INTERNATIONAL BACCALAUREATE
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MARKSCHEME

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MATHEMATICS

Higher Level

Paper 2

25 pages

1. (a) The system is $\begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & -2 \\ 2 & 1 & 3 \\ 3 & -1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -6 \\ 7 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. ***(M1)***

Therefore, the solution is $x = 1, y = -1, z = 2$. ***(G2)***

OR

The system of equations is:

	1	3	–2	–6
	2	1	3	7
	3	–1	1	6
	1	3	–2	–6
$R_2 \leftarrow R_2 - 2R_1$	0	–5	7	19
$R_3 \leftarrow R_3 - 3R_1$	0	–10	7	24
	1	3	–2	–6
	0	–5	7	19
$R_3 \leftarrow R_3 - 2R_2$	0	0	–7	–14

(M2)

Back substitution gives $x = 1, y = -1, z = 2$. ***(C1)***

OR

$x = 1, y = -1, z = 2$. ***(G3)***

[3 marks]

(b) $\boldsymbol{v} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 3 & -2 \\ 2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -2 \\ 1 & 3 \end{vmatrix}\boldsymbol{i} - \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}\boldsymbol{j} + \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\boldsymbol{k} = 11\boldsymbol{i} - 7\boldsymbol{j} - 5\boldsymbol{k}$. ***(M1)(C2)***

[3 marks]

(c) $\boldsymbol{u} = m(\boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}) + n(2\boldsymbol{i} + \boldsymbol{j} + 3\boldsymbol{k})$

$= (m + 2n)\boldsymbol{i} + (3m + n)\boldsymbol{j} + (-2m + 3n)\boldsymbol{k}$ ***(C1)***

Therefore, $\boldsymbol{v} \cdot \boldsymbol{u} = 11(m + 2n) - 7(3m + n) - 5(-2m + 3n)$

$= 11m + 22n - 21m - 7n + 10m - 15n$

$= 0$, for all m and n. ***(M1)(C1)***

That is, $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ for all values of m and n. ***(AG)***

OR

$\boldsymbol{v}$ is perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$ [from part (b)]. ***(M1)***

Therefore, $\boldsymbol{v} \cdot \boldsymbol{a} = \boldsymbol{v} \cdot \boldsymbol{b} = 0$, so $\boldsymbol{v} \cdot \boldsymbol{u} = m(\boldsymbol{v} \cdot \boldsymbol{a}) + n(\boldsymbol{v} \cdot \boldsymbol{b}) = 0$, and hence $\boldsymbol{v}$ is perpendicular to $\boldsymbol{u}$ for all values of m and n. ***(M1)(R1)(AG)***

[3 marks]

continued…

Question 1 continued

(d) The normal to the plane, $3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$, and $\boldsymbol{v}$ are both perpendicular to the required line, *l*.
Therefore, the direction of *l* is given by

$$\boldsymbol{v} \times (3\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}) = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 11 & -7 & -5 \\ 3 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -7 & -5 \\ -1 & 1 \end{vmatrix} \boldsymbol{i} - \begin{vmatrix} 11 & -5 \\ 3 & 1 \end{vmatrix} \boldsymbol{j} + \begin{vmatrix} 11 & -7 \\ 3 & -1 \end{vmatrix} \boldsymbol{k}$$

$$= -12\boldsymbol{i} - 26\boldsymbol{j} + 10\boldsymbol{k}$$ ***(M1)(C2)***

Thus, an equation for *l* is $\boldsymbol{r} = \boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k} + \lambda(6\boldsymbol{i} + 13\boldsymbol{j} - 5\boldsymbol{k})$, where λ is a scalar.
[Any form of the correct answer is quite acceptable.] ***(C1)***

[4 marks]

Total [13 marks]

2. (i) $y = \sin(kx) - kx\cos(kx)$

$\dfrac{dy}{dx} = k\cos(kx) - k\{\cos(kx) + x[-k\sin(kx)]\}$ ***(M1)(C1)***

$= k\cos(kx) - k\cos(kx) + k^2 x\sin(kx)$ ***(C1)***

$= k^2 x\sin(kx)$ ***(AG)***

[3 marks]

(ii) (a) $v(t) = t\sin\left(\frac{\pi}{3}t\right) = 0$ when $t = 0,\ t = 3$ or $t = 6$ ***(C1)(C1)(C1)***

[3 marks]

(b) (i) The required distance, $d = \int_0^3 t\sin\left(\frac{\pi}{3}t\right)dt - \int_3^6 t\sin\left(\frac{\pi}{3}t\right)dt$ ***(M1)***

(ii) $d = 2.865 + 8.594$ ***(G2)***
$= 11.459$
$= 11.5$ m. ***(A1)***

OR

(i) The required distance, $d = \int_0^6 \left| t\sin\left(\frac{\pi}{3}t\right)\right| dt$. ***(M1)***

(ii) $d = 11.5$ m. ***(G2)(A1)***

OR

(i) The required distance, $d = \int_0^3 t\sin\left(\frac{\pi}{3}t\right)dt - \int_3^6 t\sin\left(\frac{\pi}{3}t\right)dt$ ***(M1)***

(ii) $d = \dfrac{9}{\pi^2}\left\{\int_0^3 \left(\frac{\pi}{3}\right)^2 t\sin\left(\frac{\pi}{3}t\right)dt - \int_3^6 \left(\frac{\pi}{3}\right)^2 t\sin\left(\frac{\pi}{3}t\right)dt\right\}$ ***(C1)***

