## High Performance – Question 10

10. (a) Find $\int xe^{-x^2}\,dx$ .

(b)

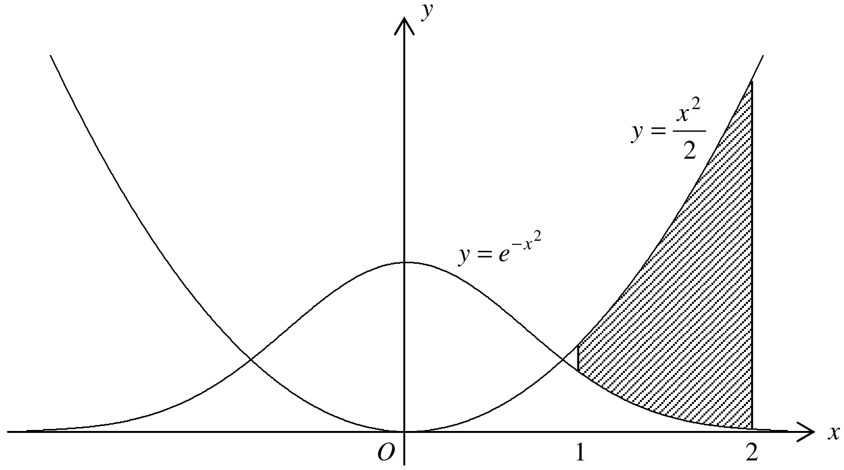


**Figure 1**

In Figure 1, the shaded region is bounded by the curves $y = \frac{x^2}{2}$ and $y = e^{-x^2}$ , where $1 \le x \le 2$ . Find the volume of the solid generated by revolving the shaded region about the $y$-axis.

(6 marks)

Start each question on

a) $\int x e^{-x^2} dx$

$= \frac{1}{2}\int e^{-x^2} d(x^2)$ 1M

$= -\frac{1}{2}\int e^{-x^2} d(-x^2)$

$= -\frac{1}{2}e^{-x^2} + C$ where $C$ is a constant ✓ 1A

b) The volume of revolution

~~$= \int_1^2 2\pi x($~~

~~$= \pi(2)^2 2 - \int_1^2 2\pi x(e^{-x^2})dx - \int_1^2 2\pi x(\frac{x^2}{2})dx$~~

~~$= \int^2$~~

✓

$= \int_1^2 2\pi x(\frac{x^2}{2})dx - \int_1^2 2\pi x(e^{-x^2})dx$ 1M+1A

$= \int_1^2 \pi x^3 dx - \int_1^2 2\pi x e^{-x^2} dx$

$= \frac{\pi x^4}{4}\Big|_1^2 + \pi e^{-x^2}\Big|_1^2$ ✓ 1M

$= \left(\frac{2^4\pi}{4} - \frac{\pi}{4}\right) + \left(\pi e^{-4} - \pi e^{-1}\right)$

$= 4\pi - \frac{\pi}{4} + \pi\left(\frac{1}{e^4} - \frac{1}{e}\right)$

$= \frac{15\pi}{16} + \pi\left(\frac{1-e^3}{e^4}\right)$

$= \left(\frac{15}{16} + \frac{1-e^3}{e^4}\right)\pi$

✗

## High Performance – Question 13

13. (a) Let $f(x)$ be an odd function for $-p \le x \le p$, where $p$ is a positive constant.

Prove that $\int_0^{2p} f(x-p)\,dx = 0$.

Hence evaluate $\int_0^{2p} [f(x-p)+q]\,dx$, where $q$ is a constant.

(4 marks)

(b) Prove that $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{1+\sqrt{3}\tan x}{2}$.

(2 marks)

(c) Using (a) and (b), or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$.

