times s}{\sqrt{n}} \right).$$
 (M1)
 Hence the confidence interval is [86.37, 87.89]. (A1)

OR

Confidence interval is [86.37, 87.89]. (G2)

[4 marks]

(c) Greater confidence implies less accuracy. (R1)
 Hence the confidence interval at the 95 % level contains the confidence level at the 90 % level since they are symmetric about the same point, (\bar{v}). (R1)

Note: Award (R0)(R1) for a convincing diagram, showing some understanding of the question.

[2 marks]

continued...

Question 6 continued

(ii) $B \sim P(2.7), S \sim P(2.5)$

(a) (i) $P(B = 2) = \frac{e^{-2.7}(2.7)^2}{2} = 0.245$ (M1)(A1)

(ii) $P(S = 3) = \frac{e^{-2.5}(2.5)^3}{6} = 0.214$ (M1)(A1)

(iii) The two events are independent.

$$P((B = 2) \cap (S = 3)) = P(B = 2) \times P(S = 3) \quad (M1)$$

$$= 0.214 \times 0.245$$

$$= 0.0524 \quad (A1)$$

[6 marks]

(b) $P(B + S = 5) = \frac{e^{-5.2}(5.2)^5}{120} \approx 0.175$ (A1)

$$P((B = 2) \cap (S = 3)) = \frac{e^{-2.7}(2.7)^2}{2} \times \frac{e^{-2.5}(2.5)^3}{6} \approx 0.245 \times 0.214 = 0.0524$$

$$P((B = 1) \cap (S = 4)) = \frac{e^{-2.7}(2.7)^1}{1} \times \frac{e^{-2.5}(2.5)^4}{24} \approx 0.181 \times 0.133 = 0.0242 \quad (A1)$$

$$P((B = 0) \cap (S = 5)) = \frac{e^{-2.7}(2.7)^0}{1} \times \frac{e^{-2.5}(2.5)^5}{120} \approx 0.067 \times 0.067 = 0.0045 \quad (A1)$$

$$P(B < S) = \frac{0.0524 + 0.0242 + 0.0045}{0.175} = \frac{0.0811}{0.175} \quad (M1)$$

$$= 0.464 \text{ (or } 0.463) \quad (A1)$$

[5 marks]

(iii) (a) H_0 : The sequence contains equal numbers of each digit. (A1)

H_1 : The sequence does not contain equal numbers of each digit. (A1)

$$\chi^2_{\text{calc}} = \frac{(9+1+25+1+25+49+1+9+4+16)}{20} = 7 \quad (M1)(A1)$$

The number of degrees of freedom is 9. (A1)

$$\chi^2_{0.95,9} = 16.919 \quad (A1)$$

$\chi^2_{\text{calc}} < 16.919$. Hence H_0 is accepted. (A1)

[7 marks]

(b) The probability of rejecting H_0 when it is true (A1)

is 0.05. (A1)

Note: Award (A1)(A1) for “the probability of a type I error is 0.05.”

[2 marks]

Total [30 marks]

7. (i) (a) f is injective since $f(x) = f(y) \Leftrightarrow 3^x = 3^y \Leftrightarrow x = y$ (M1)(R1)
 f is surjective since if $z \in \mathbb{R}^+$, $x = \log_3(z) \in \mathbb{R}$ and $z = f(x)$ (M1)(R1)

For every x, y in $(\mathbb{R}, +)$,

$$f(x + y) = 3^{(x+y)} = 3^x 3^y = f(x) \times f(y) \quad (M1)(A1)$$

[6 marks]

- (b) $f^{-1}(z) = \log_3(z)$ (A1)
 [1 mark]

- (ii) (a) MN is a matrix (A1)
 $|MN| = |M||N| \Rightarrow |MN| \neq 0$ and $MN \in G$ (A1)
 Matrix multiplication is associative. (A1)
 Identity is $I, |I| = 1 \neq 0$ $I \in G$ (A1)
 $M \in G \Rightarrow M^{-1}$ exists, and $M^{-1} \in G$ (as $|M^{-1}| \neq 0$) (A1)(A1)
 thus $(G, *)$ is a group. (AG)
 [6 marks]

- (b) G is not Abelian. (A1)
 Counter example to show that $AB \neq BA$. (A1)(A1)

e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
 $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Note: Do not award (A2) to candidates who give the reason, “matrix multiplication is not commutative”.
 Award (A1)(A0) to candidates who use an argument with general matrices.

