INTERNATIONAL BACCALAUREATE
BACCALAURÉAT INTERNATIONAL
BACHILLERATO INTERNACIONAL
IBO

MARKSCHEME

May 2003

FURTHER MATHEMATICS

Standard Level

Paper 2

12 pages

Paper 2 Markscheme

Instructions to Examiners

1 Method of marking

(a) All marking must be done using a **red** pen.

(b) Marks should be noted on candidates' scripts as in the markscheme:

- show the breakdown of individual marks using the abbreviations ***(M1)***, ***(A2)*** *etc*.
- write down each part mark total, indicated on the markscheme (for example, ***[3 marks]***) – it is suggested that this be written at the end of each part, and underlined;
- write down and circle the total for each question at the end of the question.

2 Abbreviations

The markscheme may make use of the following abbreviations:

M Marks awarded for **Method**

A Marks awarded for an **Answer** or for **Accuracy**

G Marks awarded for correct solutions, generally obtained from a **Graphic Display Calculator**, irrespective of working shown

R Marks awarded for clear **Reasoning**

AG **Answer Given** in the question and consequently marks are **not** awarded

3 Follow Through (ft) Marks

Errors made at any step of a solution can affect all working that follows. To limit the severity of the penalty, **follow through (ft)** marks should be awarded. The procedures for awarding these marks require that all examiners:

(i) penalise an error when it **first occurs**;

(ii) **accept the incorrect answer** as the appropriate value or quantity to be used in all subsequent working;

(iii) award ***M*** marks for a correct method, and ***A*(ft)** marks if the subsequent working contains no further errors.

Follow through procedures may be applied repeatedly throughout the same problem.

The following illustrates a use of the **follow through** procedure:

Markscheme		Candidate's Script	Marking	
\$ 600 × 1.02	***M1***	Amount earned = \$ 600 × 1.02	√	***M1***
= \$ 612	***A1***	= \$602	×	***A0***
\$ (306 × 1.02) + (306 × 1.04)	***M1***	Amount = 301 × 1.02 + 301 × 1.04	√	***M1***
= \$ 630.36	***A1***	= \$ 620.06	√	***A1*(ft)**

Note that the candidate made an arithmetical error at line 2; the candidate used a correct method at lines 3, 4; the candidate's working at lines 3, 4 is correct.

However, if a question is transformed by an error into a **different, much simpler question** then:

(i) **fewer** marks should be awarded at the discretion of the Examiner;

(ii) marks awarded should be followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(iii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

4 Using the Markscheme

(a) This markscheme presents a particular way in which each question may be worked and how it should be marked. **Alternative methods** have not always been included. Thus, if an answer is wrong then the working must be carefully analysed in order that marks are awarded for a different method in a manner which is consistent with the markscheme.

In this case:

(i) a mark should be awarded followed by "**(d)**" (to indicate that these marks have been awarded at the **discretion** of the Examiner);

(ii) a brief **note** should be written on the script explaining **how** these marks have been awarded.

Where alternative methods for complete questions are included, they are indicated by **METHOD 1**, **METHOD 2**, *etc*. Other alternative solutions, including graphic display calculator alternative solutions are indicated by **OR**. For example:

Mean = 7906/134 ***(M1)***
= 59 ***(A1)***

OR

Mean = 59 ***(G2)***

(b) Unless the question specifies otherwise, accept **equivalent forms**. For example: $\frac{\sin\theta}{\cos\theta}$ for $\tan\theta$.

On the markscheme, these equivalent numerical or algebraic forms will generally be written in brackets after the required answer. Paper setters will indicate the required answer, by allocating full marks at that point. Further working should be ignored, even if it is incorrect. For example: if candidates are asked to factorize a quadratic expression, and they do so correctly, they are awarded full marks. If they then continue and find the roots of the corresponding equation, do not penalize, even if those roots are incorrect ie, once the correct answer is seen, ignore further working.

(c) As this is an international examination, all **alternative forms of notation** should be accepted. For example: 1.7 , 1·7 , 1,7 ; different forms of vector notation such as $\vec{u}$, $\bar{u}$, $\underline{u}$; $\tan^{-1} x$ for arctan x.

5 Accuracy of Answers

There are two types of accuracy errors, incorrect level of accuracy, and rounding errors.

