INTERNATIONAL BACCALAURÉAT BACHILLERATO

BACCALAUREATE INTERNATIONAL - INTERNACIONAL

MARKSCHEME

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MATHEMATICS

Higher Level

Paper 2

26 pages

1.   (a)   Since the coordinates of the points P, Q and R are $(4, 1, -1)$, $(3, 3, 5)$ and $(1, 0, 2c)$, respectively, the vectors $\overrightarrow{QR}$ and $\overrightarrow{PR}$ are given by

$$\overrightarrow{QR} = -2\vec{i} - 3\vec{j} + (2c - 5)\vec{k}$$ *(M1)(A1)*

$$\overrightarrow{PR} = -3\vec{i} - \vec{j} + (2c + 1)\vec{k}$$ *(M1)(A1)*

$\overrightarrow{QR}$ is perpendicular to $\overrightarrow{PR}$ if and only if $\overrightarrow{QR} \cdot \overrightarrow{PR} = 0$ *i.e.* $6 + 3 + (2c - 5)(2c + 1) = 0$ *(M1)*

$$\Rightarrow \quad 4c^2 - 8c + 4 = 0$$

$$\Rightarrow \quad (c - 1)^2 = 0$$ *(M1)*

$$\Rightarrow \quad c = 1$$ *(A1)*

(b) $\overrightarrow{PR} = -3\vec{i} - \vec{j} + 3\vec{k}$, $\overrightarrow{PS} = -3\vec{i} + 3\vec{k}$ *(M1)(M1)*

$$\overrightarrow{PS} \times \overrightarrow{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 0 & 3 \\ -3 & -1 & 3 \end{vmatrix}$$ *(M1)*

$$= 3\vec{i} + 3\vec{k}$$ *(A1)*

(c)   The parametric equation of a line l which passes through the point $(3, 3, 5)$ and is parallel to the vector $\overrightarrow{PR}$ is given by

$$\vec{r} = (3\vec{i} + 3\vec{j} + 5\vec{k}) + t(-3\vec{i} - \vec{j} + 3\vec{k})$$ *(M1)(M1)*
$$= 3(1 - t)\vec{i} + (3 - t)\vec{j} + (5 + 3t)\vec{k} \quad (-\infty < t < \infty)$$ *(A1)*

Note:	If $-\infty < t < \infty$ is not mentioned, do not penalise.
	Also note that some candidates may give the parametric equation of the line in the form $x = 3(1 - t), y = 3 - t, z = (5 + 3t), -\infty < t < \infty$

Question 1 continued

(d) Let P_1 and P_2 be points on the line l corresponding to $t = 0$ and $t = 1$, respectively. Hence, for $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$,

$x = 3(1-t)$, $y = (3-t)$ and $z = 5 + 3t$.

Putting $t = 0$ and $t = 1$, we get the coordinates of points P_1 and P_2 as $(3,3,5)$ and $(0,2,8)$, respectively.

Vectors $\overrightarrow{SP_1}$ and $\overrightarrow{SP_2}$ are given by $\overrightarrow{SP_1} = 2\vec{i} + 2\vec{j} + 3\vec{k}$ and $\overrightarrow{SP_2} = -\vec{i} + \vec{j} + 6\vec{k}$. *(M1)*

A vector perpendicular to both $\overrightarrow{SP_1}$ and $\overrightarrow{SP_2}$ is $\overrightarrow{SP_1} \times \overrightarrow{SP_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 3 \\ -1 & 1 & 6 \end{vmatrix}$ *(M1)*

$$= 9\vec{i} - 15\vec{j} + 4\vec{k}$$ *(A1)*

Let $T(x,y,z)$ be any point of the plane π. Since $S = (1,1,2)$,

$\overrightarrow{ST} = (x-1)\vec{i} + (y-1)\vec{j} + (z-2)\vec{k}$ is a vector in π. Hence $\overrightarrow{TS} \cdot \vec{n} = 0$

i.e. $9(x-1) - 15(y-1) + 4(z-2) = 0 \Rightarrow 9x - 15y + 4z - 2 = 0$ *(A1)*

OR

$\overrightarrow{QS} = (1-3)\vec{i} + (1-3)\vec{j} + (2-5)\vec{k}$ *(M1)*

$= -2\vec{i} - 2\vec{j} - 3\vec{k}$ *(M1)*

The equation of the plane containing the line l and passing through the point S is determined by l and the vector $\overrightarrow{SQ}$. Hence, the equation is:

$$\vec{r} = \begin{pmatrix} 3 \\ 3 \\ 5 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ -1 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}$$ *(M1)(A1)*