$= \dfrac{9}{\pi^2}\left\{\left[\sin\left(\frac{\pi}{3}t\right) - \frac{\pi}{3}t\cos\left(\frac{\pi}{3}t\right)\right]_0^3 - \left[\sin\left(\frac{\pi}{3}t\right) - \frac{\pi}{3}t\cos\left(\frac{\pi}{3}t\right)\right]_3^6\right\}$

[from (i)] ***(C1)***

$= \dfrac{9}{\pi^2}(\sin\pi - \pi\cos\pi - \sin 2\pi + 2\pi\cos 2\pi + \sin\pi - \pi\cos\pi)$

$= \dfrac{36}{\pi}$ m (11.5 m). ***(A1)***

Note:	Award ***(A1)*** for $\pm\dfrac{18}{\pi}(\pm 5.73)$ which is obtained by integrating v from 0 to 6.

[4 marks]

Total [10 marks]

3. (a) The equation of (AB) is $y = mx + 1$. ***(M1)(C1)***

When $y = 0$, $x = -\frac{1}{m}$, and when $y = x$, $y = x = \frac{1}{1-m}$.

Therefore, A $= \left(-\frac{1}{m}, 0\right)$, and $\mathrm{B} = \left(\frac{1}{1-m}, \frac{1}{1-m}\right)$. ***(C2)***

[4 marks]

(b) $l^2 = \left(\frac{1}{1-m} + \frac{1}{m}\right)^2 + \left(\frac{1}{1-m}\right)^2$ ***(M1)***

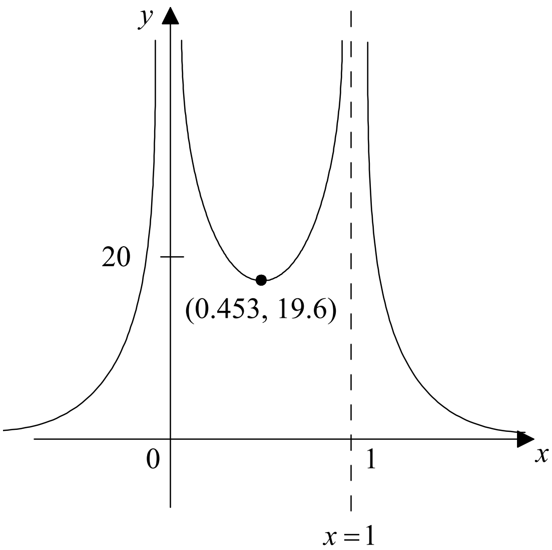
$= \frac{2}{(1-m)^2} + \frac{2}{m(1-m)} + \frac{1}{m^2}$ ***(C1)***

$= \frac{2m^2 + 2m(1-m) + (1-m)^2}{m^2(1-m)^2}$ ***(C1)***

$= \frac{m^2+1}{m^2(1-m)^2}$. ***(AG)***

[3 marks]

(c) The graph of $y = \frac{x^2+1}{x^2(1-x)^2}$ is as follows:



(A2)(A2)

Note:	Award ***(A1)*** for both asymptotes at $x = 0$ and $x = 1$, and ***(A1)*** for $y = 0$. Award ***(A1)*** for each correct coordinate for the minimum point.

[4 marks]

(d) From part (c), l is a minimum when $m = 0.453$ as $0 < m < 1$, and then the minimum value of $l = 4.43\left(\sqrt{19.63}\right)$. ***(C1)*** ***(A1)***

[2 marks]

Total [13 marks]

4. (i) Let $\mathrm{P}(n)$ be the proposition: $\boldsymbol{Q}^n = \begin{pmatrix} a_{n+1} & a_n \\ a_n & a_{n-1} \end{pmatrix}$.

$\mathrm{P}(2)$ is true since $\boldsymbol{Q}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a_3 & a_2 \\ a_2 & a_1 \end{pmatrix}$. ***(M1)(C1)***

Assume $\mathrm{P}(k)$ is true for some integer $k \ge 2$, that is $\boldsymbol{Q}^k = \begin{pmatrix} a_{k+1} & a_k \\ a_k & a_{k-1} \end{pmatrix}$. ***(M1)***

Then, $\boldsymbol{Q}^{k+1} = \boldsymbol{Q}\boldsymbol{Q}^k = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{k+1} & a_k \\ a_k & a_{k-1} \end{pmatrix}$ ***(M1)***

$= \begin{pmatrix} a_{k+1}+a_k & a_k+a_{k-1} \\ a_{k+1} & a_k \end{pmatrix}$ **(C1)**

$= \begin{pmatrix} a_{k+2} & a_{k+1} \\ a_{k+1} & a_k \end{pmatrix}$ ***(C1)***

Therefore, $\mathrm{P}(k)$ is true $\Rightarrow \mathrm{P}(k+1)$ is true, and so $\mathrm{P}(n)$ is true for all integers $n \ge 2$. ***(R1)***

[7 marks]

(ii) (a) $z^5 - 1 = (z-1)(z^4 + z^3 + z^2 + z + 1)$ ***(C2)***

[2 marks]

(b) $z^5 - 1 = 0$

$\Rightarrow z = \cos 0 + \mathrm{i}\sin 0$ (accept $z = 1$),

$\cos\left(\pm\dfrac{2\pi}{5}\right) + \mathrm{i}\sin\left(\pm\dfrac{2\pi}{5}\right), \cos\left(\pm\dfrac{4\pi}{5}\right) \pm \mathrm{i}\sin\left(\pm\dfrac{4\pi}{5}\right)$.