Unless the level of accuracy is specified in the question, candidates should be penalized **once only IN THE PAPER** for any accuracy error **(AP)**. This could be an incorrect level of accuracy **(only applies to fewer than three significant figures)**, or a rounding error. Hence, on the **first** occasion in the paper when a correct answer is given to the wrong degree of accuracy, or rounded incorrectly, maximum marks are **not** awarded, but on **all subsequent occasions** when accuracy errors occur, then maximum marks **are** awarded.

(a) **Level of accuracy**

(i) In the case when the accuracy of the answer is **specified in the question** (for example: "find the size of angle A to the nearest degree") the maximum mark is awarded **only if** the correct answer is given to the accuracy required.

(ii) When the accuracy is **not** specified in the question, then the general rule applies:

Unless otherwise stated in the question, all numerical answers must be given exactly or to three significant figures.

However, if candidates give their answers to more than three significant figures, this is acceptable

(b) **Rounding errors**

Rounding errors should only be penalized at the **final answer** stage. This does **not** apply to intermediate answers, only those asked for as part of a question. Premature rounding which leads to incorrect answers should only be penalized at the answer stage.

Incorrect answers are wrong, and should not be considered under (a) or (b).

Examples

A question leads to the answer 4.6789….

- 4.68 is the correct 3 s.f. answer.
- 4.7, 4.679 are to the wrong level of accuracy : 4.7 should be penalised the first time this type of error occurs, but 4.679 is **not** penalized, as it has more than three significant figures.
- 4.67 is incorrectly rounded – penalise on the first occurrence.
- 4.678 is incorrectly rounded, but has more than the required accuracy, do **not** penalize.

Note: All these "incorrect" answers may be assumed to come from 4.6789..., even if that value is not seen, but previous correct working is shown. However, 4.60 is wrong, as is 4.5, 4.8, and these should be penalised as being incorrect answers, not as examples of accuracy errors.

6 Graphic Display Calculators

Many candidates will be obtaining solutions directly from their calculators, often without showing any working. They have been advised that they must use mathematical notation, not calculator commands when explaining what they are doing. Incorrect answers without working will receive no marks. However, if there is written evidence of using a graphic display calculator correctly, method marks may be awarded. Where possible, examples will be provided to guide examiners in awarding these method marks.

Calculator penalties

Candidates are instructed to write the make and model of their calculator on the front cover. Please apply the following penalties where appropriate.

(i) Illegal calculators

If candidates note that they are using an illegal calculator, please report this on a PRF, and deduct 10 % of their overall mark.. Note this on the front cover.

(ii) Calculator box not filled in.

Please apply a calculator penalty (***CP***) of 1 mark if this information is not provided. Note this on the front cover.

1. (i) (a) If α and β are 1 – 1, then

if $x_1, x_2 \in S$ and $(\beta \circ \alpha)(x_1) = (\beta \circ \alpha)(x_2)$, *i.e.* $\beta(\alpha(x_1)) = \beta(\alpha(x_2))$ ***(M1)***

then $\alpha(x_1) = \alpha(x_2)$ since β is $1-1$. ***(R1)***

However, if $\alpha(x_1) = \alpha(x_2)$ then $x_1 = x_2$ since α is $1-1$, ***(R1)***

Therefore $\beta \circ \alpha$ is 1 – 1. ***(R1)***

[4 marks]

(b) If $\beta \circ \alpha$ is 1 – 1 then

if for $x_1, x_2 \in S, \alpha(x_1) = \alpha(x_2)$, then ***(M1)***

$\beta(a(x_1)) = \beta(a(x_2))$, *i.e.* $(\beta \circ \alpha)(x_1) = (\beta \circ \alpha)(x_2)$, then ***(R1)***

$x_1 = x_2$ since $\beta \circ \alpha$ is $1-1$, ***(R1)***

hence since $\alpha(x_1) = \alpha(x_2)$ implies that $x_1 = x_2$, so α is $1-1$. ***(R1)***

[4 marks]

(ii)

*	*w*	*x*	*y*	*z*
w	*y*	***z***	***w***	*x*
x	*z*	*w*	***x***	***y***
y	***w***	***x***	***y***	***z***
z	***x***	***y***	***z***	*w*

(A1)

None of the elements *w*, *x*, or *z* can be the identity since they do not give the same element when they operate on that element. ***(R1)***