(4 marks)

13a. $-f(x) = f(-x)$ ← For odd function,

$\int_0^{2p} f(x-p)\,dx$ (Let $u = x-p$, $du = dx$. When $x = 2p$, $u = p$. When $x = 0$, $u = -p$.) 1M

$= \int_{-p}^{p} f(u)\,du$

$= \int_0^p f(u)\,du + \int_{-p}^{0} f(u)\,du$

$= \int_0^p f(x)\,dx + \int_{-p}^{0} f(x)\,dx$

$= \int_0^p f(x)\,dx - \int_{-p}^{0} f(-x)\,dx$ (Let $t = -x$, $dt = -dx$. When $x = 0$, $t = 0$. When $x = -p$, $t = p$.) 1M

$= \int_0^p f(x)\,dx + \int_p^0 f(t)\,dt$

$= \int_0^p f(x)\,dx - \int_0^p f(t)\,dt$

$= \int_0^p f(x)\,dx - \int_0^p f(x)\,dx$ ✓ 1

$= 0$

$\int_0^{2p} [f(x-p)+q]\,dx$

$= \int_0^{2p} f(x-p)\,dx + \int_0^{2p} q\,dx$

$= qx\Big|_0^{2p}$

$= 2pq$ ✓ 1A

b. $\dfrac{\sqrt{3}+\tan(x-\frac{\pi}{6})}{\sqrt{3}-\tan(x-\frac{\pi}{6})} = \dfrac{\sqrt{3}+\frac{\tan x - \tan\frac{\pi}{6}}{1+\tan x\tan\frac{\pi}{6}}}{\sqrt{3}-\frac{\tan x - \tan\frac{\pi}{6}}{1+\tan x\tan\frac{\pi}{6}}}$ 1M

$= \dfrac{\sqrt{3}(1+\frac{\sqrt{3}}{3}\tan x)+\tan x-\frac{\sqrt{3}}{3}}{\sqrt{3}(1+\frac{\sqrt{3}}{3}\tan x)-\tan x+\frac{\sqrt{3}}{3}}$

$= \dfrac{\sqrt{3}+\tan x+\tan x-\frac{\sqrt{3}}{3}}{\sqrt{3}+\tan x-\tan x+\frac{\sqrt{3}}{3}}$

$= \dfrac{2\tan x+\sqrt{3}-\frac{\sqrt{3}}{3}}{\sqrt{3}+\frac{\sqrt{3}}{3}}$

$= \dfrac{6\tan x+3\sqrt{3}-\sqrt{3}}{3\sqrt{3}+\sqrt{3}}$

$= \dfrac{6\tan x+2\sqrt{3}}{4\sqrt{3}}$

$= \dfrac{3\tan x+\sqrt{3}}{2\sqrt{3}}$

$= \dfrac{\sqrt{3}\tan x+1}{2}$ ✓ 1

c. $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,dx$

$= \int_0^{\frac{\pi}{3}} \ln\left(\frac{1+\sqrt{3}\tan x}{2}\times 2\right)dx$

$= \int_0^{\frac{\pi}{3}} \left[\ln\left(\frac{1+\sqrt{3}\tan x}{2}\right)+\ln 2\right]dx$

$= \int_0^{\frac{\pi}{3}} \ln\left[\frac{\sqrt{3}+\tan(x-\frac{\pi}{6})}{\sqrt{3}-\tan(x-\frac{\pi}{6})}\right] + \int_0^{\frac{\pi}{3}} \ln 2\,dx$ 1M

$= \int_0^{\frac{\pi}{3}} \ln(\sqrt{3}+\tan(x-\frac{\pi}{6}))\,dx - \int_0^{\frac{\pi}{3}} \ln(\sqrt{3}-\tan(x-\frac{\pi}{6}))\,dx + \int_0^{\frac{\pi}{3}} \ln 2\,dx$