(C3)

$\left(\text{Accept } \cos\dfrac{2\pi}{5} \pm \mathrm{i}\sin\dfrac{2\pi}{5}, \cos\dfrac{4\pi}{5} \pm \mathrm{i}\sin\dfrac{4\pi}{5}\right)$

[3 marks]

(c) $\left(z - \cos\dfrac{2\pi}{5} - \mathrm{i}\sin\dfrac{2\pi}{5}\right)\left(z - \cos\dfrac{2\pi}{5} + \mathrm{i}\sin\dfrac{2\pi}{5}\right) = z^2 - \left(2\cos\dfrac{2\pi}{5}\right)z + 1$ ***(M1)(C1)***

$\left(z - \cos\dfrac{4\pi}{5} - \mathrm{i}\sin\dfrac{4\pi}{5}\right)\left(z - \cos\dfrac{4\pi}{5} + \mathrm{i}\sin\dfrac{4\pi}{5}\right) = z^2 - \left(2\cos\dfrac{4\pi}{5}\right)z + 1$ ***(M1)(C1)***

Thus, $z^4 + z^3 + z^2 + z + 1 = \left(z^2 - \left[2\cos\dfrac{2\pi}{5}\right]z + 1\right)\left(z^2 - \left[2\cos\dfrac{4\pi}{5}\right]z + 1\right)$ ***(C1)***

OR

$z^4 + z^3 + z^2 + z + 1 = \left(z^2 - \left[2\cos\dfrac{2\pi}{5}\right]z + 1\right)\left(z^2 + \left[2\cos\dfrac{\pi}{5}\right]z + 1\right)$ ***(C1)***

OR

$z^4 + z^3 + z^2 + z + 1 = \left(z^2 - 0.618z + 1\right)\left(z^2 + 1.618z + 1\right)$ ***(C1)***

[5 marks]

Total [17 marks]

5. (a) (i) P(Bridget wins on her first throw)

= P(Ann does not throw a '6') × P(Bridget throws a '6') ***(M1)***

$=\frac{5}{6}\times\frac{1}{6}$

$=\frac{5}{36}$ ***(C1)***

(ii) P(Ann wins on her second throw)

= P(Ann does not throw a '6') × P(Bridget does not throw a '6') × P(Ann throws a '6') ***(M1)***

$=\frac{5}{6}\times\frac{5}{6}\times\frac{1}{6}$

$=\frac{25}{216}$ ***(C1)***

(iii) P(Ann wins on her *n*th throw)

= P(neither Ann nor Bridget win on their first $(n-1)$ throws) × P(Ann throws a '6' on her *n*th throw) ***(M1)***

$=\left(\frac{5}{6}\right)^{2(n-1)}\times\frac{1}{6}.$ ***(C1)***

[6 marks]

(b) p = P(Ann wins)

= P(Ann wins on her first throw) + P(both Ann and Bridget do not win on their first throws) × P(Ann wins from then on) ***(M1)(R2)***

$=\frac{1}{6}+\left(\frac{5}{6}\right)^2\times p$ ***(C1)***

$=\frac{1}{6}+\frac{25}{36}p$ ***(AG)***

OR

p = P(Ann wins on first throw) + P(Ann wins on second throw) + P(Ann wins on third throw) + … . ***(M1)***

$=\frac{1}{6}+\left(\frac{5}{6}\right)^2\frac{1}{6}+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)+\dots$ ***(C2)***

$=\frac{1}{6}+\frac{25}{36}\left\{\frac{1}{6}+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)+\dots\right\}$ (or $\frac{\frac{1}{6}}{1-\frac{25}{36}}$) ***(C1)***

$=\frac{1}{6}+\frac{25}{36}p$, as required. ***(AG)***

[4 marks]

continued…

Question 5 continued

(c) From part (b), $\frac{11}{36}p = \frac{1}{6} \Rightarrow p = \frac{6}{11}$. ***(C1)***

Therefore, $\text{P(Bridget wins)} = 1 - p = \frac{5}{11}$. ***(C1)***

[2 marks]

(d) P(Ann wins more games than Bridget)

$= \text{P(Ann wins 4 games)} + \text{P(Ann wins 5 games)} + \text{P(Ann wins 6 games)}$ ***(M2)***

$= \binom{6}{4}\left(\frac{6}{11}\right)^4\left(\frac{5}{11}\right)^2 + \binom{6}{5}\left(\frac{6}{11}\right)^5\left(\frac{5}{11}\right) + \left(\frac{6}{11}\right)^6$ ***(M2)***

$= \frac{6^4}{11^6}(15 \times 25 + 36 \times 5 + 36)$

$= 0.432$. ***(A1)***

[5 marks]

Total [17 marks]

Note:	Different calculators may give answers which vary slightly from those obtained from tables. Accept these unless they need to be penalized under **LAP** or **RP**.

6. (i) Let W be the time Roger waits each morning $W \sim \mathrm{N}(15, 9)$.