[3 marks]

- (c) For all $M \in G, MR_H M$ since H contains the identity element I .
 Hence R is reflexive. (A1)

If M and $N \in G$ and $MR_H N$ then there exists $L \in H$ such that $M = L * N$ and therefore $N = L^{-1} * M$. (M1)
 Since $L^{-1} \in H$ (A1)

it follows that $NR_H M$ and therefore that R_H is symmetric.

If M, N and $P \in G$ and $MR_H N$ and $NR_H P$ then there exist $U, V \in H$ such that $M = U * N$ and $N = V * P$.

Hence $M = U * V * P$. (M1)
 Since $U * V \in H$, (A1)

it follows that $MR_H P$ so that R is transitive.

Therefore R_H is an equivalence relation. (AG)

[5 marks]

continued...

Question 7(ii) continued

(d) $M, N \in K \Rightarrow |M| > 0, |N| > 0 \Rightarrow |MN| > 0$ (M1)
 $\Rightarrow K$ is closed under $*$ (A1)

Identity: $|I| = 1 \Rightarrow I \in K$ (A1)

Inverse: $M \in K \Rightarrow |M| > 0$

$|M^{-1}| = \frac{1}{|M|} \Rightarrow |M^{-1}| > 0$ (M1)

i.e. $M^{-1} \in K$ (A1)

so $(K, *)$ is a subgroup of $(G, *)$. (AG)

Note: Other correct methods do exist.

[5 marks]

(e) (i) Let $M, N \in G$. Then $MR_K N$ if and only if the determinants of M and of N have the same sign. (A1)(A1)

Therefore there can be only two equivalent classes: one containing all the elements of G with positive determinants and the other containing all the elements of G with negative determinants. (A1)(AG)

(ii) Given an element M of G , it belongs to one or other equivalence class depending on the sign of the determinant. (R1)

[4 marks]

Total [30 marks]

8. (i) (a) $31 = 1 \times 17 + 14, 17 = 1 \times 14 + 3, 14 = 4 \times 3 + 2, 3 = 1 \times 2 + 1$ (M1)
 Hence: $1 = 3 - 1 \times 2 = 3 - 1 \times (14 - 4 \times 3) = 5 \times 3 - 14$
 $= 5 \times (17 - 1 \times 14) - 14 = 5 \times 17 - 6 \times 14$
 $= 5 \times 17 - 6 \times (31 - 1 \times 17) = 11 \times 17 - 6 \times 31$
 Hence $x = 11$ and $y = -6$. (A1)(A1)

[3 marks]

- (b) From the hypothesis we see that $17 \times (p - 11) + 31 \times (q + 6) = 0$.
 Therefore, since 31 is prime, it divides $p - 11$ and this is not possible for $|p| < 11$. (R1)(AG)
 Similarly, since 17 is prime, it divides $q + 6$ and this is not possible for $|q| < 6$. (R1)(AG)

[2 marks]

- (ii) The fixed point λ of the relation is given by:

$$\lambda = 2\lambda + 3 \text{ so that } \lambda = -3 \quad \text{(M1)(A1)}$$

Setting $x_n = y_n + 3$ the relation becomes:

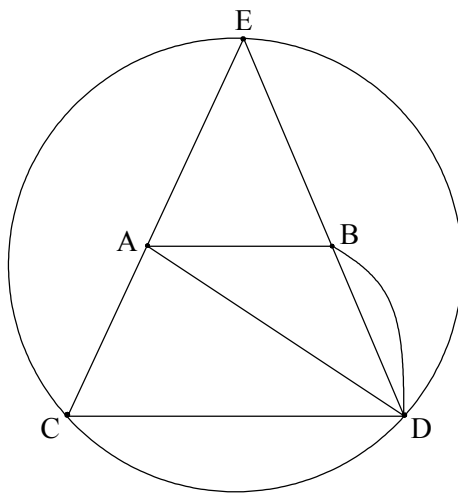
$$x_0 = 2; x_{n+1} = 2x_n \text{ for } n = 0, 1, 2, \dots \quad \text{(M1)(A1)}$$

Hence $x_n = 2 \times 2^n = 2^{n+1}$ and $y_n = 2^{n+1} - 3$ (M1)(A1)

[6 marks]

- (iii) **Note:** There are many correct diagrams in all parts of (iii).

