INTERNATIONAL BACCALAUREATE
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MARKSCHEME

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FURTHER MATHEMATICS

Standard Level

Paper 2

13 pages

1. (i) Let X, Y be the mass of one bag and 10 bags respectively.

$X \sim \mathrm{N}(100\ \mathrm{g}, 1\ \mathrm{g}^2)$ $Y = X_1 + X_2 + \ldots + X_{10}$, so $Y \sim \mathrm{N}(1000\ \mathrm{g}, 10\ \mathrm{g}^2)$ ***(M1)(A1)***

$\mathrm{P}(995 < Y < 1005) = 0.886$ (3 s.f.) ***(M1)(A1)***

OR

$\mathrm{P}(995 < Y < 1005) = 0.886$ ***(G2)***

[4 marks]

(ii) (a) $\bar{X} = \dfrac{\sum x_i f_i}{100} = 2.1$ ***(M1)(AG)***

[1 mark]

(b) $\mathrm{P}(X = x_i) = \dfrac{m^{x_i}}{x_i!}\mathrm{e}^{-m}$ ***(M1)***

$\mathrm{P}(X = 0) = \dfrac{2.1^0}{0!}\mathrm{e}^{-2.1} = 0.122 \Rightarrow a = 12.2$ ***(A1)***

$\mathrm{P}(X = 2) = \dfrac{2.1^2}{2!}\mathrm{e}^{-2.1} = 0.270 \Rightarrow b = 27.0$. Also $c\,100 \times (1 - \mathrm{P}(\times \le 5)) = 2.1$ ***(A1)***

H_0: X can be modelled by the Poisson distribution $\mathrm{Po}(2.1)$

H_1: X can not be modelled by the Poisson distribution $\mathrm{Po}(2.1)$. ***(C1)***

$$\chi^2 = \sum_{i=0}^{5} \frac{(f_o - f_e)^2}{f_e} = 2.38 \text{ (3 s.f.)}$$ ***(M1)(A1)***

OR

$\chi^2 = 2.38$ ***(G2)***

degrees of freedom $\upsilon = 4$ ***(M1)***

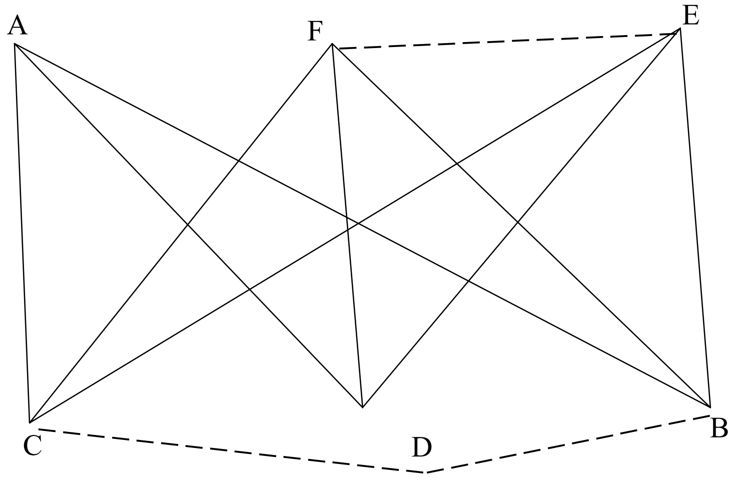
so $\chi^2_{(4,\ 5\%)} = 9.488 > 2.38$ ***(A1)***

We can not reject H_0 and conclude that we do not have enough evidence to say that the data cannot be modelled by a Poisson distribution with mean 2.1. ***(R1)***

[9 marks]

Total [14 marks]

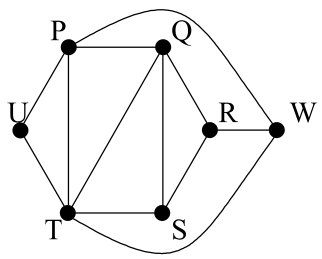
2. (i) (a) (i) Since the graph can be redrawn as follows:



(A1)

And since this graph contains $K_{3,3}$ as a sub-graph, then it cannot be planar because its sub-graph is not planar. ***(M1)(R1)(A1)***

(ii) Second graph can be drawn in the following way:



(M1)

Therefore the graph is planar. ***(C1)***

[6 marks]

(b) In the graph the orders of the vertices are 2, 4, 4, 4, 4, and 2 ***(A1)***
so by the theorem we can deduce that there is a Hamiltonian cycle. ***(M1)(C1)***
One possible cycle is $1, 2, 3, 4, 5, 6, 1$ ***(A1)***

Note:	Vertices are numbered in an anti-clockwise direction starting with the vertex of degree 2 at the top right corner.