(a) $\mathrm{P}(W > 12) = \mathrm{P}\left(Z > \frac{12-15}{3}\right)$ ***(M1)***

$= \mathrm{P}(Z > -1) = \mathrm{P}(Z < 1)$ ***(A1)***

$= 0.841$. ***(A1)***

OR

$\mathrm{P}(W > 12) = 0.841$ ***(G3)***

[3 marks]

(b) (i) $\sum_{1}^{5} W \sim \mathrm{N}(75, 45)$ ***(A1)***

$\mathrm{P}\left(\sum W < 65\right) = \mathrm{P}\left(Z < \frac{-10}{3\sqrt{5}}\right)$

$= \mathrm{P}(Z < -1.491) = 1 - \mathrm{P}(Z < 1.491)$ ***(M1)***

$= 1 - 0.9319$

$= 0.0681$ ***(A1)***

OR

$\mathrm{P}\left(\sum W < 65\right) = 0.0680$ ***(G2)***

[3 marks]

(ii) $\mathrm{P}(W < 12) = 0.159$

N – number of days on which he waits less than 12 minutes

$N \sim \mathrm{B}(5, 0.159)$ ***(M1)(A1)***

$\mathrm{P}(N \geq 3) = \mathrm{P}(N = 3) + \mathrm{P}(N = 4) + \mathrm{P}(N = 5)$ ***(M1)***

$= 0.02843 + 0.00269 + 0.000101$

$= 0.0312$ ***(A1)***

OR

$\mathrm{P}(N \geq 3) = 1 - \mathrm{P}(N \leq 2)$ ***(M1)***

$= 1 - 0.9687$

$= 0.0312$ ***(G1)***

[4 marks]

continued...

Question 6 continued

(iii) $\bar{W} \sim \mathrm{N}\left(15, \frac{9}{5}\right)$

$\mathrm{P}(\bar{W} > 13) = \mathrm{P}\left(Z > \frac{13-15}{\frac{3}{\sqrt{5}}}\right)$ ***(M1)***

$= \mathrm{P}(Z > -1.49) = \mathrm{P}(Z < 1.49)$ ***(M1)***

$= 0.932$ ***(A1)***

OR

$\mathrm{P}(\bar{W} > 13) = 0.932$ ***(G3)***

OR

$\mathrm{P}(\bar{W} > 13) = \mathrm{P}\left(\sum W > 65\right)$ ***(M1)***

$= 1 - \mathrm{P}\left(\sum W < 65\right) = 1 - 0.068$ ***(M1)***

$= 0.932$ ***(A1)***

[3 marks]

(ii) (a) 95 % C.I. for the difference of 2 means. ***(C1)***

$(\bar{x}_1 - \bar{x}_2) \pm t\sqrt{\left(\frac{(n-1)(s_1^2)+(n-1)(s_2^2)}{n+n-2}\right)\left(\frac{1}{n}+\frac{1}{n}\right)}$ ***(M1)***

$= (88.775 - 87.225) \pm 1.96 \times 0.552$ ***(M1)***

95 % C.I. for average reduction is $(0.469, 2.63)$ ***(C1)***

[4 marks]

(b) H_0 : there is no reduction in the number of passengers, that is, $\mathrm{E}(X) - \mathrm{E}(Y) = 0$

H_1 : there is a reduction, that is, $\mathrm{E}(X) - \mathrm{E}(Y) > 0$ ***(C1)***

1-tailed test, with critical value 1.645 and test statistic 2.81. ***(M1)***

Reject H_0 and conclude that the new policy does result in a reduction in the number of passengers. ***(R1)***

[3 marks]

continued…

Question 6 continued

(iii) H_0 : Day of production and quality rating are independent.
H_1 : Day of production and quality rating are not independent. ***(C1)***

This is a χ^2 test of independence where the expected frequency for each cell is
$\dfrac{\text{total of row} \times \text{total column}}{\text{total number}}$, ***(M1)***

Expected frequencies

Day Produced

Quality	M	T	W	Th	F
Superior	45.6	74.4	78.6	71.4	30
Good	15.2	24.8	26.2	23.8	10
Average	13.2	21.6	22.8	20.7	8.7
Mediocre	2	3.2	3.4	3.1	1.3

(A2)

Note: Award ***(A1)*** if one result is incorrect, ***(A0)*** if more than three are incorrect.

Since the last row contains cells with less than 5 observations, we combine the last two rows into one row: ***(R1)***

Observed frequencies (f_o)

Day Produced

Quality	M	T	W	Th	F
Superior	44	74	79	72	31
Good	14	25	27	24	10
Poor	18	25	25	23	9

Expected frequencies (f_e)

Day Produced

Quality	M	T	W	Th	F
Superior	45.6	74.4	78.6	71.4	30
Good	15.2	24.8	26.2	23.8	10
Poor	15.2	24.8	26.2	23.8	10

$$\chi^2 = \frac{\sum (f_e - f_o)^2}{f_e} = 0.920$$ ***(M1)(A1)***

With 8 degrees of freedom, $\chi_c^2 = 15.507$ ***(A1)***

Since $0.920 < 15.507$, we fail to reject the null hypothesis and conclude that there ***(R1)***
seems to be no association between the day of production and the quality of the part. ***(R1)***

Note: Do not award ***(R1)*** to candidates who fail to combine the last 2 rows.
Allow **ft** as shown below.

$$\chi^2 = \frac{\sum (f_e - f_o)^2}{f_e} = 7.86$$ ***(M1)(A1)***

With 12 degrees of freedom, $\chi_c^2 = 21.026$ ***(A1)***

Since $7.86 < 21.026$, we fail to reject the null hypothesis and conclude that there ***(R1)***
seems to be no association between the day of production and the quality of the part. ***(R1)***

[10 marks]

Total [30 marks]

7. (i) (a) Since the operation is that of matrix muliplication, then the operation is associative. ***(C1)***

$$\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & a+d & b+e+af\\0 & 1 & c+f\\0 & 0 & 1\end{bmatrix}$$ which is of the same form. ***(M1)***

The set is closed under this operation. ***(C1)***

$$\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & -a & ac-b\\0 & 1 & -c\\0 & 0 & 1\end{bmatrix}$$, which is again of the same form. ***(M2)***

Therefore, every element has an inverse in the set. ***(C1)***

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$ is an element of the set and is of the same form, so the set contains an identity element. ***(C1)***

Therefore the set is a group under matrix multiplication.

