INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL

MATHEMATICS
HIGHER LEVEL
PAPER 2

Monday 12 November 2001 (morning)

3 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all five questions from Section A and one question from Section B.
- Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures, as appropriate.
- Write the make and model of your calculator on the front cover of your answer booklets *e.g.* Casio *fx-9750G*, Sharp EL-9600, Texas Instruments TI-85.

You are advised to start each new question on a new page. A correct answer with **no** *indication of the method used will usually receive* **no** *marks. You are therefore advised to show your working. In particular, where graphs from a graphic display calculator are being used to find solutions, you should sketch these graphs as part of your answer.*

SECTION A

Answer all **five** *questions from this section.*

1. *[Maximum mark: 13]*

(a) Solve the following system of linear equations

$$\begin{aligned} x + 3y - 2z &= -6 \\ 2x + y + 3z &= 7 \\ 3x - y + z &= 6 . \end{aligned}$$

[3 marks]

(b) Find the vector $\boldsymbol{v} = (\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) \times (2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$. *[3 marks]*

(c) If $\boldsymbol{a} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$, $\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k}$ and $\boldsymbol{u} = m\boldsymbol{a} + n\boldsymbol{b}$ where m , n are scalars, and $\boldsymbol{u} \neq 0$, show that $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ for all m and n . *[3 marks]*

(d) The line l lies in the plane $3x - y + z = 6$, passes through the point $(1, -1, 2)$ and is perpendicular to $\boldsymbol{v}$. Find the equation of l . *[4 marks]*

2. *[Maximum mark: 10]*

(i) Let $y = \sin(kx) - kx\cos(kx)$, where k is a constant.

Show that $\dfrac{\mathrm{d}y}{\mathrm{d}x} = k^2 x \sin(kx)$. *[3 marks]*

(ii) A particle is moving along a straight line so that t seconds after passing through a fixed point O on the line, its velocity $v(t)\ \mathrm{m\,s^{-1}}$ is given by $v(t) = t \sin\left(\dfrac{\pi}{3} t\right)$.

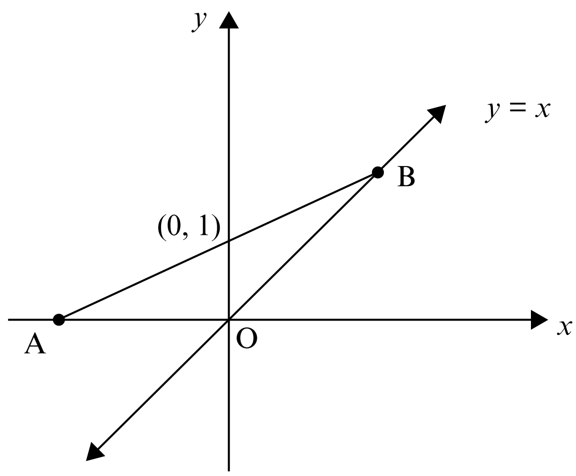
(a) Find the values of t for which $v(t) = 0$, given that $0 \le t \le 6$. *[3 marks]*

(b) (i) Write down a mathematical expression for the **total** distance travelled by the particle in the first six seconds after passing through O .

(ii) Find this distance. *[4 marks]*

3. *[Maximum mark: 13]*

The line segment [AB] has length l, gradient m, $(0 < m < 1)$, and passes through the point $(0, 1)$. It meets the x-axis at A and the line $y = x$ at B, as shown in the diagram.



(a) Find the coordinates of A and B in terms of m. *[4 marks]*

(b) Show that $l^2 = \dfrac{m^2 + 1}{m^2(1-m)^2}$. *[3 marks]*

(c) **Sketch** the graph of $y = \dfrac{x^2 + 1}{x^2(1-x)^2}$, for $x \neq 0$, $x \neq 1$, indicating any asymptotes and the coordinates of any maximum or minimum points. *[4 marks]*

(d) Find the value of m for which l is a minimum, and find this minimum value of l. *[2 marks]*

4. *[Maximum mark: 17]*

(i) Consider the sequence $\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, ...\}$ where $a_1 = a_2 = 1$ and $a_{n+1} = a_n + a_{n-1}$ for all integers $n \geq 2$.