(e) Shortest distance $\dfrac{\left| \overrightarrow{PQ} \cdot \vec{n} \right|}{|\vec{n}|} = \dfrac{\left| \begin{pmatrix} -1 \\ 2 \\ 6 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ -15 \\ 4 \end{pmatrix} \right|}{\sqrt{322}}$ *(M1)(M1)*

$$= \frac{15}{\sqrt{322}}$$ *(M1)(A1)*

OR

The distance of P from π is:

$$\frac{\left| 9(4) - 15(1) + 4(-1) - 2 \right|}{\sqrt{9^2 + 15^2 + 4^2}}$$ *(M1)(A1)*

$$= \frac{15}{\sqrt{322}}$$ *(M1)(A1)*

| **Note:** Accept 0.836 (3 s.f.) |

2.   (i)   Let the arithmetic sequence be written as $a, a+d, a+2d, \ldots$

Then $\dfrac{a+4d}{a+11d} = \dfrac{6}{13}$ *(M1)*

So $13a + 52d = 6a + 66d \Rightarrow 7a = 14d$ *(M1)*
$\Rightarrow a = 2d$.   *(A1)*

Since each term is positive, both a and d are positive. We are given $a(a+2d) = 32$,
setting $a = 2d$, we get $2d(2d+2d) = 8d^2 = 32$.   *(M1)*
$\Rightarrow d = \pm 2$.   *(A1)*

Hence, $d = 2$ and $a = 4$ and sum to 100 terms of this sequence is
$\dfrac{100}{2}\{(2)(4) + (100-1)2\}$.   *(M1)*

$= 10300$ *(A1)*

(ii) (a) Since ω is a complex number which satisfies $\omega^3 - 1 = 0$, $\omega \neq 1$. Hence,
$1 + \omega + \omega^2 = \dfrac{1 - \omega^3}{1 - \omega} = 0$.   *(M1)(A1)*

(b) $(\omega x + \omega^2 y)(\omega^2 x + \omega y) = \omega^3 x^2 + \omega^4 yx + \omega^2 xy + \omega^3 y^2$.   *(M1)*
Using $\omega^3 = 1$ and $\omega^4 = \omega$, we get,   *(M1)*
$(\omega x + \omega^2 y)(\omega^2 x + \omega y) = (x^2 + y^2) + (\omega^2 + \omega)xy$ *(M1)*
$\qquad\qquad\qquad\qquad\qquad = x^2 + y^2 - xy$, (Since $1 + \omega + \omega^2 = 0$)   *(A1)*

(iii) Let $S(n)$ be the statement: $2^{2n} - 3n - 1$ is divisible by 9.
Since $2^2 - 3 - 1 = 0$, $S(1)$ is true.   *(C1)*

Assume as the induction hypothesis $S(k)$ i.e. $2^{2k} - 3k - 1$ is divisible by 9.   *(C1)*

We shall show that $S(k+1)$ is true.
$S(k+1) = 2^{2(k+1)} - 3(k+1) - 1$ *(M1)*
$\qquad\quad = 4(2^{2k}) - 3k - 4$ *(M1)*
$\qquad\quad = 4(2^{2k} - 3k - 1) + 9k$ *(A1)*

By the induction hypothesis $2^{2k} - 3k - 1$ is divisible by 9. Since $9k$ is also divisible by
9, $S(k+1)$ is true.   *(M1)*
Thus, by mathematical induction $S(n)$ is true, $n = 1, 2, \ldots$ *(R1)*

3.   (i)   Let D be the event that the patient has the disease and S be the event that the new blood
          test shows that the patient has the disease. Let D' be the complement of D, *i.e.* the
          patient does not have the disease.
          Now the given probabilities can be written as
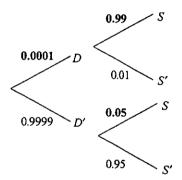 $$p(S|D) = 0.99, \ p(D) = 0.0001, p(S|D') = 0.05.$$ *(A1)(A1)(A1)*

          Since the blood test shows that the patient has the disease, we are required to find $p(D|S)$.
          By Bayes' theorem,

          $$p(D|S) = \frac{p(S|D)p(D)}{p(S|D)p(D) + p(S|D')p(D')}$$ *(M1)*

          $$= \frac{(0.99)(0.0001)}{(0.99)(0.0001) + (0.05)(1 - 0.0001)}$$ *(M1)*

          $$= 0.001976... = 0.00198 \text{ (3 s.f.)}$$ *(A1)*

OR

                                                                                         (A3)

Note: Award *(A1)* for 0.99, *(A1)* for 0.0001, *(A1)* for 0.05

          Therefore $p(S) = 0.0001 \times 0.99 + 0.9999 \times 0.05$
                              $= 0.0500939$ *(A1)*
          $$p(D|S) = \frac{0.0001 \times 0.99}{0.0500939}$$ *(M1)*
                              $= 0.00198 \text{ (3 s.f.)}$ *(A1)*