[7 marks]

(b) For this group to be Abelian,

$$\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & d & e\\0 & 1 & f\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & a & b\\0 & 1 & c\\0 & 0 & 1\end{bmatrix},$$ ***(M1)***

$$\Rightarrow\begin{bmatrix}1 & a+d & b+e+af\\0 & 1 & c+f\\0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & a+d & b+e+cd\\0 & 1 & c+f\\0 & 0 & 1\end{bmatrix},\ \Rightarrow af=cd$$ ***(M1)***

Let $f=k$, and $d=1$, then $c=ka$. ***(A1)***

Conversely, if $c=ka$ for every matrix in the group, and therefore $f=kd$, it follows that $af=cd$. ***(A1)***

[4 marks]

(ii) (a) Since the main diagonal of the matrix has ones, this means that every element is related to itself and consequently the relation is reflexive. ***(C1)***
Also, the matrix is symmetric and hence, the relation is symmetric. ***(C2)***

[3 marks]

(b) The partition of A is the set of all equivalent classes. ***(C1)***
The three classes are $\{\{a,c,e\},\{b,d\},\{f\}\}$ ***(A3)***

[4 marks]

Question 7 continued

(iii) (a) $(xy)^2 = e$ Order of $xy = 2$ ***(M1)***

$\Rightarrow (xy)(xy) = e \Rightarrow x(yx)y = e$ Associative property ***(M1)(M1)***

$\Rightarrow xx(yx)yy = xey$ Left and right-multiply ***(M1)***

$\Rightarrow e(yx)e = xy$ Order of elements given ***(M1)***

$\Rightarrow yx = xy$ ***(AG)***

OR

Since x, y and xy are self-inverses, $x^{-1} = x$, $y^{-1} = y$ and $(xy)^{-1} = xy$ ***(R1)(R1)***

Consider $xy = (xy)^{-1}$ ***(M1)***

$= y^{-1}x^{-1}$ ***(M1)***

$= yx$ ***(M1)(AG)***

[5 marks]

(b) Let *a* be any element of a group, whose identity is *e*. Let a^{-1} be an inverse of *a*, and let *b* be another inverse of *a* different from a^{-1}.

Now, $b = be = b(aa^{-1}) = (ba)a^{-1}$; identity and associativity properties, ***(M1)***

then, $b = ea^{-1} = a^{-1}$, which contradicts the assumption that $b \neq a^{-1}$, ***(M1)***

therefore there is only one inverse of *a*, namely a^{-1}. ***(R1)***

OR

Let *a* be any element of a group whose identity is *e*. Let *b* and *c* be inverses of *a*, so that $ab = ba = e$, and $ac = ca = e$. ***(M1)***

Consider $b = b(ac)$

$= (ba)c$

$= c$ ***(M1)***

Thus any two inverses are equal, so the inverse is unique. ***(R1)***

[3 marks]

(c) If G is Abelian, then $f(xy) = (xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = f(x)f(y)$ and *f* is an isomorphism. ***(M1)(R1)***

If *f* is an isomorphism, then $f(xy) = f(x)f(y)$, that is, $(xy)^{-1} = x^{-1}y^{-1} = (yx)^{-1}$

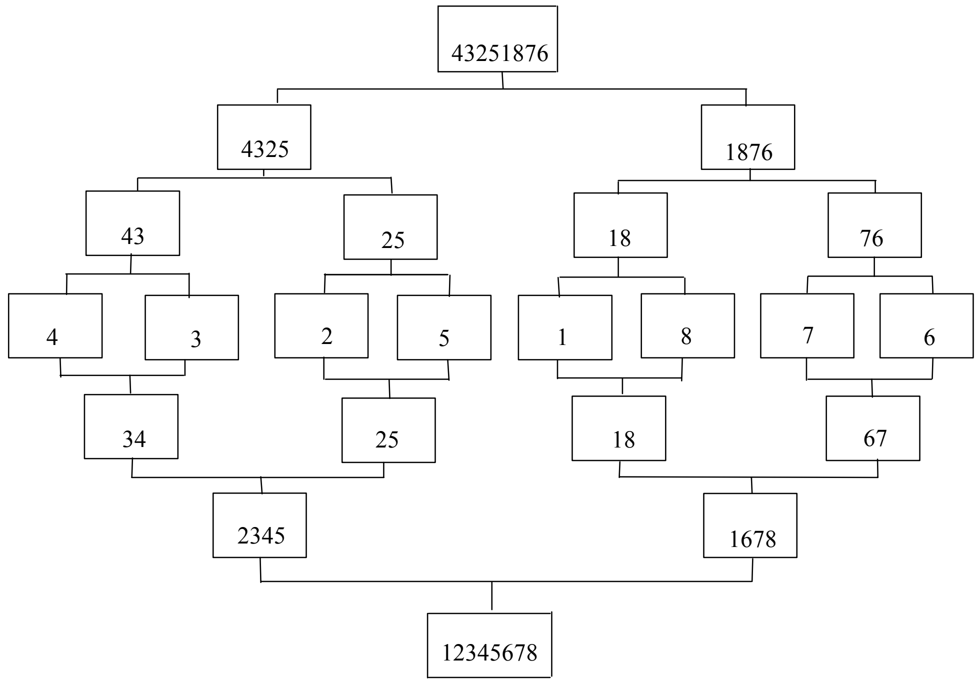
Then $xy = yx$ ***(M1)***

and hence G is Abelian. ***(R1)***

[4 marks]

Total [30 marks]

8. (i) Appropriate algorithms include 'merge sort' and 'bubble sort', the following tree and inverted tree shows how the sequence is taken apart and then put together. Candidates may also use a descriptive method to explain the steps.



