INTERNATIONAL BACCALAUREATE
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MARKSCHEME

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FURTHER MATHEMATICS

Standard Level

Paper 2

1. (a) The distribution is also normal with mean $4.02 - 4.00 = 0.02$ and a standard deviation of $\sqrt{0.1^2 + 0.1^2} = \sqrt{0.02} = 0.1\sqrt{2}$ ***(A1)*** ***(M1)(A1)***

[3 marks]

(b) $\mathrm{P}\left((B-D)>0\right) = \mathrm{P}\left(Z > \dfrac{0-0.02}{0.1\sqrt{2}}\right) = \mathrm{P}(Z > -0.1414) = 1 - 0.4438 = 0.556$ ***(M1)(M1)***

55.6 % of the rods can fit into the tubes ***(A1)***

[3 marks]

(c) After heating

Tubes will have a mean of $4.02 \times 1.05 = 4.221$ and a variance of $1.05^2 \times 0.1^2 = 0.011025$. ***(M1)***

(standard deviation of 0.105) ***(A1)***

Rods will have a mean of $4.00 \times 1.03 = 4.12$ and a variance of $1.03^2 \times 0.1^2 = 0.010609$.

(standard deviation of 0.103) ***(A1)***

$B - D$ will be a normal distribution with mean $4.221 - 4.12 = 0.101$ and a variance of $0.010609 + 0.011025 = 0.021634$. (standard deviation of 0.1471) ***(C1)***

$\mathrm{P}\left((B-D)>0\right) = \mathrm{P}\left(Z > \dfrac{0-(0.101)}{\sqrt{0.021634}}\right) = \mathrm{P}(Z > -0.68668) = 1 - 0.2461 = 0.754$ ***(M1)***

75.4 % will fit after heating. ***(A1)***

[6 marks]

(d) The new standard deviations are now 0.05 each.

The sampling distribution of the difference between two means is Normal with mean of 0.02 and the following standard deviation: ***(C1)***

$s_{\overline{B}-\overline{D}} = \sqrt{\dfrac{0.05^2}{10} + \dfrac{0.05^2}{10}} = 0.02236$ ***(M1)(A1)***

$\mathrm{P}(\overline{B} - \overline{D} > l) = \mathrm{P}\left(Z > \dfrac{l - 0.02}{0.02236}\right) = 0.9$ ***(M1)***

$\Rightarrow l - 0.02 = -1.28 \times 0.02236 \Rightarrow l = -0.0086$ ***(M1)(A1)***

[6 marks]

(e) This is a hypothesis test of the difference of two means

$\mathrm{H}_0 : \mu_B - \mu_D = 0.02$

$\mathrm{H}_1 : \mu_B - \mu_D \neq 0.02$ ***(C1)***

Since the standard deviations cannot be assumed known, then this is a 2-tailed t-test. ***(C1)***

With a 2-tailed test, we have the following results:

Assume equal variance since they come from the previous populations. ***(M1)***

$\mathrm{t} = \pm 1$ with a p-value of 0.326 and 27 degrees of freedom. ***(G1)***

Since $0.326 > 0.05$, we fail to reject the null hypothesis and conclude that we do not have enough evidence to say that the means are different by more than 0.02. ***(R1)***

[5 marks]

Total [23 marks]

2. (a) For this set to be a group, it has to be closed:

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}=\begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$, where the four elements in the product matrix are real, ***(M1)***

and the difference of the two diagonal products is

$ad(eh-fg)+bc(fg-eh)=ad(1)+bc(-1)=ad-bc=1$ ***(M1)***

Hence the product of two matrices of this set belongs to the same set and the set is therefore closed under matrix multiplication. ***(R1)***

(Candidates may argue that the determinant of a product is the product of determinants which should receive full marks.)

Matrix multiplication is associative. ***(R1)***

The identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ has the same property since $1\times1-0\times0=1$ and therefore ***(M1)*** is a member of the set. ***(R1)***

Also, for every element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}=\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\in S$ ***(M1)***

because $da-bc=1$, and $a,b,c,d\in\mathbb{R}$ ***(R1)***

Therefore $(S,\otimes)$ is a group.