     (ii) Let $n =$ number of chips $= 1000$,
          $p =$ the probability that a randomly chosen chip is defective $= 0.02$.
          Hence, the mean $np = (1000)(0.02) = 20$ and the
          variance $= np(1 - p) = (1000)(0.02)(0.98) = 19.6$.   *(A1)(A1)*
          Suppose X is the normal random variable that approximates the binomial distribution.
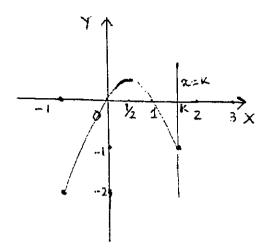   The $X \approx N(20, 19.6)$.   *(M1)(M1)*
          Thus $p(19.5 \leq X \leq 30.5) = p\left(\frac{19.5 - 20}{\sqrt{19.6}} \leq Z \leq \frac{30.5 - 20}{\sqrt{19.6}}\right)$ *(M1)(M1)*
                     $= p(-0.11 \leq Z \leq 2.37)$
                     $= 0.5349$ *(A1)*

Note: Line before last should be $p(-0.113 \leq Z \leq 2.37) = 0.536$. Accept 0.535 or 0.536
If student's work is not shown but there is evidence that he/she used the calculator to find the answer, accept the answer.

4. (i) (a)

Notes: Award *(A1)* for the correct intercepts
         (A1) for graphing over the correct interval
         (A1) for the correct x - coordinate of the maximum point.

(b)   Required area $= \int_0^1 (x - x^2)\,dx - \int_1^k (x - x^2)\,dx$ *(M1)*

$$= \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 - \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^k$$ *(A1)*

$$= \frac{1}{6} - \frac{k^2}{2} + \frac{k^3}{3} + \frac{1}{6} = \frac{1}{3} + \frac{k^3}{3} - \frac{k^2}{2}$$ *(M1)*

$$= \frac{1}{6}(2 + 2k^3 - 3k^2)$$ *(A1)*

OR

Required Area $= \left| \int_1^k (x - x^2)\,dx \right|$ *(M1)*

$$= \left| \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_1^k \right|$$ *(M1)(A1)*

$$= \frac{k^3}{3} - \frac{k^2}{2} + \frac{1}{6}$$ *(A1)*

Question 4 continued

(ii) (a) $g(t) = \dfrac{\ln t}{\sqrt{t}}$. So $g'(t) = \dfrac{2 - \ln t}{2t^{3/2}}$. *(M1)(A1)*

Hence, $g'(t) > 0$ when $2 > \ln t$ or $\ln t < 2$ or $t < e^2$. *(M1)*

Since the domain of $g(t)$ is $\{t : t > 0\}$, $g'(t) > 0$ when $0 < t < e^2$. *(A1)*

(b) Since $g'(t) = \dfrac{2 - \ln t}{2t^{3/2}}$, $g''(t) = \dfrac{-2\sqrt{t} - 3\sqrt{t}(2 - \ln t)}{4t^3}$ *(M2)*

$= -\dfrac{\sqrt{t}\,[8 - 3\ln t]}{4t^3}$ *(A1)*

Hence $g''(t) > 0$ when $8 - 3\ln t < 0$ *i.e.* $t > e^{8/3}$. *(M1)*
Similarly, $g''(t) < 0$ when $0 < t < e^{8/3}$. *(A1)*

(c) $g''(t) = 0$ when $t = 0$ or $8 = 3\ln t$.
Since, the domain of g is $\{t : t > 0\}$, $g''(t) = 0$ when $t = e^{8/3}$. *(M1)*
Since $g''(t) > 0$ when $t > e^{8/3}$ and $g''(t) < 0$ when $t < e^{8/3}$, *(M1)*
$\left(e^{8/3}, \dfrac{8}{3}e^{-4/3}\right)$ is the point of inflexion. The required value of t is $e^{8/3}$. *(A1)*

> **Note:** Award *(A1)* for evaluating t as $e^{8/3}$.

(d) $g'(t) = 0$ when $\ln t = 2$ or $t = e^2$. *(M1)*
Also $g''(e^2) = -\dfrac{\sqrt{e^2}\,[8 - 3\ln e^2]}{4e^6} = -\dfrac{1}{2e^5} < 0$. *(M1)*
Hence $t^* = e^2$ *(A1)*

(e) At $\left(t^*, g(t^*)\right)$ the tangent is horizontal. *(M1)*
So the normal at the point $\left(t^*, g(t^*)\right)$ is the line $t = t^*$. *(M1)*
Thus, it meets the t axis at the point $t = t^* = e^2$ and hence the point is $(e^2, 0)$. *(A1)*

5. **(i)** **(a)** S is the group of permutations of $\{1,2,3\}$ under the composition of permutation. *(M1)*
Since $3! = 6$, order of $S = 6$. *(R1)*

(b) Members of S are $p_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $p_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $p_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, *(AG)*

$$p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \ p_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}.$$ *(A2)*

> **Note:** Award *(A2)* for 3 correct permutations;
> *(A1)* for 2 correct permutations;
> *(A0)* for 1 correct permutation.

