香 港 考 試 及 評 核 局
HONG KONG EXAMINATIONS AND ASSESSMENT AUTHORITY

香 港 中 學 文 憑 考 試
HONG KONG DIPLOMA OF SECONDARY EDUCATION EXAMINATION

練習卷
PRACTICE PAPER

數學 延伸部分
單元二（代數與微積分）

MATHEMATICS Extended Part
Module 2 (Algebra and Calculus)

評卷參考

MARKING SCHEME

本評卷參考乃香港考試及評核局專爲本科練習卷而編寫，供教師和學生參考之用。學生不應將評卷參考視爲標準答案，硬背死記，活剝生吞。這種學習態度，既無助學生改善學習，學懂應對及解難，亦有違考試着重理解能力與運用技巧之旨。

This marking scheme has been prepared by the Hong Kong Examinations and Assessment Authority for teachers' and students' reference. This marking scheme should NOT be regarded as a set of model answers. Our examinations emphasise the testing of understanding, the practical application of knowledge and the use of processing skills. Hence the use of model answers, or anything else which encourages rote memorisation, will not help students to improve their learning nor develop their abilities in addressing and solving problems.

General Notes for Teachers on Marking

Adherence to marking scheme

1. This marking scheme has been updated, with revisions made after the scrutiny of actual samples of student performance practice papers. Teachers are strongly advised to conduct their own internal standardisation procedures before applying marking schemes. After standardisation, teachers should adhere to the marking scheme to ensure a uniform standard of marking within the school.

2. It is very important that all teachers should adhere as closely as possible to the marking scheme. In many cases, however, students may have arrived at a correct answer by an alternative method not specified in the marking scheme. In general, a correct alternative solution merits <u>all the marks</u> allocated to that part, unless a particular method has been specified in the question. Teachers should be patient in marking alternative solutions not specified in the marking scheme.

Acceptance of alternative answers

3. For the convenience of teachers, the marking scheme was written as detailed as possible. However, it is likely that students would not present their solution in the same explicit manner, e.g. some steps would either be omitted or stated implicitly. In such cases, teachers should exercise their discretion in marking students' work. In general, marks for a certain step should be awarded if students' solution indicate that the relevant concept / technique has been used.

4. In marking students' work, the benefit of doubt should be given in students' favour.

5. Unless the form of the answer is specified in the question, alternative simplified forms of answers different from those in the marking scheme should be accepted if they are correct.

6. Unless otherwise specified in the question, use of notations different from those in the marking scheme should not be penalised.

Defining symbols used in the marking scheme

7. In the marking scheme, marks are classified into the following three categories:

'M' marks	–	awarded for applying correct methods
'A' marks	–	awarded for the accuracy of the answers
Marks without 'M' or 'A'	–	awarded for correctly completing a proof or arriving at an answer given in the question.

   In a question consisting of several parts each depending on the previous parts, 'M' marks should be awarded to steps or methods correctly deduced from previous answers, even if these answers are erroneous. (I.e. Teachers should follow through students' work in awarding 'M' marks.) However, 'A' marks for the corresponding answers should <u>NOT</u> be awarded, unless otherwise specified.

8. In the marking scheme, steps which can be skipped are enclosed by dotted rectangles , whereas alternative answers are enclosed by solid rectangles .

Others

9. Marks may be deducted for poor presentation (*pp*), including wrong / no unit. Note the following points:
   (a) At most deduct 1 mark for *pp* in each section.
   (b) In any case, do not deduct any marks for *pp* in those steps where students could not score any marks.

10. (a) Unless otherwise specified in the question, numerical answers not given in exact values should not be accepted.
    (b) In case a certain degree of accuracy had been specified in the question, answers not accurate up to that degree should not be accepted. For answers with an excess degree of accuracy, deduct 1 mark for *pp*. In any case, do not deduct any marks for excess degree of accuracy in those steps where candidates could not score any marks.