[4 marks]

continued…

Question 2 continued

(ii) $n = 19x + 4$ and $n = 11y + 1$ ***(M1)***
$\Rightarrow 19x - 11y = -3$

Since $(19, 11) = 1$, the equation has an integer solution. ***(M1)***
Applying Euclid's algorithm we find:

$$\left.\begin{array}{ll} 19 = 11\times 1 + 8 & a = b + r_1 \\ 11 = 8\times 1 + 3 & b = r_1 + r_2 \\ 8 = 3\times 2 + 2 & r_1 = 2r_2 + r_3 \\ 3 = 2\times 1 + 1 & r_2 = r_3 + r_4 \end{array}\right\} \Rightarrow \begin{array}{l} r_1 = a - b \\ r_2 = 2b - a \\ r_3 = 3a - 5b \\ r_4 = -4a + 7b \end{array}$$ ***(M1)(M1)***

Since $r_4 = 1$ the particular solutions are to be found by the following:
$19\times(-4) - 11\times(-7) = 1$/ multiply by $(-3) \Rightarrow 19\times 12 - 11\times 21 = -3$ ***(M1)***
So $x_0 = 12$ and $y_0 = 21$. ***(A1)***
The general solutions are $x = 12 - 11t$, $y = 21 - 19t$, $t \in \mathbb{Z}$ ***(M1)***
$\Rightarrow n = 232 - 209t, t \in \mathbb{Z}$. ***(A1)***

For values of $t \in \{1, 0, -1\}$, the solutions are 23, 232, and 441. ***(A1)***

[9 marks]

Total [19 marks]

3. (i) Reflexive: $\boldsymbol{A} R \boldsymbol{A}$, because $\boldsymbol{A} = \boldsymbol{I}^{-1}\boldsymbol{AI}$ and $\boldsymbol{I}$ is an invertible matrix. ***(C1)(C1)***

Symmetrical: $\boldsymbol{A} R \boldsymbol{B}$, then there is an invertible matrix $\boldsymbol{X}$ such that ***(C1)***

$\boldsymbol{B} = \boldsymbol{X}^{-1}\boldsymbol{AX} \Rightarrow \boldsymbol{A} = (\boldsymbol{X}^{-1})^{-1}\boldsymbol{BX}^{-1}$, where $\boldsymbol{X}^{-1}$ is an invertible matrix, ***(M1)***

so $\boldsymbol{B} R \boldsymbol{A}$. ***(C1)***

Transitive: $\boldsymbol{A} R \boldsymbol{B}$ and $\boldsymbol{B} R \boldsymbol{C}$, means that there are invertible matrices ***(C1)***

$\boldsymbol{X}$ and $\boldsymbol{Y}$ such that $\boldsymbol{B} = \boldsymbol{X}^{-1}\boldsymbol{AX}$, $\boldsymbol{C} = \boldsymbol{Y}^{-1}\boldsymbol{BY} \Rightarrow \boldsymbol{C} = \boldsymbol{Y}^{-1}\boldsymbol{X}^{-1}\boldsymbol{AXY} = (\boldsymbol{XY})^{-1}\boldsymbol{A}(\boldsymbol{XY})$, ***(M1)***

where $\boldsymbol{XY}$ is an invertible matrix so consequently $\boldsymbol{A} R \boldsymbol{C}$. ***(C1)***

[8 marks]

(ii) Theorem: if a and b are two elements of a subgroup then ab^{-1} is also an element of the sub group. ***(M1)***

Let S_1 and S_2 be two subgroups and $S_1 \cap S_2$ be the intersection.