Since $p_3 \circ p_4 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

and $p_4 \circ p_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, *(M1)*

$p_3 \circ p_4 \neq p_4 \circ p_3$ *(R1)*

> **Note:** There are other possibilities to show that the group is not Abelian.

(c) $p_1^2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = p_2$

$p_1^3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = p_0.$ *(M1)*

(Note that p_0 is the identity of the group S.)

Hence $\{p_0, p_1, p_2\}$ form a cyclic group of order 3 under the binary operation of composition of permutations. *(R1)*

> **Note:** Some candidates may write $\{p_0, p_1, p_2\}$ is a subgroup of order 3, (award *(A1)*), and write the following table, (award *(R1)*):
>
> | $\circ$ | p_0 | p_1 | p_2 |
> |---|---|---|---|
> | p_0 | p_0 | p_1 | p_2 |
> | p_1 | p_1 | p_2 | p_0 |
> | p_2 | p_2 | p_0 | p_1 |

Question 5 continued

(ii)   (a)   Let $A = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 + b^2 \neq 0 \right\}$.

Let $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$ be two elements of A. Then

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & ad + bc \\ -bc - ad & -bd + ac \end{pmatrix}$$ *(M1)*

$$= \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \text{ if we write } \alpha = ac - bd, \beta = ad + bc.$$ *(R1)*

Since α, β are real numbers A is closed under matrix multiplication.
Associativity follows from the fact that matrix multiplication is an associative binary operation on the collection of all 2×2 matrices and the fact that A is closed with respect to the binary operation of multiplication of matrices.   *(R1)*

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in A$ and is the identity since $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ *(M1)(M1)*

for any element $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ of A.   *(R1)*

The determinant $\begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2 + b^2 \neq 0$.   *(M1)*

Hence the matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ has an inverse, which is $\dfrac{1}{a^2 + b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ *(R1)(A1)*

since $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\left(\dfrac{1}{a^2 + b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$$\frac{1}{a^2 + b^2}\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \frac{1}{a^2 + b^2}\begin{pmatrix} a^2 + b^2 & 0 \\ 0 & a^2 + b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ *(R1)*

Note: Some candidates may use $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x & y \\ -y & x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to find the inverse of $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.
Award marks as appropriate.

Question 5 (ii) continued

(b) $M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$

M contains the identity $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.   *(A1)*

The following results show that the set M is closed with respect to matrix multiplication as the binary operation.   *(R1)*

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *(M2)*

Since the product of each element with itself is the identity, every element of M is its own inverse. Hence M is a group.   *(R1)*

(c)   The order of the group M is 4. If it had a subgroup of order 3, then by the corollary of the Lagrange's theorem, 3 must be a divisor of 4.   *(M1)*
Since it is not true, M can not have a subgroup of order 3.   *(R1)*

(iii) (a)   Let $(G, \circ)$ and $(H, \bullet)$ be two groups. They are said to be isomorphic if there exists a one-to-one transformation $f : G \to H$ which is surjective (onto) with the   *(C1)*
property that for all $x, y \in G$, $f(x \circ y) = f(x) \bullet f(y)$.   *(C1)*

> **Note:** Some candidates may say that the groups $(G, \circ)$ and $(H, \bullet)$ are isomorphic if they have the same Cayley table (or group table). In that case award *(C1)*.

(b)   Since $f : G \to H$, $f(x) \in H$ for some $x \in G$.
Since e' is the identity element in H,   *(M1)*
$e' \bullet f(x) = f(x) = f(x \circ e) = f(e) \bullet f(x)$.   *(M1)(A1)*
By the right cancellation law, $e' = f(e)$.   *(R1)*

Question 5 (iii) continued

(c)   Suppose $G =< a >$, the cyclic group generated by a, *i.e.* n is the smallest positive
      integer such that $a^n = e$, the identity in G.   *(C1)*
      Let $f : G \rightarrow H$ be an isomorphism. Let $f(a) = b \in H$.

      $f(a^2) = f(a \circ a) = f(a) \bullet f(a) = \left(f(a)\right)^2$.   *(M1)*

      In general $f(a^m) = \left(f(a)\right)^m, 1 \leq m \leq n$.   *(A1)*

      By (iii) (b) $\left(f(a)\right)^n = e'$, the identity in H. Hence $b^n = e'$ and consequently H is
      a cyclic group of order n with generator b.   *(R1)*

(d) $\mathbb{Z}_4 = \{[0], [1], [2], [3]\}$. Let $\oplus_4$ be the binary operation.   *(M1)*
      $[1] \oplus_4 [1] = [2], [1] \oplus_4 [2] = [3], [1] \oplus_4 [3] = [0]$ *(M1)(A1)*
      Hence [1] is a generator of the group $(\mathbb{Z}_4, \oplus_4)$ and $\mathbb{Z}_4$ is cyclic.
      ([3] is also a generator)
      Since each element of M is of order 2, M is not cyclic.   *(M1)*
      Hence $\mathbb{Z}_4$ is not isomorphic to M.   *(R1)*

