1. (a) $(\vec{i}+\vec{j})\times(\vec{i}-2\vec{k}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 1 & 0 & -2 \end{vmatrix}$ *(M1)*

$= \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 0 \\ 1 & -2 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}\vec{k}$ *(M1)*

$= -2\vec{i} + 2\vec{j} - \vec{k}$ *(A1)*

OR $(\vec{i}+\vec{j})\times(\vec{i}-2\vec{k}) = \vec{i}\times\vec{i} - 2\vec{i}\times\vec{k} + \vec{j}\times\vec{i} - 2\vec{j}\times\vec{k}$ *(M1)*

$= 0 - 2(-\vec{j}) + (-\vec{k}) - 2\vec{i}$ *(M1)*

$= -2\vec{i} + 2\vec{j} - \vec{k}$ *(A1)*

(b) (i) $\vec{n}_1 = 6\vec{i} + 3\vec{j} - 2\vec{k}$ and $\vec{n}_2 = -2\vec{i} + 2\vec{j} - \vec{k}$ *(A2)*

(ii) If θ is the required (acute) angle, then

$\cos\theta = \dfrac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1||\vec{n}_2|}$ *(M1)*

$= \dfrac{|-12+6+2|}{(7)(3)}$

$= \dfrac{4}{21}$ *(A1)*

Therefore, the required angle is 79.0° *(A1)*

(c) The equation of L_2 is $\dfrac{x+30}{2} = \dfrac{y+39}{1} = \dfrac{z-\frac{2}{3}}{-\frac{2}{3}}$ *(M1)*

Thus, the direction of L_2 is $\vec{l}_2 = 2\vec{i} + \vec{j} - \frac{2}{3}\vec{k} = \frac{1}{3}\vec{n}_1$ *(A2)*

Therefore L_2 is normal to plane P_1 *(AG)*

continued...

Question 1 continued

(d) (i) L_2 has parametric equations

$x = 2u - 30$, $y = u - 39$, $z = \frac{2}{3}(1-u)$, where u is a parameter

L_2 meets P_1 when $6(2u-30) + 3(u-39) - 2\left(\frac{2}{3}(1-u)\right) = 12$ *(M1)*

$\Rightarrow u = 19$ *(A1)*

Thus, the required coordinates are $(8, -20, -12)$ *(A1)*

(ii) The point found in part (i) lies on L_1 since when $t = 1$, $\bar{r} = 8\bar{i} - 20\bar{j} - 12\bar{k}$ *(M1)*

This point lies on P_2 since when $\mu = 6$ and $\lambda = -20$,
$\bar{r} = 22\bar{i} - 20(\bar{i} + \bar{j}) + 6(\bar{i} - 2\bar{k}) = 8\bar{i} - 20\bar{j} - 12\bar{k}$ *(M1)*

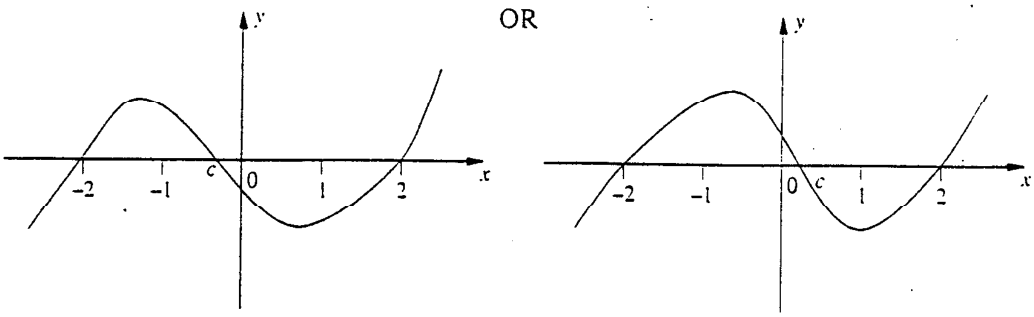
Thus, the two lines and two planes have the point $(8, -20, -12)$ in common. *(AG)*

2. (a) $f(x) = x^3 - cx^2 - 4x + 4c$

$\Rightarrow f(c) = c^3 - c(c^2) - 4c + 4c = 0$ for all c *(M2)*

(b) $f(x) = (x - c)(x^2 - 4) = (x - c)(x - 2)(x + 2)$ *(M2)(A1)*

(c)



Note: Award *(A1)* for each zero of the function. Award *(A2)* for the shape.