- (a)



(A5)(A2)

Notes: Award (A1) for each correct degree of a vertex.
 Award (A2) for a correct graph, with correct planarity.

[7 marks]

Question 8(iii) continued

- (b) The order of every vertex is equal to the sum of the numbers in the corresponding row (or column) of the adjacency table.
Therefore the order of every vertex is even so that the graph must have an Eulerian circuit.

(M1)(A1)

(A1)

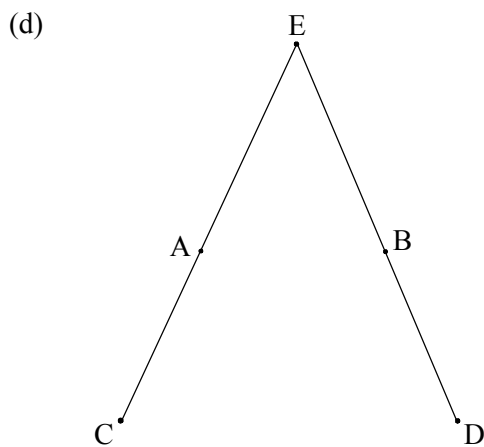
[3 marks]

- (c) $A \rightarrow E \rightarrow B \rightarrow A \rightarrow C \rightarrow D \rightarrow B \rightarrow D \rightarrow C \rightarrow E \rightarrow D \rightarrow A$

(M1)(A2)

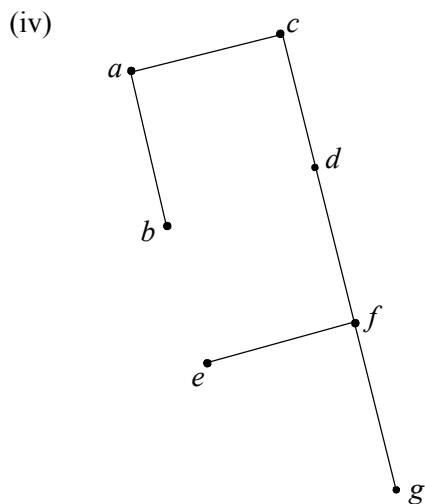
Note: Award (M1) for showing knowledge of what a Eulerian circuit is, (A2) for all edges, (A1) if only one edge is missed.

[3 marks]



(A2)

[2 marks]



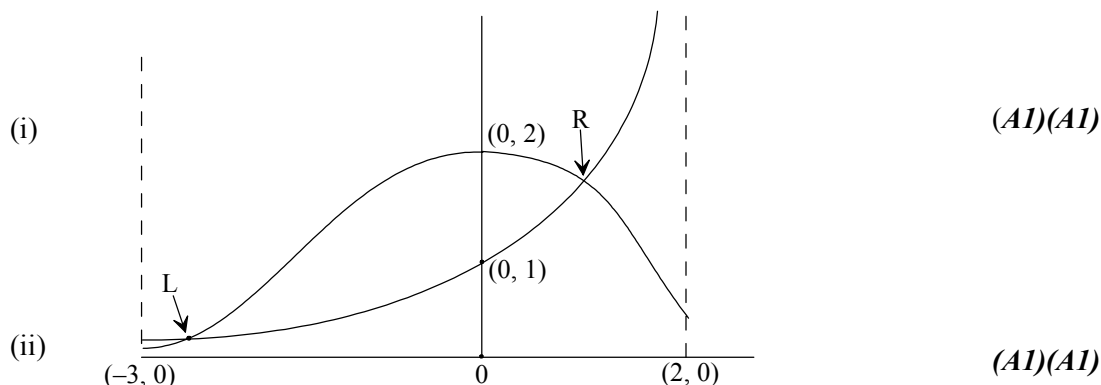
(A4)

Notes: Award (A4) for a completely correct answer.
Award (A3) if there is 1 error, (A2) if there are 2 errors,
(A1) if there are 3 errors, (A0) if there are more than 3 errors.

[4 marks]

Total [30 marks]

9. (i) (a)



Notes: The graphs and points of intersection do not need to be perfect.
 (i) Award **(AI)** for each correct graph.
 (ii) Award **(AI)** for each correct point of intersection.
 (L is approx. (-2.8, 0.06) and R is approx. (0.6, 1.8)).

