INTERNATIONAL BACCALAUREATE

MATHEMATICS

Higher Level

Thursday 7 November 1996 (morning)

Paper 2 — 2 hours 30 minutes

This examination paper consists of 2 sections, Section A and Section B.

Section A consists of 4 questions.

Section B consists of 4 questions.

The maximum mark for Section A is 80.

The maximum mark for each question in Section B is 40.

The maximum mark for this paper is 120.

This examination paper consists of 13 pages.

INSTRUCTIONS TO CANDIDATES

DO NOT open this examination paper until instructed to do so.

Answer all FOUR questions from Section A and ONE question from Section B.

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures as appropriate.

EXAMINATION MATERIALS

Required/Essential:

IB Statistical Tables
Millimetre square graph paper
Electronic calculator
Ruler and compasses

Allowed/Optional:

A simple translating dictionary for candidates not working in their own language

FORMULAE

Trigonometrical identities:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\sin\alpha - \sin\beta = 2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

$$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$

$$\cos\alpha - \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}$$

$$\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \cos^2\theta - \sin^2\theta$$

$$\text{If } \tan\frac{\theta}{2} = t \text{ then } \sin\theta = \frac{2t}{1+t^2} \text{ and } \cos\theta = \frac{1-t^2}{1+t^2}$$

Integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x$$

Standard integrals:

$$\int \frac{\mathrm{d}x}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + c$$

$$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + c \quad (|x| < a)$$

Statistics: If $(x_1, x_2, \dots, x_n)$ occur with frequencies $(f_1, f_2, \dots, f_n)$ then the mean m and standard deviation s are given by

$$m = \frac{\Sigma f_i x_i}{\Sigma f_i}, \qquad s = \sqrt{\frac{\Sigma f_i (x_i - m)^2}{\Sigma f_i}}, \qquad i = 1, 2, \dots, n$$

Binomial distribution:

$$p_x = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, 2, \dots, n$$

A correct answer with **no** *indication of the method used will normally receive* **no** *marks. You are therefore advised to show your working.*

SECTION A

Answer all FOUR questions from this section.

1. *[Maximum mark: 22]*

(i) Let two lines L, M and a plane P be given by

$$L: \frac{x-3}{2} = \frac{y+1}{4} = \frac{z-2}{-1},$$

$$M: \frac{x-6}{-4} = \frac{y}{2} = z+10, \text{ and}$$

$$P: x+3y-2z=4.$$

(a) Find the coordinates of the point of intersection of line L and plane P. *[2 marks]*

(b) Calculate the acute angle between L and P, giving your answer correct to the nearest one-tenth of a degree. *[3 marks]*

(c) Show that line M is parallel to plane P. *[2 marks]*

(d) Show that lines L and M do not meet. *[4 marks]*

(e) Calculate the shortest distance between lines L and M. *[5 marks]*

(ii) Let $I_n(x) = \int_0^x t^n e^{-t}\, dt$ where $n \in \mathbb{N}$.

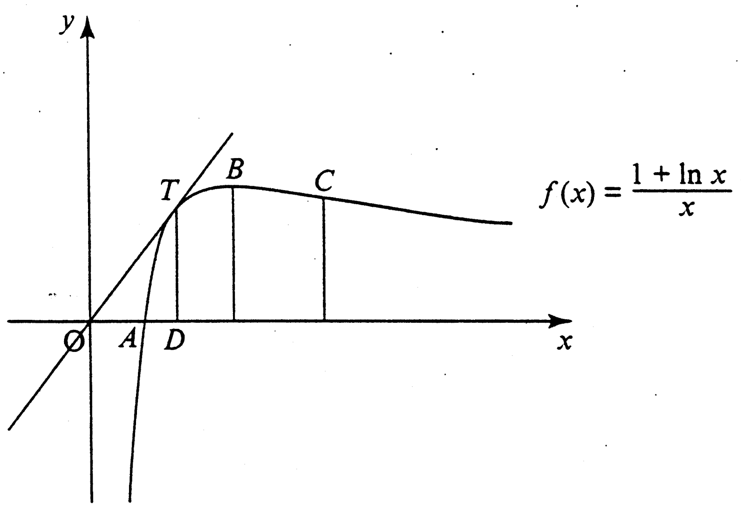
(a) Find $I_0(x)$. *[1 mark]*

(b) Show that $I_n(x) = -x^n e^{-x} + nI_{n-1}(x)$ for $n \geqslant 1$. *[3 marks]*