(A5)

(d) The required area is

$A = \int_{-2}^{c} f(x)\,dx - \int_{c}^{2} f(x)\,dx$ *(M1)(A1)*

$= \left[\frac{1}{4}x^4 - \frac{1}{3}cx^3 - 2x^2 + 4cx\right]_{-2}^{c} - \left[\frac{1}{4}x^4 - \frac{1}{3}cx^3 - 2x^2 + 4cx\right]_{c}^{2}$ *(M1)(A1)*

$= \frac{1}{4}(c^4 - 16) - \frac{1}{3}c(c^3 + 8) - 2(c^2 - 4) + 4c(c + 2)$

$\quad - \frac{1}{4}(16 - c^4) + \frac{1}{3}c(8 - c^3) + 2(4 - c^2) - 4c(2 - c)$

$= -\frac{1}{6}c^4 + 4c^2 + 8$, as required *(A1)(AG)*

continued...

Question 2 continued

(e) $\frac{dA}{dc} = -\frac{2}{3}c^3 + 8c$ *(M1)*

$= -\frac{2}{3}c(c^2 - 12)$

$= 0$ when $c = 0, \pm\sqrt{12}$ *(A1)*

But $-2 < c < 2$, so $c = 0$ only *(R1)*

Now, $\frac{d^2A}{dc^2} = 8 > 0$ when $c = 0$ *(M1)*

Thus, A is a minimum when $c = 0$ *(R1)*

(f) Required volume $= \int_0^2 \pi(x^3 - 4x)^2 dx$ *(M1)(A1)*

$= \pi\int_0^2 (x^6 - 8x^4 + 16x^2)dx$

$= \pi\left[\frac{1}{7}x^7 - \frac{8}{5}x^5 + \frac{16}{3}x^3\right]_0^2$ *(M1)*

$= \frac{64}{105}\pi[30 - 84 + 70]$

$= \frac{1024\pi}{105}$ *(A1)*

Note:	A calculator may be used to give the answer 30.6 which is correct to 3 significant figures. Award *(M2)(A2)* for this

3. (i) (a) $p(\text{neither bulb is defective}) = \dfrac{\binom{9}{2}}{\binom{12}{2}}$ *(M2)*

$= \dfrac{36}{66}$ *(A1)*

$= \dfrac{6}{11}$, as required *(AG)*

OR $p(\text{neither bulb is defective}) = \dfrac{9}{12} \times \dfrac{8}{11} = \dfrac{6}{11}$ *(M2)(A1)(AG)*

(b) $p(\text{exactly 1 bulb is defective}) = \dfrac{\binom{3}{1}\binom{9}{1}}{\binom{12}{2}}$ *(M3)*

$= \dfrac{(3)(9)}{66}$

$= \dfrac{9}{22}$ *(A1)*

OR $p(\text{exactly 1 bulb is defective}) = 2 \times p(\text{1st defective, 2nd non-defective})$ *(M2)*

$= 2 \times \dfrac{3}{12} \times \dfrac{9}{11} = \dfrac{9}{22}$ *(M1)(A1)*

(c) $p(\text{3rd bulb is non-defective} \mid \text{exactly 1 of 1st two is defective}) = \dfrac{8}{10}$ *(M1)*

$= \dfrac{4}{5}$ *(A1)*

(d) $p(\text{packet is accepted}) = \dfrac{6}{11} + \dfrac{9}{22} \times \dfrac{4}{5}$ *(M3)*

$= \dfrac{96}{110}$

$= \dfrac{48}{55}$ *(A1)*

continued...