The chart demonstrates the steps required:	
Describe the algorithm (This may be implied by the set-up)	***(M1)***
Subdivide into smaller lists	***(M1)***
Sort small lists	***(M1)***
Merge every two lists	***(M1)***
Final merge	***(M1)***

OR

Using 'bubble sort', the successive steps are:	***(M1)***
32415768	***(M1)***
23145678	***(M1)***
21345678	***(M1)***
12345678	***(A1)***

[5 marks]

continued…

Question 8 continued

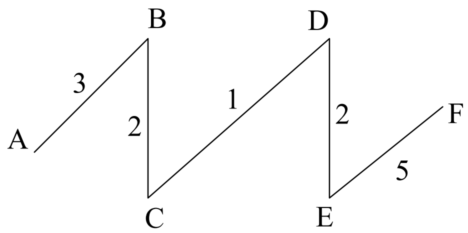
(ii)

Iteration:	Vertices:	Labels:	
First:	A;	A: 0, B: 3, C: 7, D: ∞, E: ∞, F: ∞	***(M1)***
Second:	A, B;	A: 0, B: 3, C: 5, D: 9, E: ∞, F: ∞	***(M1)***
Third:	A, B, C;	A: 0, B: 3, C: 5, D: 6, E: 11, F: ∞	
Fourth:	A, B, C, D;	A: 0, B: 3, C: 5, D: 6, E: 8, F: 14	***(M1)***
Fifth:	A, B, C, D, E;	A: 0, B: 3, C: 5, D: 6, E: 8, F: 13	

OR

Step	A	B	C	D	E	F	vertex added	
1	0( )	3(A)	7(A)	∞(A)	∞(A)	∞(A)	A	***(M1)***
2			5(B)	9(B)			B	***(M1)***
3				6(C)	11(C)		C	
4					8(D)	14(D)	D	***(M1)***
5						13(E)	E	
6							F	

Since after this iteration F is a distinguished vertex, we conclude that the shortest path is A, B, C, D, E, F and has length 13. ***(R1)(A1)***



(A1)

[6 marks]

(iii) (a) The characteristic polynomial of this equation is of the form $x^2 - px - q = 0$. ***(C1)***

Since r is a root of this equation, then

$$pa_{n-1} + qa_{n-2} = p(kr^{n-1}) + q(kr^{n-2}) \quad \textit{(M1)}$$

$$= kr^{n-2}(pr + q) \quad \textit{(M1)}$$

$$= kr^{n-2}r^2 = kr^n = a_n \quad \textit{(M1)(R1)}$$

[5 marks]

(b) $b_n = pb_{n-1} + qb_{n-2}$

$c_n = pc_{n-1} + qc_{n-2}$ ***(C1)***

$\Rightarrow a_n = b_n + c_n = p(b_{n-1} + c_{n-1}) + q(b_{n-2} + c_{n-2})$ ***(M1)(M1)***

$= pa_{n-1} + qa_{n-2}$ ***(R1)***

That is, $a_n = b_n + c_n$ satisfies the given equation. ***(AG)***

[4 marks]

continued...

Question 8 (iii) continued

(c) Since r_1 is a zero of the characteristic polynomial, then from part (a) $k_1 r_1^n$ is a solution of the difference equation. ***(R1)***

Similarly $k_2 r_2^n$ is a solution. ***(R1)***

By part (b) the sum of the solutions is also a solution. ***(R1)***

With two initial conditions a system of two equations in k_1 and k_2 will have a unique solution because $r_1 \neq r_2$. ***(R1)***

[4 marks]

(d) The characteristic polynomial of this equation is $x^2 - x - 6$ ***(M1)***

with solutions $r_1 = 3$ and $r_2 = -2$. ***(A1)***

The general solution is then $a_n = k_1 3^n + k_2(-2)^n$. ***(R1)***

The initial conditions will give

$k_1 + k_2 = 1$

$3k_1 - 2k_2 = 3$, which leads to ***(M1)***

$k_1 = 1$ and $k_2 = 0$, and the solution is ***(A1)***

$a_n = 3^n$. ***(A1)***

[6 marks]

Total [30 marks]

9. (i) (a) The derivative can be found by logarithmic differentiation. Let $y = f(x)$.

$y = x^{\frac{1}{x}} \Rightarrow \ln y = \frac{1}{x}\ln x$ ***(M1)***

$\Rightarrow \frac{y'}{y} = \frac{-1}{x^2}\ln x + \frac{1}{x} \times \frac{1}{x} = \frac{1-\ln x}{x^2}$ ***(M1)(M1)***

$\Rightarrow y' = y\left(\frac{1-\ln x}{x^2}\right)$

that is, $f'(x) = f(x)\left(\frac{1-\ln x}{x^2}\right)$ ***(AG)***

[3 marks]

(b) This function is defined for positive and real numbers only.