$a, b \in S_1 \cap S_2 \Rightarrow a, b \in S_1 \Rightarrow ab^{-1} \in S_1$ and $a, b \in S_2 \Rightarrow ab^{-1} \in S_2$ ***(M2)***

$\Rightarrow ab^{-1} \in S_1 \cap S_2$.

Therefore $S_1 \cap S_2$ is a subgroup of the same group. ***(C1)***

[4 marks]

(iii) (a) Using n to represent the equivalence class for n, the elements of $\mathbb{Z}_p \times \mathbb{Z}_p$ will be written as (i, j) where $i, j \in \{0, 1, \ldots, p\}$. ***(C1)***

Since the order of $\mathbb{Z}_3 \times \mathbb{Z}_3$ is 9, then the possible order of a subgroup is 1, 3, or 9. Obviously (0, 0) and $\mathbb{Z}_3 \times \mathbb{Z}_3$ itself are two of the subgroups. ***(R1)(C1)***

We need to take the subgroups of order 3.

Take the subgroup $\{(0, 0), (0, 1), (0, 2)\}$. We can represent it as $\langle(0, 1)\rangle$ since it is generated by (0, 1). ***(C1)***

The other groups are: $\langle(1, 0)\rangle, \langle(1, 1)\rangle$, and $\langle(1, 2)\rangle$. Each group can be generated by any of its "non-zero" elements – they are cyclic. ***(A1)(C1)***

(b) For $\mathbb{Z}_p \times \mathbb{Z}_p$ the possible orders are again 0, p^2 and p by Lagrange's Theorem. ***(R1)***

So, the only groups we need to look for are the ones with order p. ***(C1)***

Since the number of elements of $\mathbb{Z}_p \times \mathbb{Z}_p$ is p^2 there there are $p^2 - 1$ generators for the subgroups. ***(C1)***

Also for each subgroup we have $p - 1$ generators. Therefore the number of subgroups of order p is $\dfrac{p^2 - 1}{p - 1} = p + 1$, and the total number of subgroups is then $p + 3$ ***(R1)(A1)***

[11 marks]

Total [23 marks]

4. (i) (a) (i) Since $f(1) = g(1) = 1, x = 1$ is a solution. ***(M1)(A1)***

(ii) To use the Newton-Raphson method we consider the equation $h(x) = f(x) - g(x) = 0$.

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)} = x_n - \frac{3 - 2x - e^{1-x}}{e^{1-x} - 2}$$ ***(M1)(M1)(A1)***

By applying four iterations of the Newton-Raphson method we get $x = -0.256$. ***(G1)***

(iii) h is continuous and differentiable over the set of real numbers, and $h'(x) = e^{1-x} - 2$. ***(C1)***

$h'(x) = 0$ when $x = 1 - \ln 2$. So, by Rolle's theorem, h must have two zeros – one before and one after 0, which has been verified above. H cannot have solutions anywhere else, otherwise $h'(x)$ will have to have another zero which is not possible. Therefore –0.256 and 1 are the only two zeros. ***(R3)***

[10 marks]

(b) Since this function is differentiable to the fourth order over the interval [–0.256, 1], then the error of the estimate for 8 intervals satisfies the following inequality:

$$|E| \le \frac{(b-a)^5 M}{180n^4} = \frac{1.256^5}{180 \times 8^4} \cdot M \text{ with } M = \max\left|f^{(4)}(x)\right| \text{ on } [-0.256, 1].$$ ***(R2)***

Now $\left|f^{(4)}(x)\right| = e^{1-x}$ which has a max of approximately 4, when $x = -0.256$. ***(M1)(A1)***

This will give an error of 0.000017 < 0.00002. ***(G1)(AG)***

[5 marks]

(ii) Maclaurin's series for $\sin x$ requires that x be expressed in radians, hence $3^\circ = \frac{\pi}{180} \times 3 = \frac{\pi}{60}$ ***(M1)***

Also,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + R_n \Rightarrow$$

$$\sin x = 0 + x - \frac{x^3}{3!} + \ldots + \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c)$$ ***(M1)(A1)***

Since $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$, then $f^{(n+1)}(c) = 1$ ***(C1)***

