M13/5/FURMA/SP2/ENG/TZ0/XX

22137102

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FURTHER MATHEMATICS
STANDARD LEVEL
PAPER 2

Tuesday 21 May 2013 (morning)

2 hours

INSTRUCTIONS TO CANDIDATES

- Do not open this examination paper until instructed to do so.
- Answer all questions.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- A graphic display calculator is required for this paper.
- A clean copy of the ***Mathematics HL and Further Mathematics SL*** *information booklet* is required for this paper.
- The maximum mark for this examination paper is *[120 marks]*.

2213-7102

7 pages

Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. In particular, solutions found from a graphic display calculator should be supported by suitable working, for example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

1. *[Maximum mark: 9]*

The discrete random variable X follows the distribution $\text{Geo}(p)$.

(a) (i) Write down the mode of X.

(ii) Find the exact value of p if $\text{Var}(X) = \frac{28}{9}$. *[3 marks]*

Arthur tosses a biased coin each morning to decide whether to walk or cycle to school; he walks if the coin shows a head.

The probability of obtaining a head is 0.55.

(b) (i) Find the smallest value of n for which the probability of Arthur walking to school on the next n days is less than 0.01.

(ii) Find the probability that Arthur cycles to school for the third time on the last of eight successive days. *[6 marks]*

2. *[Maximum mark: 27]*

(a) (i) Use partial fractions to show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ is convergent.

(ii) Hence, by using the limit comparison test, determine whether the series $\sum_{n=1}^{\infty} \frac{e^{-n}}{n^2}$ is convergent or divergent. *[9 marks]*

(b) (i) Show that the improper integral $\int_0^{\infty} \frac{1}{x^2+1} \mathrm{d}x$ is convergent.

(ii) Use the integral test to deduce that the series $\sum_{n=0}^{\infty} \frac{1}{n^2+1}$ is convergent, giving reasons why this test can be applied. *[6 marks]*

(c) (i) Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+1}$ is convergent.

(ii) If the sum of the above series is S, show that $\frac{3}{5} < S < \frac{2}{3}$. *[6 marks]*

(d) For the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2+1}$

(i) determine the radius of convergence;

(ii) determine the interval of convergence using your answers to (b) and (c). *[6 marks]*

3. *[Maximum mark: 16]*

A random variable X has probability density function f given by:

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, \text{ for } x \geq 0 \text{ where } \lambda > 0 \\ 0, \text{ for } x < 0. \end{cases}$$

(a) (i) Find an expression for $\mathrm{P}(X > a)$, where $a > 0$.

A chicken crosses a road. It is known that cars pass the chicken's crossing route, with intervals between cars measured in seconds, according to the random variable X, with $\lambda = 0.03$. The chicken, which takes 10 seconds to cross the road, starts to cross just as one car passes.

(ii) Find the probability that the chicken will reach the other side of the road before the next car arrives.

Later, the chicken crosses the road again just after a car has passed.

(iii) Show that the probability that the chicken completes both crossings is greater than 0.5. *[6 marks]*

(b) A rifleman shoots at a circular target. The distance in centimetres from the centre of the target at which the bullet hits, can be modelled by X with $\lambda = 0.4$. The rifleman scores 10 points if $X \leq 1$, 5 points if $1 < X \leq 5$, 1 point if $5 < X \leq 10$ and no points if $X > 10$.

(i) Find the expected score when one bullet is fired at the target.

A second rifleman, whose shooting can also be modelled by X, wishes to find his value of λ.

(ii) Given that his expected score is 6.5, find his value of λ. *[10 marks]*

4. *[Maximum mark: 24]*

A group of people: Andrew, Betty, Chloe, David, Edward, Frank and Grace, has certain mutual friendships:

Andrew is friendly with Betty, Chloe, David and Edward;

Frank is friendly with Betty and Grace;

David, Chloe and Edward are friendly with one another.

(a) (i) Draw the planar graph H that represents these mutual friendships.

(ii) State how many faces this graph has. *[3 marks]*

(b) Determine, giving reasons, whether H has

(i) a Hamiltonian path;

(ii) a Hamiltonian cycle;

(iii) an Eulerian circuit;

(iv) an Eulerian trail. *[8 marks]*

(c) Verify Euler's formula for H. *[2 marks]*

(d) State, giving a reason, whether or not H is bipartite. *[2 marks]*

(e) Write down the adjacency matrix for H. *[2 marks]*

David wishes to send a message to Grace, in a sealed envelope, through mutual friends.

(f) In how many different ways can this be achieved if the envelope is passed seven times and Grace only receives it once? *[7 marks]*

5. *[Maximum mark: 21]*

The set S consists of real numbers r of the form $r = a + b\sqrt{2}$, where $a, b \in \mathbb{Z}$.

The relation R is defined on S by $r_1 R r_2$ if and only if $a_1 \equiv a_2$ (mod2) and $b_1 \equiv b_2$ (mod3), where $r_1 = a_1 + b_1\sqrt{2}$ and $r_2 = a_2 + b_2\sqrt{2}$.

(a) Show that R is an equivalence relation. *[7 marks]*

(b) Show, by giving a counter-example, that the statement $r_1 R r_2 \Rightarrow r_1^2 R r_2^2$ is false. *[3 marks]*

(c) Determine

(i) the equivalence class E containing $1+\sqrt{2}$;

(ii) the equivalence class F containing $1-\sqrt{2}$. *[3 marks]*

(d) Show that

(i) $\left(1+\sqrt{2}\right)^3 \in F$;

(ii) $\left(1+\sqrt{2}\right)^6 \in E$. *[4 marks]*

(e) Determine whether the set E forms a group under

(i) the operation of addition;

(ii) the operation of multiplication. *[4 marks]*

6. *[Total mark: 23]*

Part A *[Maximum mark: 10]*

(a) Show that the opposite angles of a cyclic quadrilateral add up to 180°. *[3 marks]*

(b) A quadrilateral ABCD is inscribed in a circle S. The four tangents to S at the vertices A, B, C and D form the edges of a quadrilateral EFGH. Given that EFGH is cyclic, show that AC and BD intersect at right angles. *[7 marks]*

Part B *[Maximum mark: 13]*

The circle C has centre O. The point Q is fixed in the plane of the circle and outside the circle. The point P is constrained to move on the circle.

(a) Show that the locus of a point P', which satisfies $\overrightarrow{\text{QP}'} = k\overrightarrow{\text{QP}}$, is a circle C', where k is a constant and $0 < k < 1$. *[6 marks]*

(b) Show that the two tangents to C from Q are also tangents to C'. *[4 marks]*

The circle C' cuts the line OQ at the points X and Y.

(c) Show that $\text{QX} \times \text{QY} = k^2 p$, where p is the power of Q with respect to the circle C. *[3 marks]*