[8 marks]

(b) Since $\frac{z}{z}=1\in\mathbb{R}$, the relation is reflexive, ***(C1)***

if $\frac{z}{w}=a\neq0$, then $\frac{w}{z}=\frac{1}{a}\in\mathbb{R}$, and the relation is symmetric. ***(C1)***

Also, if $\frac{z}{w}=a\neq0$, and $\frac{w}{u}=b\neq0$, then $\frac{z}{u}=a\times b\in\mathbb{R}$ ***(M1)***

Therefore the relation is transitive and hence an equivalence relation. ***(R1)(AG)***

[4 marks]

(c) Since the operation of *H* is the same as that for *G*, the operation is associative.

Since *H* is non-empty, then for any $x\in H$ let $a=x$ and $b=x$, then by hypothesis ***(C1)***

$a\odot b^{-1}=x\odot x^{-1}=e\in H$. ***(R1)***

Therefore the identity element belongs to *H*. ***(C1)***

As $e\in H$, let $a=e$ in the hypothesis, then $a\odot b^{-1}=e\odot b^{-1}=b^{-1}\in H$. ***(R1)***

So, whenever $b\in H$, $b^{-1}\in H$. ***(C1)***

Since we showed that whenever $b\in H$, $b^{-1}\in H$, take $a=x$ and $b=y^{-1}$. ***(R1)***

We have $x\odot y=x\odot(y^{-1})^{-1}=a\odot b^{-1}$ which is in *H*. ***(M1)***

Hence closure is verified and *H* is a subgroup of *G*. ***(C1)***

[8 marks]

Total [20 marks]

3. (i) The characteristic polynomial for this difference equation is $r^2 - 7r + 12 = 0$ with solutions $r_1 = 3$ and $r_2 = 4$. *(M1)(A1)*

Hence, the general form of the solution to this equation is $b_n = c3^n + d4^n$ which has to be solved using the initial conditions. *(C1)*

The system of equations is $3c + 4d = 1, 9c + 16d = 7$ *(C1)*

which yields a solution of $c = -1$ and $d = 1$ *(A1)*

Therefore the solution of the equation is $b_n = 4^n - 3^n$ *(A1)*

[6 marks]

(ii) The difference between Prim's Algorithm and Kruskal's is that, in the process of finding a minimum spanning tree, at every stage Prim's adds an edge of minimum weight which is connected to an edge that is already in the tree while Kruskal's adds only an edge with minimum weight regardless whether it connects an existing one. *(R1)*

Prim's	
Edge added	Weight
FG	2
GE	3
EC	3
CA	2
CB	4
BD	3
	17

Kruskal's	
Edge added	Weight
FG	2
AC	2
GE	3
BD	3
CE	3
CB	4
	17

(M2)
(A1)

Note: Please observe that in Kruskal's algorithm there is an ascending order of weights while in Prim's there is not.

[4 marks]

(iii) (a) Since $10 = 9 + 1$ *(C1)*

then $10^n = \sum_0^n \binom{n}{i} 9^{n-i} = 9^n + n \times 9^{n-1} + \cdots + 9 + 1 = 3Q + 1, Q \in \mathbb{N}$ *(M1)(R1)*

Note: Some students may use mathematical induction, please award ***(C3)***.

[3 marks]

(b) $y = a_n(3k_n + 1) + a_{n-1}(3k_{n-1} + 1) + \cdots + a_1(3 \times 3 + 1) + a_0$ *(M1)*

$= 3(a_n k_n + a_{n-1}k_{n-1} + \cdots + a_1 \times 3) + a_n + a_{n-1} + \cdots + a_1 + a_0$ *(M1)*

$= 3k + a_n + a_{n-1} + \cdots a_1 + a_0$ *(R1)*

[3 marks]

(c) If $3 | (a_n + a_{n-1} + \cdots + a_1 + a_0)$ then $3 | 3k + (a_n + a_{n-1} + \cdots + a_1 + a_0)$ and the result follows. *(R1)*

[1 mark]

Total [17 marks]

4. (i) (a) Rolle's theorem states that if f is continuous over an interval $[a, b]$ and differentiable over (a, b) and $f(a) = f(b)$, then there is a number c in (a, b) such that $f'(c) = 0$. ***(C1)*** ***(C1)***

[2 marks]

(b) (i) $f'(x) = 2x + \sin x$, and since $f''(x) = 2 + \cos x \geq 1 > 0$ for all x, then f' is increasing for all values of x. ***(M1)*** ***(R1)***