To find the exact value of the local maximum:

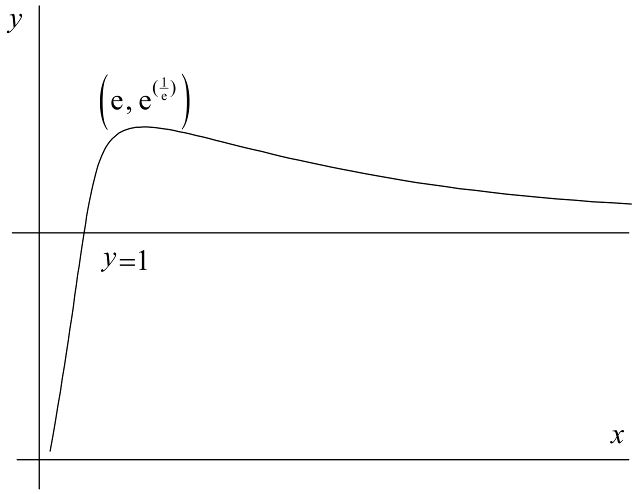
$y' = 0 \Rightarrow \ln x = 1 \Rightarrow x = \mathrm{e}$ ***(M1)***

$\Rightarrow y = \mathrm{e}^{\frac{1}{\mathrm{e}}}$ ***(A1)***

To find the horizontal asymptote:

$\lim_{x\to\infty}(y = x^{\frac{1}{x}}) \Rightarrow \lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} = 0$

$\Rightarrow \lim_{x\to\infty} y = 1$ ***(M1)(A1)***



(A1)

[5 marks]

(c) By Taylor's theorem we have

$P_2(x) = f(\mathrm{e}) + f'(\mathrm{e})(x-\mathrm{e}) + \frac{f''(\mathrm{e})}{2}(x-\mathrm{e})^2$ ***(A1)***

$f''(x) = f'(x)\left(\frac{1-\ln x}{x^2}\right) + f(x)\left(\frac{2\ln x - 3}{x^3}\right)$ ***(M1)***

Also, $f'(\mathrm{e}) = 0$, and $f''(\mathrm{e}) = 0 + f(\mathrm{e})\left(\frac{2-3}{\mathrm{e}^3}\right) = \mathrm{e}^{\frac{1}{\mathrm{e}}}\left(\frac{-1}{\mathrm{e}^3}\right) = -\mathrm{e}^{\frac{1}{\mathrm{e}}-3}$ ***(M1)(A1)***

hence $P_2(x) = \mathrm{e}^{\frac{1}{\mathrm{e}}} - \frac{\mathrm{e}^{\frac{1}{\mathrm{e}}-3}}{2}(x-\mathrm{e})^2$ which is a parabola with vertex at $x = \mathrm{e}$ and

$P_2(\mathrm{e}) = \mathrm{e}^{\frac{1}{\mathrm{e}}} = f(\mathrm{e})$ ***(R1)(AG)***

[5 marks]

continued…

Question 9 continued

(ii) This can be done using comparison with the harmonic series. ***(R1)***

Let $b_n = \frac{1}{n}$ represent the harmonic series.

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{n^{1+\frac{1}{n}}} \times \frac{n}{1} = \lim_{n\to\infty} \frac{1}{n^{\frac{1}{n}}} = 1$. ***(M1)(A1)***

Since b_n diverges, so does a_n. ***(A1)***

[4 marks]

(iii) $f(0) = f(1) = 0$ and f is continuous and differentiable on [0, 1], hence by Rolle's theorem, there is a number $c \in]0,1[$ such that $f'(c) = 0$. ***(C1)***

$f'(x) = mx^{m-1}(x-1)^n + nx^m(x-1)^{n-1} = x^{m-1}(x-1)^{n-1}\left[m(x-1)+nx\right]$, in]0, 1[***(M1)***

$f'(c) = 0 \Leftrightarrow m(c-1) + nc = 0 \Leftrightarrow c(m+n) = m$ ***(M1)***

$\Leftrightarrow c = \frac{m}{m+n}$ ***(A1)***

$\frac{c-0}{1-c} = \frac{\frac{m}{m+n}}{1-\frac{m}{m+n}} = \frac{m}{n}$ ***(R1)(AG)***

[5 marks]

(iv) (a) Since $h(x)$ is periodic with period 2π, the values will repeat every period, and hence the area under the curve equals the area under one cycle of the function multiplied by the number of cycles covered. Hence the required result. ***(R2)***

[2 marks]

(b) The maximum error is given by $\frac{(b-a)^3}{12n^2} h''(c)$, and $h''(x) = e^{\cos x}(\sin^2 x - \cos x)$. ***(A1)***

$f''(x)$ takes its maximum absolute value when $x = 0$ and $|h''(\text{o})| = e$. ***(R1)(A1)***

For the total error to be 0.15, the error over [0, 2π] must be less than 0.01. ***(M1)***

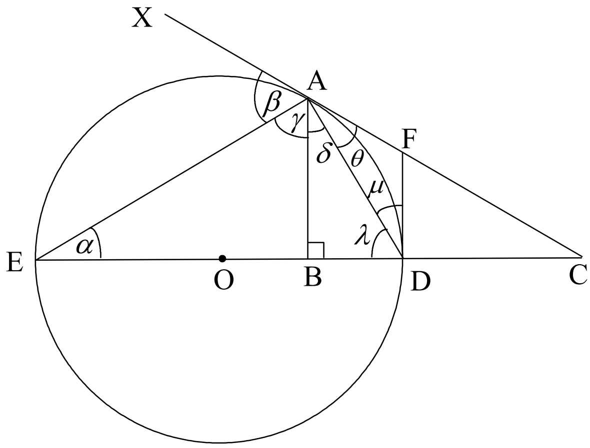
$\frac{(2\pi)^3 e}{12n^2} < 0.01 \Rightarrow n^2 > \frac{800\pi^3 e}{12} \Rightarrow n > 74.96$ ***(M1)***

$\Rightarrow n = 75$ ***(A1)***

[6 marks]

Total [30 marks]

10.