So, *y* must be the identity element. ***(C1)***

This fills the *y*-column and *y*-row *w*, *x*, *y*, *z* in that order. ***(A1)***

Since the set must be closed, then in the first row $wx = z$. ***(R1)(A1)***

Similarly in the second row, $xz = y$. ***(A1)***

Also, $xz = y$, and $zw = x$ for the same reason. ***(A1)***

This leaves us with *z* without an inverse in the last row. But *x* and *z* are inverses of each other, from last column and hence $zx = y$. ***(R1)***

[9 marks]

Total [17 marks]

2. (i) (a) To apply Prim's algorithm we start with E, choose edges with minimum length connected to a vertex in the tree such that no cycle is formed.
Hence, Prim's algorithm suggests the following sequence of steps.
(Tree is not unique, set-up may differ.) ***(A2)(M3)(A2)***

Edge added	Tree	Weight added
EF	EF	10
ED	EFD	12
(Any of DB, DG, FG)		
DG	EFDG	13
(Any of DB, FG)		
DB	EFDGB	13
DC	EFDGBC	16
CA	EFDGBCA	20
	Length of minimum tree	84

Notes: Award ***(A2)*** for the correct order, ***(M3)*** for the correct description and set up and ***(A2)*** for the answer.
If a student gives the correct answer without showing appropriate work, then award ***[2 marks]*** only.

[7 marks]

(b) Kruskal's algorithm requires that we choose an edge of minimum length regardless of whether it is incident to an existing vertex. We stop after $n-1$ edges have been chosen and no cycles created.
Kruskal's will imply the following sequence: ***(R3)***

Edge added	Tree	Weight added
EF	EF	10
DE	EFD	12
Any of DB, DG, FG		12
DG	EFDG	13
Any of DB, FG		
DB	EFDGB	13
FG cannot be added since it makes a cycle		
BE cannot be added		
DC	EFDGBC	16
CA	EFDGBCA	20
	Length of minimum tree	84

[3 marks]

(ii) The characteristic polynomial corresponding to the equation is $r^2 - 9 = 0$, ***(M1)***
with solutions $r = \pm 3$. ***(A1)***
Hence the general solution is of the form
$a_n = A(3)^n + B(-3)^n$
$a_0 = 6 \Rightarrow A + B = 6$
$a_2 = 54 \Rightarrow 9A + 9B = 54 \Rightarrow A + B = 6$ ***(M1)(A2)***
Hence there is no unique solution, and the general solution would be
$a_n = k(3)^n + (6-k)(-3)^n$ with $k \in \mathbb{R}$. ***(M1)(A1)***

[7 marks]

Total [17 marks]

3. (i) We use the ratio test first.

Since $\left(\frac{k+1}{k}\right)^2$ approaches 1 as k approaches infinity *(M1)(R1)*

$$\left|\frac{\frac{3(k+1)^2}{e^{k+1}}\cdot x^{k+1}}{\frac{3k^2}{e^k}\cdot x^k}\right| = \left(\frac{k+1}{k}\right)^2\cdot\frac{1}{e}|x| \to \frac{1}{e}|x|$$ *(M2)*

$$\Rightarrow \frac{1}{e}|x|<1 \Rightarrow |x|<e \Rightarrow x\in\,]-e,e[$$ *(M1)(A1)*

The series will diverge when $x=\pm e$ since $\frac{3k^2}{e^k}\cdot x^k = \frac{3k^2}{e^k}\cdot e^k = 3k^2$ or *(R2)*

$\frac{3k^2}{e^k}\cdot x^k = \frac{3k^2}{e^k}\cdot(-1)^k e^k = (-1)^k 3k^2$ which is also divergent, so the interval of convergence is $]-e,e[$. *(R2)*

[8 marks]

(ii) (a) If $\sum a_k$ converges, then $a_k \to 0$, *i.e.* $a_k < 1$, *(M1)*

hence $a_k^2 < a_k \Rightarrow \sum a_k^2$ converges by comparison with $\sum a_k$. *(M1)(M1)*