(c) Evaluate $I_3(1)$. *[2 marks]*

2. *[Maximum mark: 24]*

The graph of $f(x) = \dfrac{1 + \ln x}{x}$ is given below.



The line (OT), where O is the origin and T is on the curve, is the tangent at T. The graph of $f(x)$ meets the x-axis at A.

(a) Find the coordinates of A. *[2 marks]*

(b) Find the coordinates of the point B where $f(x)$ attains its maximum value. *[4 marks]*

(c) Find the coordinates of C, the point of inflexion of $f(x)$. *[5 marks]*

(d) If the x-coordinate of T is x_0, find, in terms of x_0, the equation of the tangent line (OT). *[2 marks]*

(e) Find the value of x_0. *[3 marks]*

(f) Prove that the x-coordinates of the points A, T, B and C, as defined above (and shown in the diagram), are four consecutive terms of a geometric sequence. State the common ratio of this sequence. *[3 marks]*

(g) Calculate the area of the region OTA. *[5 marks]*

3. *[Maximum mark: 16]*

(i) A particular experiment has two possible outcomes, a success which occurs with probability p, $(0 < p < 1)$, and a failure which occurs with probability $(1 - p)$. The outcomes of successive performances of this experiment are independent.

(a) Find the value of p if the probability of obtaining exactly two successes in six experiments is **three times** the probability of obtaining exactly three successes in six experiments. *[4 marks]*

(b) In another similar experiment the value of p was found to be $\frac{1}{5}$.

Find the least number of times the experiment must be performed in order that the probability of obtaining at least one success is greater than 0.99 . *[5 marks]*

(ii) Six per cent (6%) of electric bulbs produced in a given factory are defective. The manufacturer chooses 200 bulbs from a day's production.

(a) If X represents the number of defective bulbs found in the sample, find the mean and standard deviation of X. *[2 marks]*

(b) Determine the probability that 20 or more defective bulbs are found in the sample of 200 bulbs. *[5 marks]*

4. *[Maximum marks: 18]*

(i) Let z_1 and z_2 be the roots of the equation

$$2z^2 - (2 - 2i)z - 5i = 0 .$$

(a) Using the quadratic formula, show that

$$z_1 = \frac{1}{2}(1 - i) + \sqrt{2i} \quad \text{and} \quad z_2 = \frac{1}{2}(1 - i) - \sqrt{2i} .$$

[3 marks]

(b) Express each of z_1 and z_2 in the form $a + ib$, where a and b are real numbers. *[4 marks]*

(ii) (a) If $z = \cos\theta + i\sin\theta$, show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$. *[3 marks]*

(b) Given that each of the four roots of the equation

$$5z^4 - 11z^3 + 16z^2 - 11z + 5 = 0$$

has a modulus equal to 1 , find these roots.

[Hint: Divide the equation through by z^2 .] *[8 marks]*

SECTION B

Answer ONE question from this section.