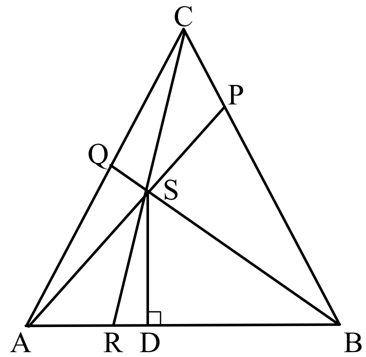
Thus $|R_n| \le \frac{\left|\frac{\pi}{60}\right|^{n+1}}{(n+1)!} \le 0.000005$, then by trial and error we find that the minimum $n = 3$, this yields

$$\sin 3^\circ \approx \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} \approx 0.05234$$ ***(M1)(A1)(A1)***

[7 marks]

Total [22 marks]

5. (i) (a)



Area $\Delta(\text{ARS}) = \dfrac{\text{AR} \times \text{SD}}{2}$

and Area $\Delta(\text{RBS}) = \dfrac{\text{RB} \times \text{SD}}{2}$ ***(M1)***

$\dfrac{\text{Area } \Delta(\text{ARS})}{\text{Area } \Delta(\text{RBS})} = \dfrac{\text{AR}}{\text{RB}}$ ***(A1)(AG)***

[2 marks]

(b) $\dfrac{\text{AR}}{\text{RB}} \times \dfrac{\text{BP}}{\text{PC}} \times \dfrac{\text{CQ}}{\text{QA}} = \dfrac{\text{Area } \Delta(\text{ARS})}{\text{Area } \Delta(\text{RBS})} \times \dfrac{\text{Area } \Delta(\text{BPS})}{\text{Area } \Delta(\text{PCS})} \times \dfrac{\text{Area } \Delta(\text{CQS})}{\text{Area } \Delta(\text{QAS})}$ ***(M1)(C1)***

$= \dfrac{\text{SA} \times \text{SR} \times \sin(\text{RSA})}{\text{SR} \times \text{SB} \times \sin(\text{BSR})} \times \dfrac{\text{SB} \times \text{SP} \times \sin(\text{BSP})}{\text{SP} \times \text{SC} \times \sin(\text{PSC})} \times \dfrac{\text{SC} \times \text{SQ} \times \sin(\text{CSQ})}{\text{SQ} \times \text{SA} \times \sin(\text{QSA})} = 1$ ***(M1)(C1)(R1)***

i.e. $\dfrac{\text{AR}}{\text{RB}} = \dfrac{\text{BP}}{\text{PC}} = \dfrac{\text{CQ}}{\text{QA}} = 1$ (Ceva's theorem) ***(AG)***

[5 marks]

(ii) (a) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} = \dfrac{\cos t}{-4\sin t} \Rightarrow \dfrac{\text{gradient}}{\text{tangent}} = \dfrac{-1}{4\tan t}$ ***(M1)(A1)***

For the line MP, the gradient $m = 4\tan t$.

Therefore $y - \sin t = 4\tan t(x - 4\cos t)$ ***(M1)***

$\Rightarrow \quad y = 4x\tan t - 15\sin t$. ***(A1)***

The diameter through the point N goes through the origin (centre of the circle) so its equation is ***(M1)***

$y = x\tan t$. ***(A1)***

Now we have to solve the system $\begin{cases} y = 4x\tan t - 15\sin t \\ y = x\tan t \end{cases}$, ***(M1)***

$\Rightarrow x = 5\cos t,\ y = 5\sin t$. ***(A1)***

Which is the parametric equation of a circle with radius 5. ***(R1)(A1)***

[10 marks]

continued…

Question 5 (ii) continued

(b) The diameter through the point R has equation $y = -x\tan t$. ***(A1)***

Solving the system $\begin{cases} y = 4x\tan t - 15\sin t \\ y = -x\tan t \end{cases}$ ***(M1)***

the coordinates of the point Q are found to be $(3\cos t, -3\sin t)$. ***(A1)***

Since $t \in \left]0, \frac{\pi}{2}\right[$ the locus is an arc of the circle, ***(A1)***

the centre at the origin, and radius 3. The arc goes from the point $(0, -3)$ to the point $(3, 0)$, excluding the endpoints. ***(R1)***

[5 marks]

Total [22 marks]