Question 3 continued

(e) Required probability $= \dfrac{\frac{9}{22} \times \frac{4}{5}}{\frac{48}{55}}$ *(M3)*

$= \dfrac{3}{8}$ *(A1)*

(ii) (a) $\sum_{x=1}^{n} p(X = x) = 1 \Rightarrow k \sum_{x=1}^{n} x = 1$ *(M1)(A1)*

$\Rightarrow k \dfrac{n}{2}(n+1) = 1$ *(M1)*

$\Rightarrow k = \dfrac{2}{n(n+1)}$ as required *(AG)*

(b) $E(X) = \sum_{x=1}^{n} x \times p(X = x)$ *(M1)*

$= k \sum_{x=1}^{n} x^2$ *(M1)*

$= \dfrac{2}{n(n+1)} \times \dfrac{1}{6} n(n+1)(2n+1)$ *(M1)*

$= \dfrac{(2n+1)}{3}$ *(A1)*

4. (a) $z^3 = 1 = \cos 0 + i\sin 0$ *(M1)*

Thus, the required solutions are $z = \cos\left(0 + \frac{2\pi n}{3}\right) + i\sin\left(0 + \frac{2\pi n}{3}\right)$, $n = 0, \pm 1$ *(M1)*

i.e. $z = \cos 0 + i\sin 0, \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}, \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}$

or $z = 1, \frac{1}{2}\left(-1 + i\sqrt{3}\right), \frac{1}{2}\left(-1 - i\sqrt{3}\right)$ *(A2)*

(b) $\omega = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \frac{1}{2}\left(-1 + i\sqrt{3}\right)$ *(A1)*

$\omega^2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \frac{1}{2}\left(-1 - i\sqrt{3}\right)$ *(A1)*

Thus, $1 + \omega + \omega^2 = 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = 0$, as required *(M1)(AG)*

Alternative method: $\omega^3 - 1 = 0 \Rightarrow (\omega - 1)(\omega^2 + \omega + 1) = 0$ *(M1)*

Since $\omega \neq 1$, $\omega^2 + \omega + 1 = 0$ *(R1)(A1)*

(c) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$

$= \begin{pmatrix} 3 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+2\omega^3 & 1+\omega^2+\omega^4 \\ 1+\omega+\omega^2 & 1+\omega^2+\omega^4 & 1+2\omega^3 \end{pmatrix}$ *(M2)*

$= \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ since $1 + \omega + \omega^2 = 0$, $\omega^3 = 1$ and $\omega^4 = \omega$ *(R2)(A1)*

continued...

Question 4 continued

(d) The system is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ *(M1)*

Thus, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}^{-1}\begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}\begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$ (from part (c)) *(M1)*

$= \begin{pmatrix} 1-1-1 \\ 1-\omega^2-\omega \\ 1-\omega-\omega^2 \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 2-(1+\omega+\omega^2) \\ 2-(1+\omega+\omega^2) \end{pmatrix}$

$= \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}$ *(M1)(A1)*

Alternative Method using row operations:

The system is

1	1	1	3	
1	ω	ω^2	-3	
1	ω^2	ω	-3	
--	---	---	--	------------
1	1	1	3	
0	$\omega-1$	ω^2-1	-6	$R_2 \leftarrow R_2 - R_1$
0	ω^2-1	$\omega-1$	-6	$R_3 \leftarrow R_3 - R_1$
--	---	---	--	------------
1	1	1	3	
0	$\omega-1$	ω^2-1	-6	
0	0	3ω	6ω	$R_3 \leftarrow R_3 - (\omega+1)R_2$
--	---	---	--	------------