Abstract Algebra

5. *[Maximum mark: 40]*

(i) On a set $\varepsilon = \{a, b, c, d\}$, the operations $\otimes$ and $*$ are defined by the following tables:

$\otimes$	a	b	c	d
a	a	c	d	a
b	b	a	c	d
c	d	b	a	c
d	c	d	b	a

$*$	a	b	c	d
a	b	c	d	a
b	c	d	a	b
c	d	a	b	c
d	a	b	c	d

Solve for x in each of the following:

(a) $a * (x \otimes b) = c$;

(b) $(a \otimes x) * b = d$. *[6 marks]*

(ii) Given a multiplicative group in which $a^3 = b^2 = e$, where e is the identity element and $ab = ba^2$, show that

(a) $a^2ba^2b = e$;

(b) $a^2ba = ab$. *[6 marks]*

(iii) Let G be the group of all non-singular 2×2 matrices of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a, b \in \mathbb{R}$, under multiplication of matrices. Let H be the group of all non-zero complex numbers under multiplication of complex numbers. Show that G and H are isomorphic. *[10 marks]*

(This question continues on the following page)

(Question 5 continued)

(iv) Let $\mathbb{Z}_n$ denote the group of integers $\{0, 1, \ldots, n-1\}$ with binary operation addition modulo n.

Now, $\mathbb{Z}_n \times \mathbb{Z}_m = \{(e, f) \mid e \in \mathbb{Z}_n \text{ and } f \in \mathbb{Z}_m\}$ forms a group under a binary operation $*$ defined by

$$(e_1, f_1) * (e_2, f_2) = (e_1 +_n e_2, f_1 +_m f_2)$$

where $+_n$ and $+_m$ denote addition of integers modulo n and m, respectively.

(a) State the order of $(\mathbb{Z}_4 \times \mathbb{Z}_5, *)$ and evaluate $(3, 2) * (1, 4)$. *[5 marks]*

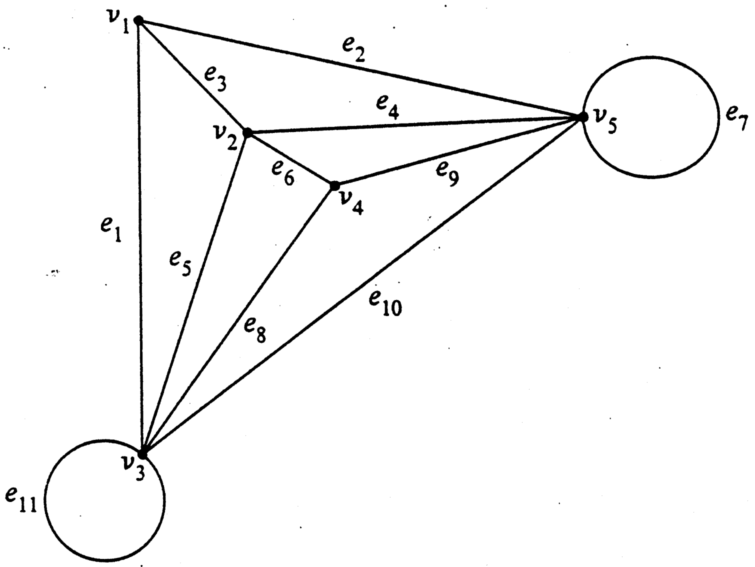
(b) Show that $(\mathbb{Z}_2 \times \mathbb{Z}_3, *)$ is cyclic, and list all possible generators. *[6 marks]*

(c) How many elements of $(\mathbb{Z}_2 \times \mathbb{Z}_4, *)$ have order 4? *[7 marks]*

Graphs and Trees

6. *[Maximum mark: 40]*

(i) (a) Write down the **adjacency matrix** A and the **incidence matrix** B of the following graph.



[4 marks]

(b) Evaluate A^2, and explain what the entries of this matrix reveal about the given graph. *[3 marks]*

(c) What does the sum of the entries in each column of A reveal about the given graph? *[2 marks]*

(d) What does the sum of the entries in each column of B reveal about the given graph? *[2 marks]*

(ii) (a) Let $G = (V, E)$ be an undirected, simple (no loops and no multiple edges) graph with a set of vertices V and edges E. If v and e denote the number of vertices and the number of edges of the graph G, respectively, show that

$$2e \leqslant v^2 - v\ .$$

[4 marks]

(b) Find the corresponding inequality for the case where G is a loop-free, directed graph. *[2 marks]*

(This question continues on the following page)

(Question 6 continued)

(iii) (a) A complete graph with n vertices, κ_n, is a loop-free, undirected graph with an edge joining every pair of vertices. How many paths of length 4 are there in the complete graph κ_n, $n > 4$?