$= \int_0^{\frac{\pi}{3}} \ln 2\,dx$ ? ? ✓

$= \ln 2 \times \Big|_0^{\frac{\pi}{3}}$

$= \frac{\pi}{3}\ln 2$ ✓ 1A

## 表現中等 — 第五題

5. (a) 已知對任意實數 $x$， $\cos(x+1)+\cos(x-1)=k\cos x$ 。求 $k$ 的值。

(b) 不用計算機，求 $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}$ 的值。

（6 分）

5.(a) $\cos(x+1)+\cos(x-1)=k\cos x$

$2\cos\frac{(x+1)+(x-1)}{2}\cos\frac{(x+1)-(x-1)}{2}=k\cos x$ ✓ 1M

$2\cos\frac{2x}{2}\cos\frac{2}{2}=k\cos x$

$2\cos x\cos(1)=k\cos x$

$2\cos(1)=k$ ✓ 1A

$k=2.000$ X

(b) $\begin{vmatrix} \cos 1 & \cos 2 & \cos 3 \\ \cos 4 & \cos 5 & \cos 6 \\ \cos 7 & \cos 8 & \cos 9 \end{vmatrix}=\cos 1(\cos 5\cos 9-\cos 6\cos 8)$

$-\cos 2(\cos 4\cos 9-\cos 6\cos 7)$

$+\cos 3(\cos 4\cos 8-\cos 5\cos 7)$

$=\frac{\cos 1}{2}\left[(\cos 14+\cos 4)-(\cos 14+\cos 2)\right]$ 1M

$-\frac{\cos 2}{2}\left[(\cos 13+\cos 5)-(\cos 13+\cos 1)\right]$

$+\frac{\cos 3}{2}\left[(\cos 12+\cos 4)-(\cos 12+\cos 2)\right]$

$=\frac{\cos 1}{2}(\cos 4-\cos 2)-\frac{\cos 2}{2}(\cos 5-\cos 1)$

$+\frac{\cos 3}{2}(\cos 4-\cos 2)$ X

$=\frac{(\cos 4-\cos 2)(\cos 1+\cos 3)}{2}$

寫於邊界以外的答案，將不予評閱。

## Mid Performance – Question 12

12.

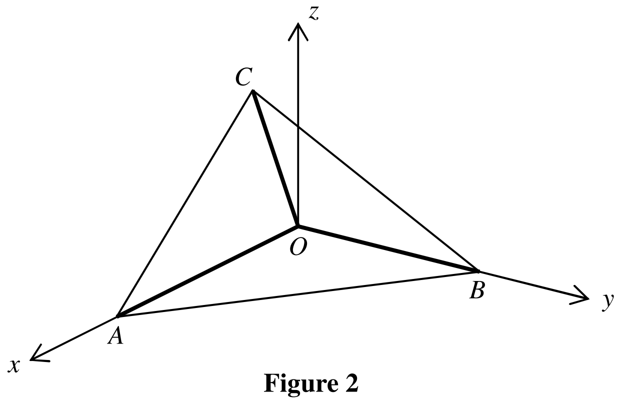


**Figure 2**

Let $\overrightarrow{OA} = \mathbf{i}$ , $\overrightarrow{OB} = \mathbf{j}$ and $\overrightarrow{OC} = \mathbf{i}+\mathbf{j}+\mathbf{k}$ (see Figure 2). Let $M$ and $N$ be points on the straight lines $AB$ and $OC$ respectively such that $AM : MB = a : (1-a)$ and $ON : NC = b : (1-b)$ , where $0 < a < 1$ and $0 < b < 1$ . Suppose that $MN$ is perpendicular to both $AB$ and $OC$ .

(a) (i) Show that $\overrightarrow{MN} = (a+b-1)\mathbf{i} + (b-a)\mathbf{j} + b\mathbf{k}$ .

(ii) Find the values of $a$ and $b$ .

(iii) Find the shortest distance between the straight lines $AB$ and $OC$ .

(8 marks)

(b) (i) Find $\overrightarrow{AB} \times \overrightarrow{AC}$ .

(ii) Let $G$ be the projection of $O$ on the plane $ABC$ , find the coordinates of the intersecting point of the two straight lines $OG$ and $MN$ .