6. **(i)** **(a)** If G is a directed graph with n vertices, then an adjacency matrix of G is an $n \times n$ matrix whose (i, j) entry is the number of directed edges from the vertex v_i to the vertex v_j, or zero if there is no directed edge connecting v_i and v_j. *(C2)*
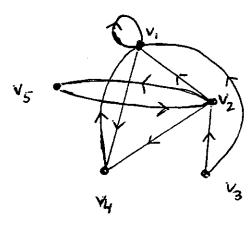
(b) The sum of the entries in row i of the adjacency matrix of the directed graph G is equal to the outdegree of the vertex v_i of G, *i.e.* the number of directed edges from vertex v_i. *(C2)*

(c) The sum of the entries in column j equals the indegree of the vertex j *i.e.* the number of directed edges to vertex v_j. *(C2)*

(ii) **(a)**

Graph H *(A3)*

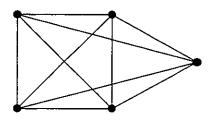| **Note:** Award *(A3)* for a correct graph, *(A2)* for a graph with 1 error; *(A1)* for a graph with 2 errors, *(A0)* for a graph with 3 or more errors |
| --- |

(b) Since the element in the second row and the fourth column of A^2 is 1, there is only one path of length 2 from v_2 to v_4. *(M1)(R1)*
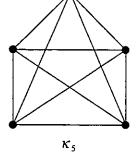
(iii) **(a)** **(i)** A complete graph κ_5 is a graph on 5 vertices where every vertex is connected to every other vertex. *(A2)*

Example: or

κ_5 κ_5 *(A1)*
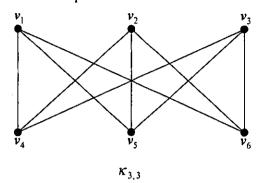
Question 6 (iii) continued

(ii) In $\kappa_{3,3}$ vertices are divided into two disjoint sets A and B, each having three vertices such that no two vertices in A are adjacent and no two vertices in B are adjacent, but every vertex of A is adjacent to every vertex of B. *(A1)*
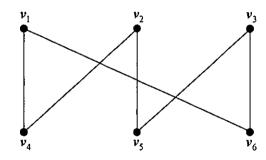
Example:

v_1 v_2 v_3

v_4 v_5 v_6

$\kappa_{3,3}$

(A1)

(b) To find a Hamiltonian circuit in $\kappa_{3,3}$, we show that there is a circuit which goes through each vertex exactly once. Such a circuit is shown below: *(M1)*
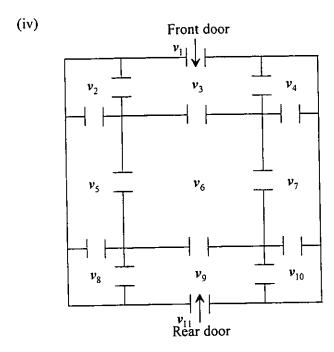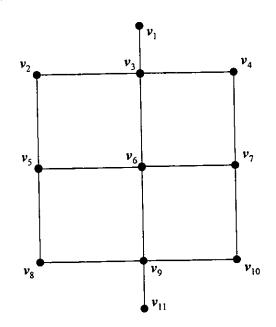
v_1 v_2 v_3

v_4 v_5 v_6

(M1)(A1)

Note:	Please check the circuit drawn by the candidate and award *(M2)(A1)* for a correct answer. Also note that some candidates may not show the Hamiltonian circuit but mention that each vertex of $\kappa_{3,3}$ has degree $3 = \dfrac{6}{2}$. Hence there is a Hamiltonian circuit. In this case award *(M2)(A1)*.

Question 6 continued

(iv)

Front door

Let the front door, rear door and each room of the house be labelled as shown. We take each of these as a vertex, and draw edges when they are connected by a door. The graph of the floor plan is: *(M1)*

The degree sequence for this graph $v_1, v_2, \ldots, v_{11}$ is $1, 2, 4, 2, 3, 4, 3, 2, 4, 2, 1$. *(M1)(A1)*

The degree of more than two vertices is odd and hence there is no Eulerian path. *(R1)*

Hence it is not possible to enter the house through the front door and exit at the rear door, travelling through the house going through each internal doorway exactly once. *(R1)*

continued...

Question 6 continued

(v) (a) The degree sequence yields the number of edges at each vertex and an isomorphism preserves adjacency of vertices. *(M1)*
Hence, if two groups are isomorphic and the degree sequences are not the same then the isomorphism can not preserve the adjacency of vertices contradicting our observation. *(M1)*
Thus an isomorphism must preserve degree sequences. *(R1)*

(b) The degree sequences of the graphs G and H are $2,3,3,3,3,4$ and $2,3,3,3,5,2$ respectively. *(M1)(M1)*
Since they are not the same, the two graphs are not isomorphic by (a). *(R1)*

(vi) We start with point A and write S as the set of vertices and T as the set of edges. The weights on each edge will be used in applying Prim's algorithm.
Initially, $S = \{A\}, T = \phi$. In each subsequent stage, we shall update S and T.