(M3)

where $\omega - 1 - (\omega+1)(\omega^2 - 1) = 2\omega - \omega^2 - \omega^3 = 3\omega - (1+\omega+\omega^2) = 3\omega$
and $-6 - (\omega+1)(-6) = -6(1-\omega-1) = 6\omega$

Therefore, back substitution gives $z = 2$
$(\omega - 1)\,y = -6 - 2(\omega^2 - 1) = 2(-\omega^2 - 2) = 2(\omega - 1)$ giving $y = 2$
and $x + 2 + 2 = 3$ giving $x = -1$ *(A1)*

5. (i) (a)

*	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

(A2)

Since each entry in the table is an element of S, S is **closed** under $*$. *(R1)*

6 is the **identity** since the 3rd row and 3rd column are identical to the row at the top and the column on the left, respectively. *(R1)(A1)*

4 and 6 are their own inverses; 2 and 8 are each other's inverses. *(A1)*

Thus, $(S,*)$ is a group. *(AG)*

(b) $2^2=4$, $2^3=8$, $2^4=6$ (or $8^2=4$, $8^3=2$, $8^4=6$) *(M2)*
Thus, $(S,*)$ is cyclic, and 2 (or 8) is a generator. *(A1)(AG)*

(ii) (a) Let $a,b\in T$. Then, $a\neq -1$ and $b\neq -1$
Suppose that $a+ab+b=-1$ *(M1)*
$\Rightarrow a+1+b(a+1)=0$ *(M1)*
$\Rightarrow (a+1)(b+1)=0$ *(M1)*
$\Rightarrow a=-1$ or $b=-1$ which is a contradiction *(M1)*
$\Rightarrow a+ab+b\neq -1$ *(M1)*

(b) T is **closed** under $\circ$ (from part (a)). *(R1)*

Let $a,b\in T$.
Then, $a\circ(b\circ c) = a\circ(b+bc+c)$
$= a+a(b+bc+c)+(b+bc+c)$
$= a+b+c+ab+ac+bc+abc$

Also, $(a\circ b)\circ c = (a+ab+b)\circ c$
$= (a+ab+b)+(a+ab+b)c+c$
$= a+b+c+ab+ac+bc+abc$
$= a\circ(b\circ c)$

Therefore, T is associative under $\circ$ *(M3)*

Let e be the identity (if it exists)
Then, for all $a\in T$, $a\circ e=e\circ a=a$
$\Rightarrow a+ae+e=a$
$\Rightarrow e(a+1)=0$ *(M1)*
$\Rightarrow e=0$ since $a\neq -1$ and $0\in T$ *(A1)*

Thus, 0 is the identity since $a\circ 0=0\circ a=a$ *(R1)*

continued...

Question 5 continued

Let a^{-1} be the inverse of a (if it exists)

Then, $a \circ a^{-1} = a^{-1} \circ a = e$

$\Rightarrow a + aa^{-1} + a^{-1} = 0$ *(M1)*

$\Rightarrow \quad a^{-1}(a+1) = -a$

$\Rightarrow \quad a^{-1} = -\dfrac{a}{a+1}$ since $a \neq -1$ and $-\dfrac{a}{a+1} \in T$ *(A1)*

Thus, the inverse of $a = -\dfrac{a}{a+1}$ since

$a \circ \left(-\dfrac{a}{a+1}\right) = \left(-\dfrac{a}{a+1}\right) \circ a = e = 0$ *(R1)*

Therefore, $(T, \circ)$ is a group *(AG)*

(c) $2 \circ (x \circ (-3)) = 5$

Pre-'multiply' each side by $2^{-1} = -\dfrac{2}{3}$ *(M1)*

$\Rightarrow \quad x \circ (-3) = \left(-\dfrac{2}{3}\right) \circ 5$

$\Rightarrow \quad = \left(-\dfrac{2}{3}\right) + \left(-\dfrac{2}{3}\right)(5) + 5$