Now, since $f(0) < 0$ while $f(1)$ and $(f - 1)$ are both positive, f must have at least two zeros. However, if it has more than two zeros, according to Rolle's theorem f' must have at least two zeros. This cannot be true since f' is an increasing function and can only cross the x-axis once. ***(R1)*** ***(R1)*** ***(R1)***

[5 marks]

(ii) Taylor's expansion to the second term for $\cos x$ is

$$P_2(x) = 1 - \frac{x^2}{2}$$ ***(A1)***

The approximate solution can be achieved by solving the following equation ***(M1)(M1)***

$$x^2 = 1 - \frac{x^2}{2} \Rightarrow 3x^2 = 2 \Rightarrow x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$$ ***(AG)***

[3 marks]

(iii) In the situation above, the equation can be written as

$$x^2 = 1 - \frac{x^2}{2} + R(x), \text{ where } |R(x)| \leq \frac{|x|^3}{6}$$ ***(C1)***

and since $|x| \leq 1, \left|3x^2 - 2\right| = 2|R(x)| \leq \frac{1}{3}$ ***(R1)(M1)***

This in turn gives $\frac{\sqrt{5}}{3} \leq |x| \leq \frac{\sqrt{7}}{3}$. ***(A2)***

[5 marks]

(iv) Since $f'(x) = 2x + \sin x$ ***(M1)***

Then $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ and with $x_0 = 1, x_1 = 0.838$ and after three iterations the value of the expression is 0.824. ***(M1)(C1)*** ***(A1)***

[4 marks]

(ii) $$\left|\frac{a_{n+1}}{a_n}\right| = \frac{\left|\left[(n+1)x\right]^{n+1}\right|}{(n+1)!} \times \frac{n!}{\left|(nx)^n\right|} = \frac{(n+1)^{n+1}\left|x^{n+1}\right|}{(n+1)!} \times \frac{n!}{n^n\left|x^n\right|}$$ ***(M1)***

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}|x|}{(n+1)} \times \frac{1}{n^n} = \frac{(n+1)^n|x|}{n^n} = \left(\frac{n+1}{n}\right)^n |x|$$ ***(M1)***

$$\Rightarrow \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n |x| = e|x|$$ ***(M1)(A1)***

$$\Rightarrow |x| < \frac{1}{e}$$

$\Rightarrow$ radius is $\frac{1}{e}$ ***(A1)***

[5 marks]

Total [24 marks]

5. (a) $\dfrac{\text{CD}}{\text{DB}} = \dfrac{\text{AC}}{\text{AB}} = \dfrac{8}{11}$ ***(M1)(A1)***

[2 marks]

(b) Since $\dfrac{\text{CD}}{\text{DB}} = \dfrac{8}{11}$, then $\text{DB} = \dfrac{11}{8} \times \text{CD} = 11k$ ***(M1)(A1)***

Also, $\text{CM} = \text{CD} + 3 = 8k + 3$ ***(A1)***

Note: Accept $\text{CM} = 9\frac{1}{2}k$

[3 marks]

(c) $\text{BC} = 2\text{CM} = 16k + 6$ ***(M1)***

$\text{BC} = \text{CD} + \text{DB} = 8k + 11k = 19k$ ***(M1)***

then $16k + 6 = 19k$ and $k = 2$ ***(R1)***

[3 marks]

(d) The bisector of angle C divides the segment [AB] in the ratio $\text{AE} : \text{EB} = \text{AC} : \text{CB}$. ***(M1)***

Since $\text{CB} = 19k = 38$, then $\text{AE} : \text{EB} = 24 : 38 = 12 : 19$. ***(C1)(A1)***

Now, let $\text{AE} = 12m$ and $\text{EB} = 19m$, and $\text{FA} = y$. So, by Menelaus' Theorem: ***(M1)***

$\dfrac{\text{BM}}{\text{MC}} \times \dfrac{\text{CF}}{\text{FA}} \times \dfrac{\text{AE}}{\text{EB}} = -1 \Rightarrow \dfrac{19}{19} \times \dfrac{24 + y}{-y} \times \dfrac{12m}{19m} = -1 \Rightarrow$ ***(R1)(R1)***

$19y = 288 + 12y \Rightarrow y = \dfrac{288}{7}$ ***(M1)(A1)***

[8 marks]

Total [16 marks]