Step 1:	Add edge h:	So $S = \{A, D\}$,	$T = \{h\}$
Step 2:	Add edge e:	So $S = \{A, D, E\}$	$T = \{h, e\}$
Step 3:	Add edge d:	Then $S = \{A, D, E, F\}$	$T = \{h, e, d\}$
Step 4:	Add edge a:	Then $S = \{A, D, E, F, B\}$	$T = \{h, e, d, a\}$
Step 5:	Add edge i:	Then $S = \{A, D, E, F, B, G\}$	$T = \{h, e, d, a, i\}$
Step 6:	Add edge g:	Then $S = \{A, D, E, F, B, G, C\}$	$T = \{h, e, d, a, i, g\}$

(M4)(A3)

> **Notes:** Award *(M4)(A3)* for all 6 correct, *(M4)(A2)* for 5 correct;
> *(M3)(A2)* for 4 correct, *(M3)(A1)* for 3 correct;
> *(M1)(A1)* for 2 correct, *(M1)(A0)* for 1 correct
> **OR**
> *(M2)* for the correct definition of Prim's algorithm,
> *(M2)* for the correct application of Prim's algorithm,
> *(A3)* for the correct answers at the last three stages.

Now S has all the vertices and the minimal spanning tree is obtained.
The weight of the edges in T is $5 + 3 + 5 + 7 + 5 + 6$
$$= 31$$ *(A1)*

7.

> **Note:** Throughout the whole question, students may be using their graphic display calculators and should not be penalized if they do not show as much work as the marking scheme.

(i) (a) Let X denote the number of flaws in one metre of the wire. Then $E(X) = 2.3$ *(M1)(M1)*

flaws and $P(X = 2) = e^{-2.3} \dfrac{(2.3)^2}{2!}$

$= 0.265$.   *(A1)*

> **Note:** Award *(C3)* for a correct answer from a graphic display calculator.

(b) Let Y denote the number of flaws in two metres of wire. Then Y has a Poisson distribution with mean $E(Y) = 2 \times 2.3 = 4.6$ flaws for 2 metres.   *(M1)*
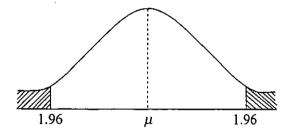
Hence, $P(Y \geq 1) = 1 - P(Y = 0) = 1 - e^{-4.6}$ *(M1)*

$= 0.990$ (3 s.f.)   *(A1)*

> **Note:** Accept $1 - e^{-4.6}$.

(ii) Let $\bar{x}$ be the sample mean, μ be the population mean and σ be the population standard deviation.

Hence, $\sigma_{\bar{x}}$, the standard error is given by $\sigma_{\bar{x}} = \dfrac{\sigma}{\sqrt{n}} = \dfrac{4}{\sqrt{n}}$, where n is the sample size   *(M1)(A1)*

which is required.

The 95 % confidence interval of the population mean μ is given by $(\bar{x} - \mu) \pm 1.96 \sigma_{\bar{x}}$.   *(M1)(A1)*

Hence, we need to find n so that $(\sigma_{\bar{x}})(1.96) \leq 0.25$ *(M1)(R1)*

$\Rightarrow \left(\dfrac{4}{\sqrt{n}} \right)(1.96) \leq 0.25$

$\Rightarrow \left(\dfrac{1}{\sqrt{n}} \right) \leq \dfrac{0.25}{4(1.96)} \Rightarrow \sqrt{n} \geq \dfrac{4(1.96)}{0.25} = 31.36$ *(M1)(M1)*

Hence, $n \geq 983.44$ *(A1)*

The sample size required is 984.   *(R1)*

Question 7 continued

(iii) Let μ_1 and μ_2 be the weekly mean wage of science and humanities students respectively. The hypotheses to be tested are

$H_0: \mu_1 - \mu_2 = 0$ *(A1)*

$H_1: \mu_1 - \mu_2 \neq 0$ *(A1)*

It is a two-tailed test. The test statistic is $\bar{x}_1 - \bar{x}_2$. The two random variables are independent *i.e.* probability of selection of a science student is not affected by the selection of a humanity student.
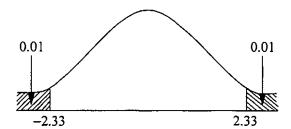
$$\mu_{\bar{x}_1 - \bar{x}_2} = 0, \sigma_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where σ_1, σ_2 are the population standard deviations of science and humanity students respectively. Since the samples are large one can approximate σ_1, σ_2 by s_1, s_2, respectively. The resulting estimated standard error is given by

$$s_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$$ *(M2)*

$$= \sqrt{\frac{4^2}{100} + \frac{(4.5)^2}{200}}$$ *(A1)*

$$= 0.51...$$ *(A1)*

From the normal distribution table we obtain the decision rule that if the computed value of $z > 2.33$ or $z < -2.33$ then we reject H_0. *(M1)(A1)*

But $z = \dfrac{\bar{x}_1 - \bar{x}_2}{s_{\bar{x}_1 - \bar{x}_2}}$ *(M1)*

$$= \frac{\$ 120.5 - 115}{0.51}$$ *(M1)*

$$= 10.784 > 2.33$$ *(A1)*

So we reject H_0. *(R1)*

Thus, we conclude the wages of science and humanity students are not the same.