$\Rightarrow \quad = 1$ *(M1)(A1)*

Post-'multiply' each side by $(-3)^{-1} = -\dfrac{3}{2}$ *(M1)*

$\Rightarrow \quad x = 1 \circ \left(-\dfrac{3}{2}\right)$

$\Rightarrow \quad = 1 + (1)\left(-\dfrac{3}{2}\right) + \left(-\dfrac{3}{2}\right)$

$\Rightarrow \quad x = -2$ *(M1)(A1)*

OR $\quad 2 \circ (x \circ (-3)) = 5$

$\Rightarrow \quad 2 \circ (-2x - 3) = 5$ *(M1)(A1)*

$\Rightarrow 2 + 2(-2x - 3) - 2x - 3 = 5$ *(M1)(A1)*

$\Rightarrow \quad x = -2$ *(A2)*

(iii) (a) Two groups $(G, \#)$ and $(H, \bullet)$ are **isomorphic** if there exists a one-to-one correspondence $\phi : G \rightarrow H$ such that for all $x,\ y \in G$, $\phi(x \# y) = \phi(x) \bullet \phi(y)$. *(A4)*

(b) Let x, y be any two real numbers. *(M1)*

Then, $\phi(x) = e^x$ and $\phi(y) = e^y$. *(M1)*

Now, $\phi(x+y) = e^{(x+y)} = (e^x)(e^y) = \phi(x)\phi(y)$. *(M3)(A1)*

Therefore, the group of real numbers under addition is isomorphic to the group of positive real numbers under multiplication. *(AG)*

6. (i) (a) The adjacency matrix A is given by

$$A = \begin{pmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

(A4)

Note: Deduct *1 mark* for each error, to a maximum of 4.

(b) The required incidence matrix is given by

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

(A4)

Notes: Deduct *1 mark* for each error, to a maximum of 4. Under some definitions of the vertex-edge incidence matrix, the 1 on the 5th row and 9th column would be a 2.

(c)

$$A^2 = \begin{pmatrix} 6 & 2 & 2 & 2 & 4 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 10 & 2 \\ 4 & 2 & 2 & 2 & 5 \end{pmatrix}$$

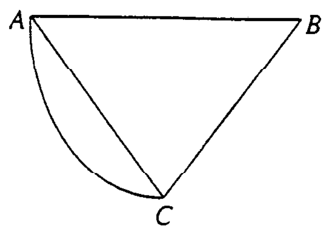
(A1)

Thus, the entry in the 4th row and 5th column of A^4 is
$2\times4+2\times2+2\times2+10\times2+2\times5=46$
Thus, the required number of walks is 46 *(M2)(A1)*

continued...

Question 6 continued

(ii) (a) The situation under consideration when represented by a multigraph with the edges representing bridges is:

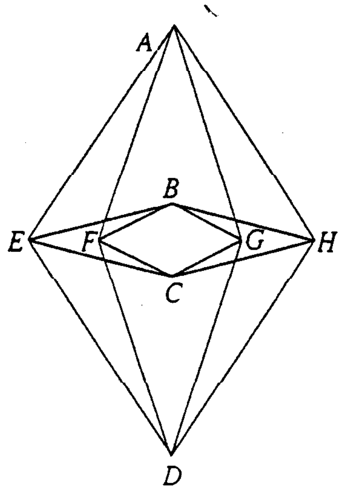


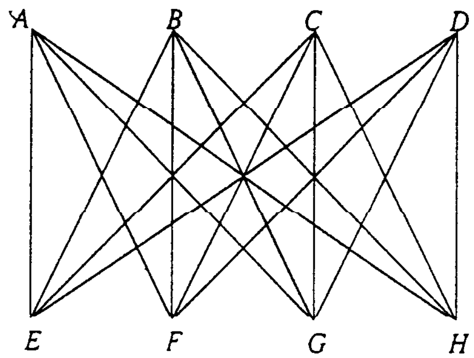
The problem is to add edges, as few as necessary, so that the graph becomes Eulerian. That is, the vertices all have even degree. Thus, the manager should add a new bridge between A and C. *(M2)(A2)*