[Note that paths such as $v_1 \to v_2 \to v_3 \to v_4 \to v_5$ and $v_5 \to v_4 \to v_3 \to v_2 \to v_1$ are the same path.] *[2 marks]*

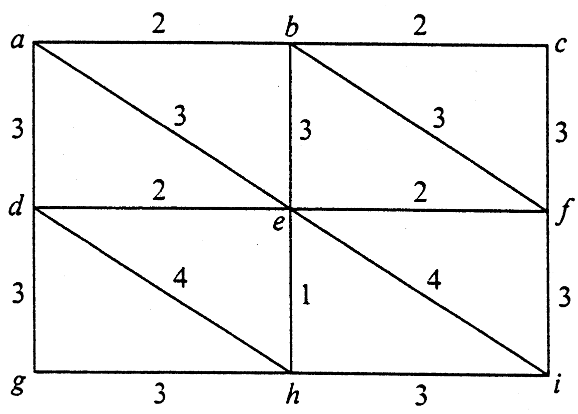
(b) An undirected, simple graph G with n vertices is said to be self-complementary if it is isomorphic to its own complement $\overline{G}$. If G is self-complementary then how many edges must it have? *[4 marks]*

(c) Give an example of a self-complementary graph on four vertices. *[2 marks]*

(d) If G is a self-complementary graph on n vertices, where $n > 1$, show that $n = 4k$ or $n = 4k + 1$, for some positive integer k. *[4 marks]*

(iv) Explain why it is not possible to draw a loop-free, connected, undirected graph with 8 vertices, where the degrees of the vertices are 1, 1, 1, 2, 3, 4, 5, and 7. *[5 marks]*

(v) The following graph represents nine cities $a, b, c, \ldots, i$ which are to be connected by a system of highways. The numbers represent the cost in millions of dollars for constructing each section. Use Prim's algorithm, or any other suitable algorithm, to find the minimum cost of constructing a road system which provides a route connecting any two of the cities.



[6 marks]

Statistics

7. *[Maximum mark: 40]*

(i) On weekdays at a certain small airport, airplanes arrive at an average rate of three per hour for the one-hour period between 13:00 and 14:00. If these arrivals are distributed according to a Poisson probability distribution, determine the probabilities that

(a) no airplanes will arrive between 13:00 and 14:00 pm next Tuesday; *[4 marks]*

(b) either one or two airplanes will arrive between 13:00 and 14:00 next Tuesday. *[4 marks]*

(ii) A testing service has designed a new aptitude test for insurance companies to screen applicants for vacant positions within the organisations. To estimate the mean score achieved by all future applicants who will eventually take the test, a psychologist associated with the testing service administered it to 9 persons. The sample mean $\bar{x}$, and the sample standard deviation s were found to be $\bar{x} = 83.7$ and $s = 12.9$, respectively. Determine a 95% confidence interval for the mean score. *[10 marks]*

(iii) A career-counselling group claims that at least 40% of all engineers employed by aerospace firms change jobs within three years of being hired. The alternative hypothesis is that the rate of changing jobs is below 40%. At a significance level of 0.01, should the claim be accepted or rejected if the sample results show that 25 out of 100 engineers changed their jobs? *[10 marks]*

(iv) A researcher wants to estimate the mean lifetime of a new type of tyre compared with an existing tyre. In a test, ten taxis from the same company were equipped with two of each type of tyre. Each taxi's total mileage was recorded until both tyres of the same type had to be replaced. For each car, the replacement mileage for the present tyre (B) was subtracted from the corresponding figure for the new tyre (A). The mean difference between mileages was 2000 miles, with a standard deviation of 2000 miles.