[4 marks]

Note: For parts (b), (c) and (d), if candidates choose a different starting point (x_0) , which leads to the correct answer, award marks as indicated. There is no need to check the calculations/iterations.

(b) The Newton-Raphson method relies on defining a sequence $\{x_n\}$ such that

$$x_{n+1} = -\frac{e^{x_n} - \cos x_n - 1}{e^{x_n} + \sin x_n} + x_n; n = 0, 1, \dots \quad (M1)$$

Negative solution:

Choose x_0 in such a way that for every n , $e^{x_n} + \sin x_n \neq 0$, e.g. $x_0 = -2$.

- $x_0 = -2$ (M1)
- $x_1 = -2.579508809$ (G1)
- $x_2 = -2.750227876$
- $x_3 = -2.787065714$
- $x_4 = -2.789123109$
- $x_5 = -2.789129646$
- $x_6 = -2.789129646$ (A1)
- $\Rightarrow x = -2.78913$ (5 d.p.) (A1)

Positive solution:

Choose x_0 in such a way that for every n , $e^{x_n} + \sin x_n \neq 0$, e.g. $x_0 = 0.5$.

- $x_0 = 0.5$ (M1)
- $x_1 = 0.6075401801$ (G1)
- $x_2 = 0.60136792$
- $x_3 = 0.601346768$
- $x_4 = 0.6013467677$
- $x_5 = 0.6013467677$ (A1)
- $\Rightarrow x = 0.60135$ (5 d.p.) (A1)

[9 marks]

Question 9(i) continued

- (c) (i) The fixed point method relies on finding a starting point x_0 for the sequence $\{x_n\}$ where $x_{n+1} = h(x_n)$, $n = 0, 1, 2$, e.g. for the negative solution, chose $x_0 = -2$. (M1)

$$\begin{aligned}
 x_0 &= -2 \\
 x_1 &= -2.244851788 && \text{(G1)} \\
 x_2 &= -2.592811137 \\
 x_3 &= -2.664840917 \\
 x_4 &= -2.70674027 \\
 x_5 &= -2.733053978
 \end{aligned}$$

.....
 converges to -2.789129646 (seen after approximately 30 iterations)
 $x = -2.78913$ (5 d.p.) (A1)

- (ii) When $x = -2.789129$, $|h(x)| < 1$ (A1)(R1)
(i.e. fixed-point method works due to the fact that around the value $x = -2.789129$, $|h(x)| = |e^x + \sin x + 1| < 1$).

[5 marks]

- (d) (i) For the positive solution, chose $x_0 = 0.5$. Then
 $x_0 = 0.5$
 $x_1 = 0.271138708649$ (G1)
 $x_2 = -0.3808708649$
 $x_3 = -1.625946009$
 $x_4 = -2.374098796$

.....
 converges to -2.789129646 (seen after approximately 63 iterations) (M1)
 The sequence does not converge to the positive solution. (A1)

Note: Award (M1)(A1) for an appropriate example of this behaviour *i.e.* such that x_0 is close enough to the solution and therefore **should** – if the method worked – generate a sequence converging to the solution – but in fact, **does not**.

- (ii) When $x = 0.601$, $|h(x)| > 1$, *i.e.* fixed-point method cannot work (or equivalent). (M1)
(R1)

[5 marks]

Question 9 continued

$$(ii) \quad (a) \quad S_{2n} = \sum_{k=1}^{2n} u_k = \sum_{k=1}^n (u_{2k-1} + u_{2k}) \quad (A1)$$

$$= \sum_{k=1}^n \frac{3}{2k+1} - \frac{1}{2k} \quad (A1)$$

$$= \sum_{k=1}^n \frac{4k-1}{2k(2k+1)} \quad (A1)$$

[3 marks]

(b) (By limit comparison test)

$$\lim_{k \rightarrow \infty} \left(\frac{4k-1}{2k(2k+1)} : \frac{1}{k} \right) = 1 \quad (M1)(A1)$$

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ is divergent.} \quad (A1)$$

Therefore $S_{2n} \rightarrow \infty$ as $n \rightarrow \infty$, hence the given series is divergent. (A1)

Notes: The series is alternating and its general term converges to 0 but is not decreasing in absolute value so that no conclusion is possible using this criterion.
Award **(R1)** to candidates who observed this but did not otherwise get full marks on this question.