Solution	Marks	
1. The general term of $(2-x)^9$ is $C_r^9 2^{9-r}(-x)^r$	1M	
$= C_r^9 2^{9-r}(-1)^r x^r$	1A	
Alternative Solution $(2-x)^9 = 2^9 - C_1^9 2^8 x + C_2^9 2^7 x^2 - C_3^9 2^6 x^3 + C_4^9 2^5 x^4 - C_5^9 2^4 x^5 + \cdots$	1M+1A	
Hence the coefficient of x^5 is $-C_5^9 2^4$	1M	
$= -2016$	1A	
	(4)	
2. If the system of homogeneous equations has non-trivial solutions, then		
$\begin{vmatrix} 1 & -7 & 7 \\ 1 & -k & 3 \\ 2 & 1 & k \end{vmatrix} = 0$	1M+1A	
$-k^2 + 7 - 42 + 14k + 7k - 3 = 0$	1M	
$k^2 - 21k + 38 = 0$		
$k = 19$ or 2	1A	
	(4)	
3. For $n=1$,		
$4^1 + 15(1) - 1 = 18$ which is divisible by 9.		
$\therefore$ the statement is true for $n=1$.	1	
Assume $4^k + 15k - 1$ is divisible by 9, where k is a positive integer.	1	
i.e. let $4^k + 15k - 1 = 9N$, where N is an integer.		Withdraw the last mark if "N is an integer" was omitted
$\therefore\ 4^k = 9N - 15k + 1$		
$4^{k+1} + 15(k+1) - 1$		
$= 4(9N - 15k + 1) + 15k + 15 - 1$ (by induction assumption)	1	
$= 36N - 45k + 18$		
$= 9(4N - 5k + 2)$ which is divisible by 9	1	
Hence the statement is true for $n = k+1$.		
By the principle of mathematical induction, the statement is true for all positive integers n.	1	Follow through
	(5)	
4. (a) $\dfrac{2x}{1+x^2} = \dfrac{2\tan\theta}{1+\tan^2\theta}$		
$= \dfrac{2\tan\theta}{\sec^2\theta}$	1M	
$= 2\dfrac{\sin\theta}{\cos\theta}\cdot\cos^2\theta$		
$= \sin 2\theta$	1	
(b) $\dfrac{(1+x)^2}{1+x^2} = \dfrac{1+x^2+2x}{1+x^2}$		
$= 1 + \dfrac{2x}{1+x^2}$	1M	
Since x is real, we can let $x = \tan\theta$ for some θ.		
$\therefore\ \dfrac{(1+x)^2}{1+x^2} = 1 + \sin 2\theta$ by (a)	1M	

Solution	Marks	Remarks
Since the maximum value of $\sin 2\theta$ is 1 , the maximum value of $\frac{(1+x)^2}{1+x^2}$ is 2 .	1A	
	(5)	
5. (a) $\cos(x+1)+\cos(x-1)=2\cos\frac{x+1+x-1}{2}\cos\frac{x+1-x+1}{2}$	1M	OR $\cos x\cos 1-\sin x\sin 1$ $+\cos x\cos 1+\sin x\sin 1$
$=2\cos 1\cos x$		
Alternative Solution $\cos(0+1)+\cos(0-1)=k\cos 0$	1M	
i.e. $k=2\cos 1$	1A	
(b) $\begin{vmatrix}\cos 1 & \cos 2 & \cos 3\\ \cos 4 & \cos 5 & \cos 6\\ \cos 7 & \cos 8 & \cos 9\end{vmatrix}=\begin{vmatrix}\cos 1+\cos 3 & \cos 2 & \cos 3\\ \cos 4+\cos 6 & \cos 5 & \cos 6\\ \cos 7+\cos 9 & \cos 8 & \cos 9\end{vmatrix}$	1M	For column (or row) operations
$=\begin{vmatrix}2\cos 1\cos 2 & \cos 2 & \cos 3\\ 2\cos 1\cos 5 & \cos 5 & \cos 6\\ 2\cos 1\cos 8 & \cos 8 & \cos 9\end{vmatrix}$ by (a)	1M	For using (a) or sum-to-product formula of cosine
$=2\cos 1\begin{vmatrix}\cos 2 & \cos 2 & \cos 3\\ \cos 5 & \cos 5 & \cos 6\\ \cos 8 & \cos 8 & \cos 9\end{vmatrix}$	1M	
$=0$	1A	
	(6)	
6. $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{x}\right)=\lim_{h\to 0}\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$	1M+1A	
$=\lim_{h\to 0}\frac{x-x-h}{h(x+h)x}$		
$=\lim_{h\to 0}\frac{-1}{(x+h)x}$	1A	
$=\frac{-1}{x^2}$	1A	
	(4)	
7. (a) $\mathrm{f}(x)=e^x(\sin x+\cos x)$		
$\mathrm{f}'(x)=e^x(\sin x+\cos x)+e^x(\cos x-\sin x)$		
$=2e^x\cos x$	1A	
$\mathrm{f}''(x)=2e^x\cos x-2e^x\sin x$		
$=2e^x(\cos x-\sin x)$	1A	
(b) $\mathrm{f}''(x)-\mathrm{f}'(x)+\mathrm{f}(x)=0$		
$2e^x(\cos x-\sin x)-2e^x\cos x+e^x(\sin x+\cos x)=0$	1M	
$e^x(\cos x-\sin x)=0$		
$\sin x=\cos x$ or $e^x=0$ (rejected)	1A	
$\tan x=1$		
$x=\frac{\pi}{4}$ for $0\le x\le\pi$	1A	
	(5)	