(b) The problem is to add edges, as few as necessary, so that C and B have odd degrees, and A has even degree. One new bridge between A and B makes this possible. *(M2)(A2)*

(iii) (a) A complete bipartite graph, $\kappa_{m,n}$, has $m+n$ vertices which can be divided into 2 sets of m and n each such that every vertex in one set is adjacent to every vertex in the other set but not adjacent to any vertex in its own set. *(R3)*

(b) Each of the following is a graphic representation of $\kappa_{4,4}$.





(A3)

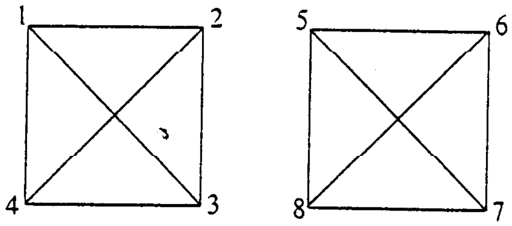
continued...

Question 6 continued

(iv) (a) The handshaking lemma assures us that the sum of the degrees of the vertices in the proposed graph must be even. The sum of the members of the given sequence is 17 which is odd. Thus there is no graph of which the vertices can have the given sequence for its degree sequence. That is, the given sequence is not graphic. *(M2)(A1)*

(b) The given sequence is not graphic since a vertex of degree 5 must connect 5 other vertices and there are only 4 other vertices in the graph. *(M2)(A1)*

(v) (a) Consider the following graph in which the vertices are the people and the edges represent the acquaintances.



The above graph meets the requirements of the problem, but it does not contain a Hamiltonian circuit since the graph is not connected. Thus, it is not necessarily possible to accomplish the desired seating arrangement. *(M3)(A1)*

(b) If each person is acquainted with at least 4 other people then a graph with vertices representing people and edges representing acquaintanceship must be connected. Now, if G is a connected graph with n vertices where $n \geq 3$, then G has a Hamiltonian circuit if the degree of each vertex is at least $\frac{n}{2}$. In this problem, the graph is connected, $n = 8$, $\frac{n}{2} = 4$ and $\deg(v) \geq 4$ for all vertices v. Thus, the graph has a Hamiltonian circuit.

It is possible to seat the group so that no-one is beside a stranger. *(M3)(A1)*

7. (i) Let X represent the number of trucks arriving at the warehouse to be unloaded. Then X has a Poisson distribution with an average of 3 trucks per hour. For a 2-hour period, $\lambda = 6$. *(M1)(A1)*

Therefore, the probability that exactly 7 trucks will arrive between 09:00 and 11:00 on a given Monday

$$= \frac{6^7(e^{-6})}{7!}$$ *(M1)*

$$= 0.138$$ *(A1)*

(ii) Here we use the central limit theorem according to which the sampling distribution of the mean is approximately normal with sample size $n = 40$. The probability that the mean area covered by the sample of 40 cans lies between 510.0 and 520.0 square metres is

$$p\left(\frac{510.0 - 513.3}{\frac{31.5}{\sqrt{40}}} < z < \frac{520.0 - 513.3}{\frac{31.5}{\sqrt{40}}}\right)$$ *(M2)(A2)*

$$= p(-0.663 < z < 1.345)$$

$$= 0.657$$ *(M1)(A1)*

(iii) (a) $$\text{mean} = \frac{\sum f_i x_i}{\sum f_i} = 20.32$$ *(M1)(A1)*

$$\text{variance} = \frac{\sum f_i (x_i - m)^2}{\sum f_i} = 17.34$$ *(M1)(A1)*

$$\text{standard deviation} = \sqrt{17.34} = 4.16$$ *(M1)(A1)*

(b) The 95 % confidence interval for the population mean is

$$\left(20.32 - 1.96\frac{4.16}{\sqrt{100}},\ 20.32 + 1.96\frac{4.16}{\sqrt{100}}\right)$$
$$= (19.5, 21.1)$$ *(M2)(A2)*

continued...