(a) Construct a 95% confidence interval estimate for the difference between mean tyre-lifetimes. *[10 marks]*

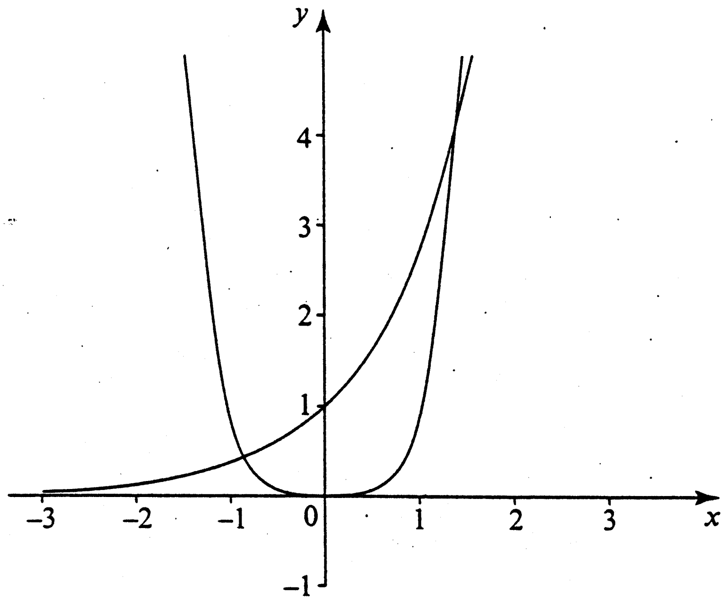
(b) Should the null hypothesis, that the mean tyre-lifetimes are identical, be accepted or rejected at the 5% significance level? *[2 marks]*

Analysis and Approximation

8. *[Maximum mark: 40]*

(i) The diagram below shows the graphs of $y = e^x$ and $y = x^4$.

(Note that the graphs meet a third time when $x \approx 8$.)



(a) Show that the iteration formula $x_{n+1} = 4 \ln x_n$ can be used to estimate the x-coordinate of the point of intersection near $x = 8$, and that the iteration formula $x_{n+1} = e^{x_n/4}$ can be used to estimate the x-coordinate of the point of intersection near $x = 1$.

Use **one** of these formulae with $x_0 = 8$ or $x_0 = 1$, respectively, to find the x-coordinate of one point of intersection correct to three decimal places. *[8 marks]*

(b) The x-coordinate of the point of intersection near $x = -1$, can be approximated using the Newton-Raphson method with an initial approximation $x_0 = -1$. Find, in simplified form and in terms of e, the value of the next approximation x_1. *[4 marks]*

(c) Find the exact value of the area between the curves and the lines $x = 0$ and $x = 1$. *[3 marks]*

(d) Use Simpson's Rule with 4 subintervals to estimate the area between the curves and the lines $x = 0$ and $x = 1$. Give your answer correct to four decimal places. *[3 marks]*

(This question continues on the following page)

(Question 8 continued)

(e) Write down an expression for the error if

$$\int_a^b f(x)\,\mathrm{d}x$$

is approximated using Simpson's Rule with n subintervals of equal length, where n is even.

Use this to find the maximum error, correct to four decimal places, if the area in part (c) is approximated by Simpson's Rule with $n = 4$. Show that your answers to parts (c) and (d) are consistent with this maximum error. *[7 marks]*

(f) Using Simpson's Rule, what value of n is needed to approximate

$$\int_0^1 (\mathrm{e}^x - x^4)\,\mathrm{d}x$$

to an accuracy of 0.000 005? *[5 marks]*

(ii) (a) Using the integral test, show that

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$$

is convergent. *[6 marks]*

(b) Show that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$$

is conditionally convergent. *[4 marks]*