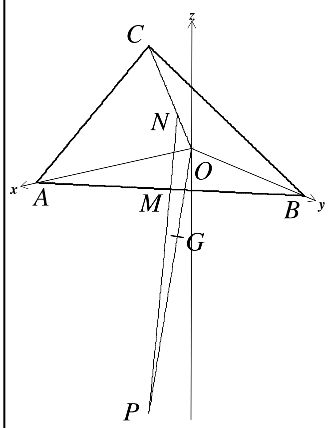
Solution	Marks	
8. (a) Let $x = 2\sin\theta$.	1M	OR $x =$
$\mathrm{d}x = 2\cos\theta\,\mathrm{d}\theta$		
$\int \frac{\mathrm{d}x}{\sqrt{4-x^2}} = \int \frac{2\cos\theta}{\sqrt{4-4\sin^2\theta}}\,\mathrm{d}\theta$		
$= \int 1\,\mathrm{d}\theta$	1A	
$= \theta + C$		
$= \sin^{-1}\frac{x}{2} + C$	1A	
(b) $\int \ln x\,\mathrm{d}x = x\ln x - \int x\,\mathrm{d}\ln x$	1M	
$= x\ln x - \int x \cdot \frac{1}{x}\mathrm{d}x$		
$= x\ln x - x + C$	1A	
	(5)	
9. $x^2 - xy - 2y^2 - 1 = 0$ -- (*)		
$2x - x\frac{\mathrm{d}y}{\mathrm{d}x} - y - 4y\frac{\mathrm{d}y}{\mathrm{d}x} = 0$	1A	
For the tangents parallel to $y = 2x+1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$.		
$\therefore\ 2x - x(2) - y - 4y(2) = 0$	1M	
$y = 0$	1A	
By (*), $x^2 - 1 = 0$	1M	
$x = \pm 1$		
Hence the tangents are $y - 0 = 2[x - (\pm 1)]$	1M	
i.e. $y = 2x+2$ and $y = 2x-2$	1A	For both
	(6)	
10. (a) $\int xe^{-x^2}\mathrm{d}x = \int e^{-x^2}\frac{1}{2}\mathrm{d}x^2$	1M	OR $\int e^{-x^2}\frac{-1}{2}\mathrm{d}(-x^2)$
$= \frac{-1}{2}e^{-x^2} + C$	1A	
(b) The volume of the solid		
$= 2\pi\int_1^2 x\left(\frac{x^2}{2} - e^{-x^2}\right)\mathrm{d}x$	1M+1A	1M for $V = 2\pi\int xy\,\mathrm{d}x$
$= 2\pi\int_1^2 \left(\frac{x^3}{2} - xe^{-x^2}\right)\mathrm{d}x$		
$= 2\pi\left[\frac{x^4}{8} + \frac{1}{2}e^{-x^2}\right]_1^2$	1M	For using (a)
$= \left(\frac{15}{4} + e^{-4} - e^{-1}\right)\pi$	1A	
	(6)	