(a) Since [FA] and [FD] are tangents from the same point, then they are equal. ***(M1)***
Hence triangle FAD is isosceles. ***(R1)***

[2 marks]

(b) (i) From (a), $\theta = \mu$, also, $\delta = \mu$ since they are alternate interior angles for the two parallel lines (AB) and (FD). ***(M1)(R1)***
Hence $\delta = \theta$ and (AD) bisects angle BAC. ***(C1)***

[3 marks]

(ii) Triangle EAD is right angled at A because it is inscribed in a semi circle, hence γ and δ are complementary. ***(R1)***
$\beta + \gamma + \delta + \theta = \pi, \Rightarrow \beta + \theta$ are complementary. ***(M1)***
Hence $\beta = \gamma$ since they are complements to the equal angles, θ and δ. ***(R1)***
So, (AE) bisects angle BAX.

[3 marks]

(c) Since (AD) bisects angle BAC, then $\frac{\text{BD}}{\text{DC}} = \frac{\text{AB}}{\text{AC}}$ by the bisector theorem. ***(R1)(C1)***

Also, (AE) bisects angle BAX, then $\frac{\text{AB}}{\text{AC}} = \frac{\text{BE}}{\text{EC}}$ ***(C1)***

Hence $\frac{\text{BD}}{\text{DC}} = \frac{\text{BE}}{\text{EC}}$, and the result follows. ***(C1)***

[4 marks]

(d) $\frac{\text{BD}}{\text{DC}} = \frac{\text{BE}}{\text{EC}} \Rightarrow \frac{p}{\text{BC} - p} = \frac{q}{q + \text{BC}}$ ***(M1)***

$q\text{BC} - pq = pq + p\text{BC}$ ***(M1)***

$\Rightarrow \text{BC}(q - p) = 2pq$ ***(A1)***

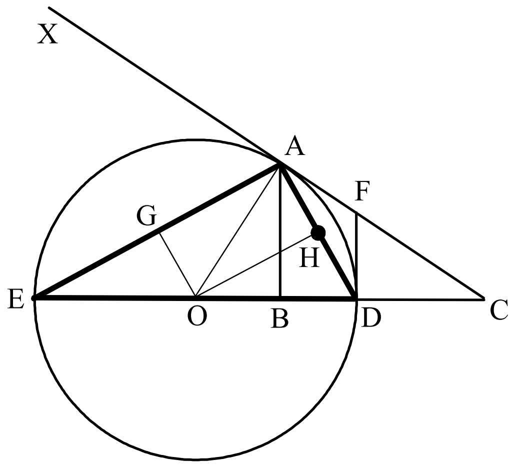
$\Rightarrow \text{BC} = \frac{2pq}{q - p}$ ***(AG)***

[3 marks]

continued…

Question 10 continued

(e) The nine-point circle must pass through the midpoints of the sides, the feet of the altitudes and the midpoints of the segments joining the orthocentre to the vertices. ***(C1)***
Let G and H be the midpoints of [AE] and [AD] respectively.



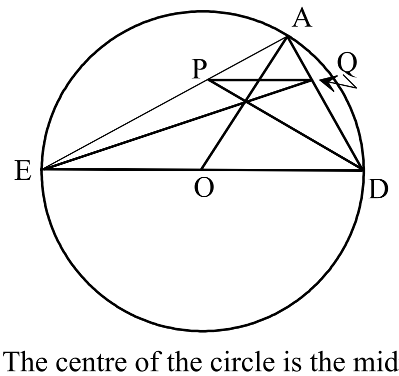
Since [AO] is the diameter, the midpoint O of one side of DAED is on this circle, the line joining the midpoints of two sides of a triangle is parallel to the third side. ***(C1)***
Therefore $(\text{OG}) \parallel (\text{DA})$.

Hence $\text{O}\hat{\text{G}}\text{A}$ is a right angle, and G is on the circle. H is also on the circle for the same reason. ***(R1)(C2)***

Since $\text{O}\hat{\text{B}}\text{A}$ is a right angle, B is on the circle. The other two altitudes are [DA] and [EA] so their feet are both at A and therefore on the circle. ***(R1)(C1)***
Since A is the orthocentre, the midpoint of A and A is A, and hence is on the circle.
The other two midpoints coincide with H and G, and hence are on the circle. ***(R1)***
Therefore the circle passes through all nine points as required.

[8 marks]

(f)



The centre of the circle is the midpoint of [OA]. ***(R1)***
By drawing the lines as requested, the lines (AO), (DP), and (EQ) will be Cevians of triangle AED. ***(R1)***
By Ceva's theorem,

$$\frac{\text{DO}}{\text{OE}} \times \frac{\text{EP}}{\text{PA}} \times \frac{\text{AQ}}{\text{QD}} = 1 \Rightarrow 1 \times \frac{\text{EP}}{\text{PA}} \times \frac{\text{AQ}}{\text{QD}} = 1$$, since O is the centre of circle. ***(R1)(M2)***

$$\Rightarrow \frac{\text{AQ}}{\text{QD}} = \frac{\text{AP}}{\text{PE}}$$ ***(C1)***

Therefore, (PQ) is parallel to (DE) ***(R1)***

[7 marks]

Total [30 marks]