Given the matrix $\boldsymbol{Q} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, use the principle of mathematical induction

to prove that $\boldsymbol{Q}^n = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}$ for all integers $n \geq 2$. *[7 marks]*

(ii) (a) Express $z^5 - 1$ as a product of two factors, one of which is linear. *[2 marks]*

(b) Find the zeros of $z^5 - 1$, giving your answers in the form $r(\cos\theta + \mathrm{i}\sin\theta)$ where $r > 0$ and $-\pi < \theta \leq \pi$. *[3 marks]*

(c) Express $z^4 + z^3 + z^2 + z + 1$ as a product of two real quadratic factors. *[5 marks]*

5. *[Maximum mark: 17]*

Two women, Ann and Bridget, play a game in which they take it in turns to throw an unbiased six-sided die. The first woman to throw a '6' wins the game. Ann is the first to throw.

(a) Find the probability that

(i) Bridget wins on her first throw;

(ii) Ann wins on her second throw;

(iii) Ann wins on her n^{th} throw. *[6 marks]*

(b) Let p be the probability that Ann wins the game. Show that $p = \frac{1}{6} + \frac{25}{36}p$. *[4 marks]*

(c) Find the probability that Bridget wins the game. *[2 marks]*

(d) Suppose that the game is played six times. Find the probability that Ann wins more games than Bridget. *[5 marks]*

SECTION B

Answer **one** *question from this section.*

Statistics

6. *[Maximum mark: 30]*

(i) Roger uses public transport to go to school each morning. The time he waits each morning for the transport is normally distributed with a mean of 15 minutes and a standard deviation of 3 minutes.

(a) On a specific morning, what is the probability that Roger waits more than 12 minutes? *[3 marks]*

(b) During a particular week (Monday–Friday), what is the probability that

(i) his total waiting time does not exceed 65 minutes? *[3 marks]*

(ii) he waits less than 12 minutes on at least three days of the week? *[4 marks]*

(iii) his average daily waiting time is more than 13 minutes? *[3 marks]*

(ii) An airline is considering banning smoking on its short-distance flights, but is concerned about losing customers. They therefore decide to introduce a 'no-smoking' policy for a trial period of six months.

Random samples of 200 flights were chosen during the six months before the trial and during the six month trial period, and the number of passengers (n) on each flight was recorded. The results are given in the following table. (You may assume that the number of passengers per flight is normally distributed).

number of passengers (n)	$71 \le n \le 75$	$76 \le n \le 80$	$81 \le n \le 85$	$86 \le n \le 90$	$91 \le n \le 95$	$96 \le n \le 100$
number of flights before trial	0	15	47	52	64	22
number of flights during trial	2	20	50	74	43	11

(a) Calculate a 95% confidence interval for the average reduction in the number of passengers after the introduction of the policy. *[4 marks]*

(b) Is the new policy producing a reduction in the average number of passengers per flight? Use a 5% level of significance. *[3 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) In a computer assembly shop it is claimed that the quality of a part depends on the day it is produced. To test this claim, a random sample of 500 parts is selected, and each part is classified according to the day it was produced and its quality rating. The data is shown in the table below.

	Day Produced				
	Monday	**Tuesday**	**Wednesday**	**Thursday**	**Friday**
Quality					
Superior	44	74	79	72	31
Good	14	25	27	24	10
Average	15	20	20	23	9
Mediocre	3	5	5	0	0

Test at the 5% level of significance whether the claim is valid. Write down your assumptions and mention any restrictions you may need to impose on the data. *[10 marks]*

Sets, Relations and Groups

7. *[Maximum mark: 30]*

(i) (a) Prove that the set of matrices of the form $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$,

where a, b, $c \in \mathbb{R}$, is a group under matrix multiplication. *[7 marks]*

(b) Show that this group is Abelian if and only if there exists a real constant k such that $c = ka$. *[4 marks]*

(ii) Let $A = \{a, b, c, d, e, f\}$, and R be a relation on A defined by the matrix below.