Question 7 continued.

(c) Let the null hypothesis be $H_0: \mu = 20$
and the alternative hypothesis $H_1: \mu \neq 20$ *(M1)*

mean = 20.32 standard deviation = 4.16

$$z = \frac{20.32 - 20}{\frac{4.16}{\sqrt{100}}} = 0.769$$ *(M2)(A1)*

Since $-1.96 < z < 1.96$, we cannot reject H_0 *(A1)(R1)*

(d) If the distribution is Poisson with mean 20, then the expected frequencies are
$100 \times e^{-20} \times \frac{20^x}{x!}$ *(M1)*

x	≤11	12–14	15–17	18–20	21–23	24–26	27–29	≥30
Observed frequencies, f_o	0	7	21	27	19	18	8	0
Expected frequencies, f_e	2.2	8.3	19.2	26.2	22.8	13.5	5.6	2.2
$\frac{(f_o - f_e)^2}{f_e}$	1.167		0.169	0.024	0.633	1.5	0.005	

(A6)

Note: The first and last columns must be combined with their neighbours since the number of observations in each is less than 5

Let the null hypothesis be
H_0: the distribution is Poisson with $\lambda = 20$
and the alternative hypothesis
H_1: the distribution is not Poisson with $\lambda = 20$ *(M1)*

$$\chi^2 = \sum \frac{(f_o - f_e)^2}{f_e} = 3.498.$$ *(A1)*

There are 5 degrees of freedom, and $\chi^2_{(0.05, 5)} = 11.07$ *(M2)*

Since $3.498 < 11.07$, we cannot reject H_0 *(M1)(C1)*

Therefore, there is no evidence that the distribution is not Poisson with $\lambda = 20$ *(A1)*

8. (i) (a) $S_{2n} = \frac{h}{3}\left[f(x_0) + 4\sum_{k=1}^{n} f(x_{2k-1}) + 2\sum_{k=1}^{n-1} f(x_{2k}) + f(x_{2n})\right]$ *(M1)(A1)*

Error $= \int_a^b f(x)\,dx - S_{2n} = -\frac{(b-a)h^4}{180} f^{(4)}(c)$, where $a < c < b$ *(M1)(A1)*

(b) From part (a) with $h = \frac{5}{8} = 0.625$, we have

$$S_8 = \frac{h}{3}[f(2) + 4f(2.625) + 2f(3.25) + 4f(3.875) + 2f(4.5) + 4f(5.125) + 2f(5.75) + 4f(6.375) + f(7)]$$ *(M1)*

$$= \frac{0.625}{3}[0.5 + 4(0.3810 + 0.2581 + 0.1951 + 0.1569) + 2(0.3077 + 0.2222 + 0.1739) + 0.1429]$$ *(M1)*

$= 1.253$ *(A1)*

Since $\int_2^7 \frac{1}{x}dx = \ln 7 - \ln 2 = 1.2528$, the estimate S_8 is accurate to three decimal places. *(M1)(A1)(AG)*

(c) Taking $f(x) = \frac{1}{x}$, we get $f^{(4)}(x) = \frac{24}{x^5}$ *(M1)(A1)*

When $2 \le x \le 7$, $\frac{24}{x^5} \le \frac{3}{4}$ *(M1)(A1)*

$|\text{Error}| = \left|\frac{-(7-2)}{180}\left(\frac{5}{8}\right)^4 f^{(4)}(c)\right| \le \frac{1}{36}\left(\frac{5}{8}\right)^4\left(\frac{3}{4}\right) \le 0.00318$ *(A1)*