(b) (i) Since $\sum a_k$ has non-negative terms and $k > 0$, then

$\left(a_k - \frac{1}{k}\right)^2 < a_k^2 + \frac{1}{k^2}$, and since $\sum\left(a_k^2+\frac{1}{k^2}\right)$ is a converging series,

then $\sum\left(a_k-\frac{1}{k}\right)^2$ converges by comparison. *(R2)*

(ii) Since $\sum\left(a_k-\frac{1}{k}\right)^2$ converges, and

$\sum\left(a_k-\frac{1}{k}\right)^2 = \sum a_k^2 + \sum\frac{1}{k^2} - 2\sum\frac{a_k}{k}$ with $\sum a_k^2$ and $\sum\frac{1}{k^2}$

both converging, then $2\sum\frac{a_k}{k}$ must also converge and so does $\sum\frac{a_k}{k}$. *(R3)*

[8 marks]

(iii) (a) Maclaurin's series for e^x is $e^x = \sum_0^\infty \frac{x^k}{k!}$ with a remainder of

$|R_n(x)| \le \frac{M}{(n+1)!}x^{n+1}$, and since all derivatives of e^x are e^x, we have *(M2)*

(b) $|R_n(0.2)| \le \frac{e^{0.2}}{(n+1)!}0.2^{n+1} < 0.0005$, and since we know that $e<3 \Rightarrow e^{0.2} < e^{0.5} < 2$,

then $|R_n(0.2)| \le \frac{2}{(n+1)!}0.2^{n+1} < 0.0005$, which by trial and error will give us $n = 3$. *(M1)(A1)*

(c) So, $e^{0.2} \approx 1 + 0.2 + \frac{0.2^2}{2!} + \frac{0.2^3}{3!} = 1 + 0.2 + 0.02 + 0.001 = 1.221$. *(A1)*

[5 marks]

Total [21 marks]

4. (i) (a) For $f(x)$ to be a density function then

$$\int_0^{\infty} k\mathrm{e}^{-\frac{x}{4}}\mathrm{d}x = 1 \Rightarrow \lim_{a\to\infty}\int_0^{a} k\mathrm{e}^{-\frac{x}{4}}\mathrm{d}x = \lim_{x\to\infty}\left[-4k\mathrm{e}^{-\frac{x}{4}}\right]_0^a = 1$$ ***(M1)***

$$0 + 4k = 1 \Rightarrow k = \frac{1}{4}$$ ***(M1)(C1)***

(b) This is a cumulative probability $\mathrm{P}(X > 4)$.

$$\frac{1}{4}\int_4^{\infty} \mathrm{e}^{-\frac{x}{4}}\mathrm{d}x = \left[-\mathrm{e}^{-\frac{x}{4}}\right]_4^{\infty} = \mathrm{e}^{-1} \ (= 0.368)$$ ***(M1)(A1)***

(c) This requires that we find a number *m* such that

$$\int_m^{\infty} \frac{1}{4}\mathrm{e}^{-\frac{x}{4}}\mathrm{d}x = 0.05 \Rightarrow \mathrm{e}^{-\frac{m}{4}} = 0.05 \Rightarrow m = -4\ln 0.05$$ ***(M1)***

So, $m = 11.98$, *i.e.* they should stock 12.0 tons. ***(M1)(A1)***

[8 marks]

(ii) (a) (i) The required estimate is the point estimate of the difference, which simply is $\bar{x} - \bar{y} = 26400 - 25100 = 1300$ kilometres. ***(A1)***

(ii) The standard error is $\sigma_{\bar{x}-\bar{y}} = \sqrt{\dfrac{\sigma_x^2}{n_x} + \dfrac{\sigma_y^2}{n_y}}$, however, we do not know the population's variances, but since the sample sizes are large, the we use the sample variances instead, so ***(C1)***

$$\sigma_{\bar{x}-\bar{y}} = \sqrt{\frac{s_x^2}{n_x} + \frac{s_y^2}{n_y}} = \sqrt{\frac{1200^2}{100} + \frac{1400^2}{100}} = 184.4 \text{ km}.$$ ***(A1)***

so, the error $2\sigma_{\bar{x}-\bar{y}} = 368.8$ km and the estimate will be between 931.2 and 1668.8 km. (*i.e.* 931 – 1670 to 3 s.f.). ***(A1)***

(iii) We need to find the probability for the number to lie between 931.2 and 1668.8 km. This is a normal distribution calculation which yields the level at 95.5 %. ***(M1)(A1)***