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Note that a '1' in the matrix signifies that the element in the corresponding row is related to the element in the corresponding column, for example dRb because there is a '1' on the intersection of the d-row and the b-column).

(a) Assuming that R is transitive, verify that R is an equivalence relation. *[3 marks]*

(b) Give the partition of A corresponding to R. *[4 marks]*

(iii) (a) In any group, show that if the elements x, y, and xy have order 2, then $xy = yx$. *[5 marks]*

(b) Show that the inverse of each element in a group is unique. *[3 marks]*

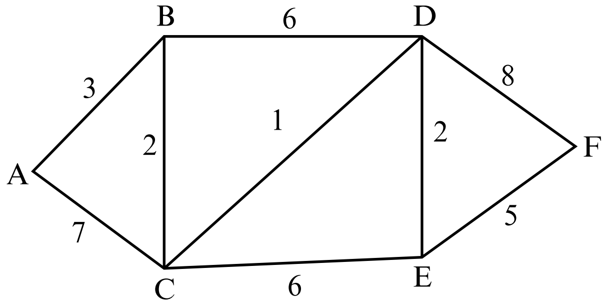
(c) Let G be a group. Show that the correspondence $x \leftrightarrow x^{-1}$ is an isomorphism from G onto G if and only if G is **abelian**. *[4 marks]*

Discrete Mathematics

8. *[Maximum mark: 30]*

(i) Use an appropriate **sorting** algorithm to sort {4, 3, 2, 5, 1, 8, 7, 6} into ascending order. Show all the steps used by the algorithm. *[5 marks]*

(ii) The diagram shows a weighted graph with vertices A , B , C , D , E and F .



Use **Dijkstra's Algorithm** to find the length of the shortest path between the vertices A and F . Show all the steps used by the algorithm and draw the path. *[6 marks]*

(iii) Let $a_n = pa_{n-1} + qa_{n-2}$, $n \ge 2$, be a second-order difference equation with constant coefficients p and q .

(a) Show that if r is a root of the characteristic polynomial and k is any constant, then $a_n = kr^n$ satisfies the given difference equation for $n \ge 2$. *[5 marks]*

(b) Show that if b_n and c_n both satisfy the given difference equation for $n \ge 2$, then $a_n = b_n + c_n$ also satisfies the given equation. *[4 marks]*

(c) Use the results above to prove that if r_1 and r_2 are the zeros of the polynomial $x^2 - px - q$, where $r_1 \ne r_2$, then the solution of the given difference equation is

$$a_n = k_1 r_1^{\,n} + k_2 r_2^{\,n} .$$ *[4 marks]*

(d) Solve the difference equation $a_n = a_{n-1} + 6a_{n-2}$, $n \ge 2$, with $a_0 = 1$, and $a_1 = 3$. *[6 marks]*

Analysis and Approximation

9. *[Maximum mark: 30]*

Consider the function $f(x) = x^{\frac{1}{x}}$, where $x \in \mathbb{R}^{+}$.

(i) (a) Show that the derivative $f'(x) = f(x)\left(\frac{1 - \ln x}{x^2}\right)$. *[3 marks]*

(b) Sketch the function $f(x)$, showing clearly the local maximum of the function and its horizontal asymptote. You may use the fact that $\lim_{x \to \infty} \frac{\ln x}{x} = 0$. *[5 marks]*

(c) Find the Taylor expansion of $f(x)$ about $x = \mathrm{e}$, up to the second degree term, and show that this polynomial has the same maximum value as $f(x)$ itself. *[5 marks]*

(ii) Determine whether the series $\sum_{n=1}^{\infty}\left(\frac{1}{n^{1+\frac{1}{n}}}\right)$ converges. *[4 marks]*

(iii) Let $g(x) = x^m (x - 1)^n$, where m and n are positive integers. Show using Rolle's theorem that there is a number $c \in \,]0\,,\,1[$ which divides the interval $[0\,,\,1]$ in the ratio $m : n$. *[5 marks]*

(iv) Let $h(x) = \mathrm{e}^{\cos x}$.