[4 marks]

Total [30 marks]

10. (i) (a) $x^2 + 4y^2 - 4x - 8y - 24 = 0$ can be written as $(x - 2)^2 + 4(y - 1)^2 = 32$ (M1)
 and also as $\frac{(x - 2)^2}{(\sqrt{32})^2} + \frac{(y - 1)^2}{(\sqrt{8})^2} = 1$. (A1)
 Hence the conic is an ellipse. (A1)

[3 marks]

- (b) The gradient of the given line is $\frac{1}{2}$. (A1)

Hence the equation of the tangents to the ellipse can be written as $y = \frac{x}{2} + c$ (M1)

$\Rightarrow 2x^2 + (4c - 8)x + (4c^2 - 8c - 24) = 0$. (M1)

The values of c corresponding to the desired tangent to the conic are those for which the quadratic equation in x has two equal roots. In other words, the values of c for which the discriminant $(4c - 8)^2 - 8(4c^2 - 8c - 24) = 0$. (M1)

This gives $c = \pm 4$. (A1)

Hence the desired tangent lines are defined by the two equations:

$y = \frac{x}{2} + 4$ (A1)

and $y = \frac{x}{2} - 4$. (A1)

Note: There are other possible correct methods.

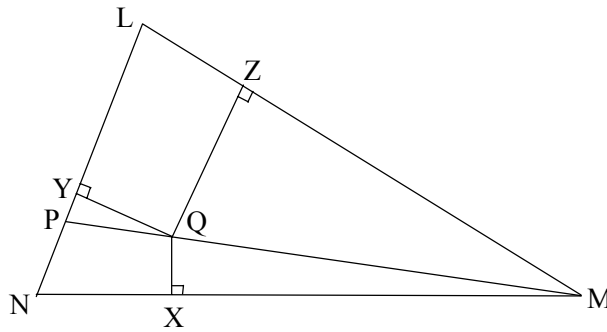
[8 marks]

- (ii) Consider $\triangle BAE$ and $\triangle CAD$.
 $\hat{BAE} = \hat{CAD}$
 $\hat{ABE} = \hat{ACD}$ (they both span AD) (A1)
 Hence $\triangle BAE$ and $\triangle CAD$ are similar. (A1)
 Therefore $\frac{BA}{CA} = \frac{BE}{CD}$ (A1)
 And hence $BA \times CD = CA \times BE$ ① (A1)
 Consider $\triangle ABC$ and $\triangle ADE$.
 $\hat{BAC} = \hat{EAD}$ (being the sum of two equal angles) (A1)
 $\hat{ADE} = \hat{BCA}$ (they both span AB) (A1)
 Hence $\triangle BAC$ and $\triangle ADE$ are similar. (A1)
 Therefore $\frac{BC}{DE} = \frac{CA}{AD}$ (A1)
 And hence $AD \times BC = DE \times CA$ ②
 Adding ① and ② (M1)
 $AB \times CD + AD \times BC = CA \times BE + DE \times CA$ (A1)
 $= CA \times (BE + DE) = CA \times BD = AC \times BD$ (AG)

Note: Award a maximum of [3 marks] to candidates who merely say that this is proved because it is Ptolemy's theorem.

[10 marks]

Question 10 continued



- (iii) Then $\hat{LQP} = \hat{QLM} + \hat{LMQ}$ ③
- since \hat{LQP} is the exterior angle of $\triangle LMQ$. (M1)(A1)
- The quadrilateral $LYQZ$ is cyclic
- since $\hat{QYL} = \hat{LZQ} = \frac{\pi}{2}$ ④ (M1)(A1)
- Thus $\hat{QLM} = \hat{QLZ} = \hat{QYZ}$. (A1)
- From ③ it follows that $\hat{LQP} = \hat{QYZ} + \hat{LMQ}$. (A1)
- Similarly $\hat{PQN} = \hat{XYQ} + \hat{QMN}$ ⑤ (A1)
- Adding ④ and ⑤
- $\hat{LQN} = \hat{LQP} + \hat{PQN}$ (A1)
- $= (\hat{QMN} + \hat{LMQ}) + (\hat{XYQ} + \hat{QYZ})$ (A1)
- i.e.* $\hat{LQN} = \hat{LMN} + \hat{XYZ}$ (AG)

[9 marks]

Total [30 marks]