(b) Here we have to test hypothesis for difference between means

$\mathrm{H}_0 : \mu_x - \mu_y = 1000$

$\mathrm{H}_0 : \mu_x - \mu_y > 1000$

$$z = \frac{\left((26400 - 25100) - 1000\right)}{184.4} = 1.627$$ ***(R1)***

Since $1.627 < z_c = 1.645$, we fail to reject the null hypothesis, and hence we conclude that we do not have enough evidence to support Type X manufacturer's claim. ***(R1)(A1)***

[9 marks]

continued…

Question 4 continued

(iii) H_0 : Data fit a Poisson distribution.
H_1 : Population is not Poisson. ***(C1)***

We need to estimate the Poisson parameter from the data first:

$$\lambda \approx \bar{x} = \frac{\sum x_i f(x_i)}{n} = \frac{976}{400} = 2.44$$ ***(M1)(A1)***

We need to calculate the expected frequencies, so for each cell $E(n_i) = np_i$, by applying it to our data we get ***(M1)***

number of colonies	n_i	P_i	$E(n_i)$
0	55	0.087	34.86
1	104	0.2127	85.07
2	80	0.2595	103.78
3	62	0.2110	84.41
4	42	0.1287	51.49
5	27	0.0628	25.13
6	9	0.0255	10.22
7 or more	21		5.04

(A2)

Notice that we had to combine the last cells since the count dropped below 5. ***(R1)***

Then $\chi^2 = \frac{(55-34.86)^2}{34.86} + \ldots + \frac{(21-5.04)^2}{5.04} = 79.82$. ***(A1)***

The rejection region based on $k-2=6$ degrees of freedom is $\chi^2 > 12.59$.

Hence the null hypothesis is rejected. Our data does not fit the Poisson distribution. ***(R2)***

[10 marks]

Total [27 marks]

5. (i) Let the point of intersection, the midpoint of MN be Z. Therefore MZ = NZ.

In circle $\text{PQN}: \text{PZ}\cdot\text{ZQ} = \text{NZ}\cdot\text{ZY}$ ***(R1)***

In circle $\text{PQM}: \text{MZ}\cdot\text{ZX} = \text{PZ}\cdot\text{ZQ}$

Therefore $\text{NZ}\cdot\text{ZY} = \text{MZ}\cdot\text{ZX}$ ***(R1)***

Since $\text{MZ} = \text{NZ} \Rightarrow \text{ZY} = \text{ZX}$

Therefore $\text{NZ} - \text{ZX} = \text{MZ} - \text{ZY}$ ***(R2)***

$\Rightarrow \text{NX} = \text{MY}$ ***(AG)***

[4 marks]

(ii) (a) $\text{AE} = \text{AF}, \text{BF} = \text{BD}, \text{CD} = \text{CE}$ since they are tangents to the circle from one point outside the circle. ***(R1)***

Hence $\frac{\text{AF}}{\text{FB}}\cdot\frac{\text{BD}}{\text{DC}}\cdot\frac{\text{CE}}{\text{EA}} = 1$ since the equal segments will cancel out. ***(R1)***

By converse of Ceva's theorem, the lines must be concurrent. ***(R1)***

[3 marks]

(b) Let I be the point in question. If I is the incentre, then it lies on the bisectors of the angles of the triangle. ***(R1)***

(AD) is also perpendicular to (BC). Since (BC) is tangent to the circle and [ID] is a radius. ***(R1)***

With (AD) as a bisector of angle A, and perpendicular to the base,

$\Delta\text{ABD} \cong \Delta\text{ACD}$ by AAS. ***(R1)***

Hence $\text{AB} = \text{AC}$. ***(M1)***

With a similar argument, $\text{CB} = \text{CA}$. ***(C1)***

Therefore ΔABC is equilateral.

[5 marks]

(iii) $\text{KA} = \text{KB}$ ***(R1)***

Similarly, $\text{GB} = \text{GI} = m + n$ ***(C1)***

Also, $\text{AH} = \text{HI} = n$ ***(C1)***

Hence $\text{KG} + \text{KH} = \text{GB} - \text{KB} + \text{KA} + \text{AH}$

$= \text{GB} + \text{AH} = \text{GI} + \text{HI}$

$= m + n + n = m + 2n$ ***(R2)***

Locus of K is then an ellipse with G and H as foci and $m + 2n$ as focal distance. ***(R1)***

[6 marks]

Total [18 marks]