Notes:	An alternative approach by calculating critical values for a rejection region is possible. Please check the working and award marks accordingly. Award full marks for answers obtained using a graphic display calculator with a p-value of 5.4×10^{-27} (exponent is -27), a z-value of 10.761, and a correct conclusion.

Question 7 continued

(iv) H_0: the cost of the automobile is independent of the number of complaints.
       H_1: the cost of an automobile is not independent of the number of complaints. *(A1)*

The observed frequencies are first totalled, and then the expected frequencies under H_0 are calculated from the formula:

$$\text{expected frequency} = \frac{(\text{row total})(\text{column total})}{\text{grand total}}$$

Cost	Number of complaints			Total
	≤ 5	6–10	≥ 11	
$\geq \$30001$	**60**	**120**	**20**	200
$\$15001 - \$30000$	**138**	**276**	**46**	460
$\leq \$15000$	**102**	**204**	**34**	340
Total	300	600	100	1000

(A4)

Table of expected frequencies

Note:	Award *(A4)* for 9 correct bold entries
	(A3) for 7 or 8 correct bold entries
	(A2) for 5 or 6 correct bold entries
	(A1) for 4 correct bold entries
	(A0) for 3 or less bold entries

$$\chi^2 = \sum \left\{ \frac{(f_o - f_e)^2}{f_e} \right\}$$

(M1)

We shall calculate χ^2 from a table where we shall list $f_o, f_e, (f_o - f_e)^2, \dfrac{(f_o - f_e)^2}{f_e}$, where f_o is the observed number of complaints and f_e is the expected number of complaints.

Question 7 (iv) continued

f_o	f_e	$f_o - f_e$	$(f_o - f_e)^2$	$\dfrac{(f_o - f_e)^2}{f_e}$
100	60	40	1600	26.67
90	120	−30	900	7.5
10	20	−10	100	5.0
150	138	12	144	1.04
260	276	−16	256	0.93
50	46	4	16	0.35
50	102	52	2704	26.51
250	204	46	2116	10.37
40	34	6	36	1.06
Total 1000	1000			$\chi^2 = \sum \dfrac{(f_o - f_e)^2}{f_e} = 79.43$

(A3)

Calculation of χ^2

v = number of degrees of freedom = $(3-1)(3-1) = 4$ *(M1)*

At the 5 % level of significance, $\chi^2_{0.05}$ with four degrees of freedom, $\chi^2_{0.05} = 9.49$ *(A1)*

<u>Decision rule:</u> If $\chi^2 > 9.49$, reject H_0. If $\chi^2 \leq 9.49$, then accept H_0.

Our computed $\chi^2 = 79.43 > 9.49$ and hence according to our decision rule, we reject H_0.

<u>Conclusion:</u> The cost of the automobile is not independent of the number of complaints.   *(R1)*

Note: Award *(M1)(R1)* for stating: $\chi^2_{0.05} = 9.49 < 79.43$, hence reject H_0.

8. **(i)** **(a)** Let $f(x) = x^3 - 3x - 5$. Hence $f(1) = -7 < 0$ and $f(3) = 13 > 0$. *(M1)*

Hence, by the intermediate value theorem there is one zero of $f(x) = x^3 - 3x - 5$ *(R1)*
in the interval $1 \le x \le 3$.

(b) $f(x) = x^3 - 3x - 5$
$f'(x) = 3x^2 - 3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 3x_n - 5}{3x_n^2 - 3}$$ *(M1)(A1)*

n	x_n	x_{n+1}	$\lvert x_{n+1} - x_n \rvert$
0	2	2.333333	0.333333
1	2.333333	2.280556	0.052777
2	2.280556	2.279020	0.001536
3	2.279020	2.279019	1×10^{-6}
4	2.279019	2.279019	1×10^{-6}

(A3)

Note: Award *(A3)* for any indication that the Newton-Raphson method has been correctly used.