(a) Show that the area A enclosed by the curve $h(x)$, the x-axis, and the lines $x = -10\pi$ and $x = 20\pi$ is equal to $15\int_0^{2\pi} h(x)\mathrm{d}x$. *[2 marks]*

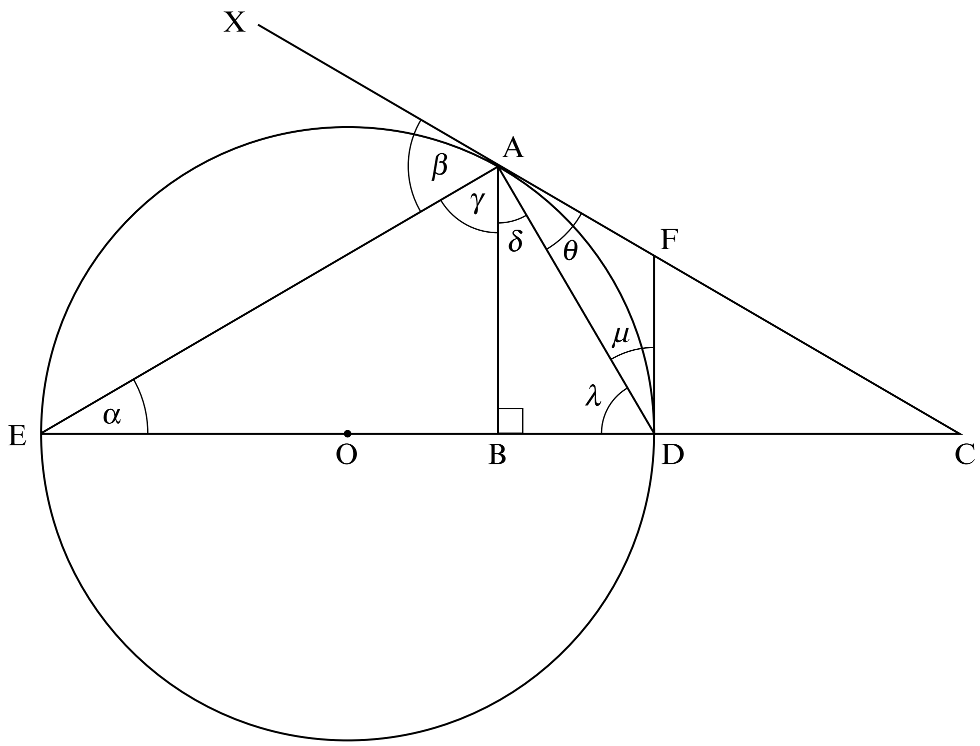
(b) Find the number of sub-intervals required for the trapezium rule to yield an error less than 0.15 when evaluating A .

(You may assume that the error in evaluating $\int_a^b h(x)\mathrm{d}x$ by the trapezium rule with n sub-intervals is of the form $\frac{(b-a)^3}{12n^2} h''(c)$, where $a < c < b$.) *[6 marks]*

Euclidean Geometry and Conic Sections

10. *[Maximum mark: 30]*

The following diagram shows a circle, centre O and diameter [ED]. (CAX) and (FD) are tangents to the circle at A and D respectively. (AB) is perpendicular to (DE).



Let $A\widehat{E}O = \alpha$, $E\widehat{A}X = \beta$, $E\widehat{A}B = \gamma$, $B\widehat{A}D = \delta$, $D\widehat{A}F = \theta$, $B\widehat{D}A = \lambda$, $A\widehat{D}F = \mu$.
Also, $BD = p$, $EB = q$, where $q > p$.

(a) Show that triangle AFD is isosceles. *[2 marks]*

(b) Show that

(i) (AD) bisects angle BAC;

(ii) (AE) bisects angle BAX. *[6 marks]*

(c) Show that {E, D, B, C} is a harmonic division. *[4 marks]*

(d) Prove that $BC = \dfrac{2pq}{q-p}$. *[3 marks]*

(e) Prove that the circle with diameter [AO] is the nine-point circle of triangle AED. *[8 marks]*

(f) The lines joining the points D and E to the centre of the nine-point circle intersect the lines (AE) and (AD) at P and Q respectively. Show that (PQ) is parallel to (DE). *[7 marks]*