(d) For an approximation which is correct to four decimal places,

$|S_{2n}| < \frac{1}{2} \times 10^{-4}$ *(M1)(A1)*

$$\Rightarrow \frac{(7-2)\left(\frac{5}{2n}\right)^4}{180}\left(\frac{3}{4}\right) < \frac{1}{2} \times 10^{-4}$$

$$\Rightarrow \frac{1}{48}\left(\frac{625}{16n^4}\right) < \frac{1}{2} \times 10^{-4}$$

$$\Rightarrow n^4 > \frac{125 \times 10^5}{768}$$

$$\Rightarrow n > \sqrt[4]{\frac{125 \times 10^5}{768}} \approx 11.3$$ *(M1)(A1)*

Since n is an integer, $n \ge 12$. *(R1)*

Therefore, 24 sub-intervals are required. *(A1)*

continued...

Question 8 continued

(ii) Let $u_k = \dfrac{1}{2k-1}$

Then, $\lim\limits_{k\to\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to\infty} \dfrac{2k-1}{2k+1} = \lim\limits_{k\to\infty} \dfrac{2-\frac{1}{k}}{2+\frac{1}{k}} = 1$ *(M2)(A1)*

Since this is 1, we cannot say whether the sequence converges or diverges. *(R1)*

Let $f(x) = \dfrac{1}{2x-1}$ for $x \geq 1$

Then, $f(x)$ is continuous, positive and decreasing for $x \geq 1$

Therefore, the integral test may be used *(R1)*

Now, $\displaystyle\int_1^{\infty} \frac{1}{2x-1}\,\mathrm{d}x = \lim_{b\to\infty}\left(\int_1^b \frac{1}{2x-1}\,\mathrm{d}x\right)$ *(M1)*

$\displaystyle = \lim_{b\to\infty}\left(\left[\frac{1}{2}\ln(2x-1)\right]_1^b\right)$ *(M1)*

$\displaystyle = \lim_{b\to\infty}\left(\frac{1}{2}\ln(2b-1)\right)$, which does not exist *(A1)*

Therefore the series diverges. *(R1)*

continued...

Question 8 continued

(iii) Mean value theorem: If $f(x)$ is continuous for $a \le x \le b$ and differentiable for $a < x < b$, then there exists c, $a < c < b$, such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ *(A2)*

Put $a = 0.5$, $b = 0.6$, and let $f(x) = \arcsin x$ *(M1)*

Then, $f(x)$ is continuous for $0.5 \le x \le 0.6$ and differentiable for $0.5 < x < 0.6$ with

$f'(x) = \dfrac{1}{\sqrt{1 - x^2}}$ *(A1)*

Therefore, there exists c, $0.5 < c < 0.6$, such that

$$f'(c) = \frac{\arcsin 0.6 - \arcsin 0.5}{0.6 - 0.5} = \frac{\arcsin 0.6 - \frac{\pi}{6}}{0.1}$$ *(A2)*

But if $0.5 < c < 0.6$, then $\dfrac{1}{\sqrt{1 - 0.5^2}} < f'(c) < \dfrac{1}{\sqrt{1 - 0.6^2}}$

That is, $\dfrac{1}{\frac{\sqrt{3}}{2}} < f'(c) < \dfrac{1}{0.8}$ *(A2)*

Thus, $\dfrac{2}{\sqrt{3}} < \dfrac{\arcsin 0.6 - \frac{\pi}{6}}{0.1} < \dfrac{1}{0.8}$ *(A1)*

$\Rightarrow \dfrac{1}{5\sqrt{3}} < \arcsin 0.6 - \dfrac{\pi}{6} < \dfrac{1}{8}$ *(A1)*

$\Rightarrow \dfrac{\pi}{6} + \dfrac{\sqrt{3}}{15} < \arcsin 0.6 < \dfrac{\pi}{6} + \dfrac{1}{8}$ *(A1)*