(5 marks)

a $\overrightarrow{MN} = \overrightarrow{ON} - \overrightarrow{OM}$

$= (\vec{i}+\vec{j}+\vec{k})b - \dfrac{(1-a)(\vec{i}) + (a)\vec{j}}{1-a+a}$ 1M

$= b\vec{i} + b\vec{j} + b\vec{k} - (1-a)\vec{i} - a\vec{j}$

$= (a+b-1)\vec{i} + (b-a)\vec{j} + b\vec{k}$ // ✓ 1

(ii) $\because \overrightarrow{MN} \perp$ to both $\overrightarrow{AB}$ and $\overrightarrow{OC}$.

$\overrightarrow{MN} \cdot \overrightarrow{AB} = 0$ and $\overrightarrow{MN} \cdot \overrightarrow{OC} = 0$ 1M+1M

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$ $[(a+b-1)\vec{i} + (b-a)\vec{j} + b\vec{k}] \cdot [\vec{j} - \vec{i}] = 0$ — (1)

$= \vec{j} - \vec{i}$ $[(a+b-1)\vec{i} + (b-a)\vec{j} + b\vec{k}] \cdot [\vec{i} + \vec{j} + \vec{k}] = 0$

(1) : $1 - a - \cancel{b} + \cancel{b} - a = 0$

$-2a = -1$

$a = \frac{1}{2}$ // ✓ 1A

(2) $a + b - 1 + b - a + b = 0$

$3b - 1 = 0$

$b = \frac{1}{3}$ // ✓ 1A

(iii) The distance is $\overrightarrow{OM}$

$\left| (1 - \frac{1}{2})\vec{i} + \frac{1}{2}\vec{j} \right|$

$= \left| \frac{1}{2}\vec{i} + \frac{1}{2}\vec{j} \right|$

$= \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2}$

$= 5$ ✗

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$ b i $\overrightarrow{AB} \times \overrightarrow{AC}$

$= \vec{j} + \vec{k}$ $= (-\vec{i} + \vec{j}) \times (\vec{j} + \vec{k})$

$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$

$= \vec{i} + \vec{j} - \vec{k}$ // 1A

**表現稍遜 — 第三題**

3. 以數學歸納法，證明對所有正整數 $n$，$4^n+15n-1$ 可被 9 整除。

(5

設命題为 P(n)。

当 n = 1 时，

$4^{(1)}+15(1)-1=18$ (它能整除9)

∴ P(1) 為真 1

即 $4^k+15(k)-1=9M$ (M為正整數) 1

假設 P(k) 為真。

$4^k+15(k)-1=9M$

当 ~~k=~~ n = k+1 时，

$4^{k+1}+15(k+1)-1$

$= ~~4^k+4~~ +15k+15-1-1+1$

$=4^k+15k+4+15-1-1+1$

$=4^k+15k-1+4+15-1+1$

$=9M+18+9\times\frac{1}{9}$ 1

$=9(M+2)+9\times\frac{1}{9}$

$=9(M+2+\frac{1}{9})$

$=9(M+\frac{19}{9})$ (它能整除9)

∴ P(2) 為真

∴ 利用數学归纳法 P(n) 為真，对于所有正整数 n，$4^n+15n-1$ 可被9整除。

**表現稍遜 — 第一題**

1. 求在 $(2-x)^9$ 的展式中 $x^5$ 項的係數。

（4分）

$1.\ (2-x)^9$

$=[C^9_0 2^9 + C^9_1 2^8 x + C^9_2 2^7 x^2 + C^9_3 2^6 x^3 + C^9_4 2^5 x^4 + C^9_5 2^4 x^5 + \ldots]$ 1M

$=[2^9 + 9(2^8)x + 36(2^7)x^2 + 84(2^6)x^3 + 126(2^5)x^4 + 126(2^4)x^5 + \ldots]$

$=[512 + 2304x + 4608x^2 + 5376x^3 + 4032x^4 + 2016x^5 + \ldots]$