Since we need accuracy 10^{-5}, we need $\lvert x_n - x_{n+1} \rvert < 5 \times 10^{-6}$.
So $x = 2.27902$ is the solution of $f(x) \approx 0$. *(R1)*

(ii) **(a)** Formula for the trapezium rule with $n = 6$ gives,

$$\int_0^1 g(x)\,dx = \frac{1}{12}[g(x_0) + 2g(x_1) + 2g(x_2) + 2g(x_3) + 2g(x_4) + 2g(x_5) + g(x_6)]$$ *(M2)*

$$= 0.745$$ *(A1)*

Formula for Simpson's rule gives,

$$\int_0^1 g(x)\,dx = \frac{1}{18}[g(x_0) + 4g(x_1) + 2g(x_2) + 4g(x_3) + 2g(x_4) + 4g(x_5) + g(x_6)]$$ *(M2)*

$$= 0.747$$ *(A1)*

(b) $\lvert \text{Error by the Simpson's rule} \rvert \le \max_{0 \le x \le 1} \lvert g^{(4)}(x) \rvert \left(\frac{1}{180}\right)\left(\frac{1}{6}\right)^4$ *(M1)(A1)*

Question 8 (ii) continued

(c)   Let $S_n(g)$ be the approximation to $\int_0^1 g(x)\mathrm{d}x$ by Simpson's rule over n intervals of equal length.

So $\left| \int_0^1 g(x)\mathrm{d}x - S_n(g) \right| \le \dfrac{6}{(180)n^4}$ *(M1)*

$= \dfrac{1}{30n^4}$,   *(A1)*

Since $\left| g^{(4)}(x) \right| \le 6$ for $0 \le x \le 1$.

To be correct to 5 d.p., we need to find n so that   *(M1)*

$\dfrac{1}{30n^4} \le 5 \times 10^{-6}$ *(M1)*

i.e. $30n^4 \ge \dfrac{10^6}{5}$ or $n^4 \ge \dfrac{10^6}{150}$ or $n \ge \sqrt[4]{\dfrac{10^6}{150}} = 9.036$ *(M1)*

Hence, we need 10 intervals.   *(A1)*

(iii) (a) If $g(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$, then there exists a ξ in $a < \xi < b$ so that   *(A1)*
$g(b) - g(a) = (b-a)g'(\xi), a < \xi < b$.   *(A1)*

(b) By the mean value theorem $h(x) - h(0) = h'(c)(x - 0)$, for $0 < c < 7$.   *(M1)*

Therefore $\left| h(x) - h(0) \right| \le 10x$, for $0 \le x \le 7$, since $\left| h'(x) \right| \le 10$.   *(A1)*

Thus $-70 + h(0) \le h(x) \le h(0) + 70$ *(M1)*

$\Rightarrow \qquad -70 - 4 \le h(x) \le -4 + 70$

$\Rightarrow \qquad\quad -74 \le h(x) \le 66$

Hence $h(x) \ge -74$, for $0 \le x \le 7$.   *(A1)*

(iv) (a) Let $u_k = \dfrac{k+1}{3^k}$.

Then $\lim_{k \to \infty} \dfrac{u_{k+1}}{u_k} = \lim_{k \to \infty} \dfrac{(k+2)}{3^{k+1}} \times \dfrac{3^k}{(k+1)} = \dfrac{1}{3} < 1.$ *(M2)*

and hence the series converges by the ratio test.   *(R1)*

Question 8 (iv) continued

(b)   Let $f(x) = \dfrac{1}{x(\ln x)^3}$

Since $x(\ln x)^3$ is an increasing function of x for $x > 1$, $f(x)$ is a positive decreasing function of x for $x > 1$.   *(M1)*

Further, $f(k) = \dfrac{1}{k(\ln k)^3}$

Hence, by the integral test, $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k(\ln k)^3}$ converges or diverges according to the

convergence or divergence of $\displaystyle\int_{2}^{\infty} \frac{1\,dx}{x(\ln x)^3} = \lim_{R \to \infty} \int_{2}^{R} \frac{dx}{x(\ln x)^3}$ *(M1)*

$= \displaystyle\lim_{R \to \infty} \left[\frac{(\ln x)^{-2}}{-2} \right]_{2}^{R} = \lim_{R \to \infty} \left\{ -\frac{1}{2(\ln R)^2} + \frac{1}{2(\ln 2)^2} \right\} = \frac{1}{2(\ln 2)^2}$ *(A1)*

So the series converges, by the integral test.   *(R1)*

(c) $\displaystyle\sum_{k=1}^{\infty} (-1)^{k+1} \frac{k}{k^2 + 1}$ is an alternating series.

When $f(x) = \dfrac{x}{x^2 + 1}$, $f'(x) = \dfrac{1 - x^2}{(x^2 + 1)^2} \le 0$,   *(M1)(A1)*

when $x \ge 1$, the sequence $\left\{ \dfrac{k}{k^2 + 1} \right\}$ is a decreasing sequence.

Also $\displaystyle\lim_{k \to \infty} \frac{k}{k^2 + 1} = \lim_{k \to \infty} \frac{1}{k + \dfrac{1}{k}} = 0$.   *(M1)(A1)*

Hence the series converges by the alternating series test.   *(R1)*