Solution	Marks	
11. (a) $A^2 = \begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix}$		
$= \begin{pmatrix} (\alpha+\beta)^2-\alpha\beta & -\alpha\beta(\alpha+\beta) \\ \alpha+\beta & -\alpha\beta \end{pmatrix}$	1A	Either one
$(\alpha+\beta)A-\alpha\beta I = (\alpha+\beta)\begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - \alpha\beta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		
$= \begin{pmatrix} (\alpha+\beta)^2-\alpha\beta & -\alpha\beta(\alpha+\beta) \\ \alpha+\beta & -\alpha\beta \end{pmatrix}$		
i.e. $A^2 = (\alpha+\beta)A-\alpha\beta I$	1	
	(2)	
(b) $(A-\alpha I)^2 = A^2-2\alpha A+\alpha^2 I$		
$= (\alpha+\beta)A-\alpha\beta I-2\alpha A+\alpha^2 I$ by (a)	1M	
$= (\beta-\alpha)A+(\alpha^2-\alpha\beta)I$		
$= (\beta-\alpha)(A-\alpha I)$	1	
Alternative Solution		
$(A-\alpha I)^2 = \left(\begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)^2$		
$= \begin{pmatrix} \beta & -\alpha\beta \\ 1 & -\alpha \end{pmatrix}\begin{pmatrix} \beta & -\alpha\beta \\ 1 & -\alpha \end{pmatrix}$		
$= \begin{pmatrix} \beta^2-\alpha\beta & \alpha^2\beta-\alpha\beta^2 \\ \beta-\alpha & \alpha^2-\alpha\beta \end{pmatrix}$	1A	
$(\beta-\alpha)(A-\alpha I) = (\beta-\alpha)\left(\begin{pmatrix} \alpha+\beta & -\alpha\beta \\ 1 & 0 \end{pmatrix} - \alpha\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$		
$= (\beta-\alpha)\begin{pmatrix} \beta & -\alpha\beta \\ 1 & -\alpha \end{pmatrix}$		
$= \begin{pmatrix} \beta^2-\alpha\beta & \alpha^2\beta-\alpha\beta^2 \\ \beta-\alpha & \alpha^2-\alpha\beta \end{pmatrix}$		
i.e. $(A-\alpha I)^2 = (\beta-\alpha)(A-\alpha I)$	1	
By interchanging α and β, we have $(A-\beta I)^2 = (\alpha-\beta)(A-\beta I)$.	1	
Alternative Solution 1		
$(A-\beta I)^2 = A^2-2\beta A+\beta^2 I$		
$= (\alpha+\beta)A-\alpha\beta I-2\beta A+\beta^2 I$ by (a)		
$= (\alpha-\beta)A+(\beta^2-\alpha\beta)I$		
$= (\alpha-\beta)(A-\beta I)$	1	

Solution	Marks	
Alternative Solution 2 $(A-\beta I)^2=\left(\begin{pmatrix}\alpha+\beta & -\alpha\beta\\ 1 & 0\end{pmatrix}-\beta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)^2$ $=\begin{pmatrix}\alpha & -\alpha\beta\\ 1 & -\beta\end{pmatrix}\begin{pmatrix}\alpha & -\alpha\beta\\ 1 & -\beta\end{pmatrix}$ $=\begin{pmatrix}\alpha^2-\alpha\beta & \alpha\beta^2-\alpha^2\beta\\ \alpha-\beta & \beta^2-\alpha\beta\end{pmatrix}$ $(\alpha-\beta)(A-\beta I)=(\alpha-\beta)\left(\begin{pmatrix}\alpha+\beta & -\alpha\beta\\ 1 & 0\end{pmatrix}-\beta\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right)$ $=(\alpha-\beta)\begin{pmatrix}\alpha & -\alpha\beta\\ 1 & -\beta\end{pmatrix}$ $=\begin{pmatrix}\alpha^2-\alpha\beta & \alpha\beta^2-\alpha^2\beta\\ \alpha-\beta & \beta^2-\alpha\beta\end{pmatrix}$		
i.e. $(A-\beta I)^2=(\alpha-\beta)(A-\beta I)$	1	
	(3)	
(c) (i) $A=X+Y$ $\begin{pmatrix}\alpha+\beta & -\alpha\beta\\ 1 & 0\end{pmatrix}=s\begin{pmatrix}\beta & -\alpha\beta\\ 1 & -\alpha\end{pmatrix}+t\begin{pmatrix}\alpha & -\alpha\beta\\ 1 & -\beta\end{pmatrix}$ $=\begin{pmatrix}s\beta+t\alpha & -\alpha\beta(s+t)\\ s+t & -s\alpha-t\beta\end{pmatrix}$	1M	
Comparing the entries, we have $\begin{cases}s\beta+t\alpha=\alpha+\beta\\ s+t=1\\ s\alpha+t\beta=0\end{cases}$		
Solving, $s=\dfrac{\beta}{\beta-\alpha}$ and $t=\dfrac{\alpha}{\alpha-\beta}$	1A	For both
(ii) Consider the statement " $X^n=\dfrac{\beta^n}{\beta-\alpha}(A-\alpha I)$ and $Y^n=\dfrac{\alpha^n}{\alpha-\beta}(A-\beta I)$ ".		
When $n=1$, $X=\dfrac{\beta}{\beta-\alpha}(A-\alpha I)$ and $Y=\dfrac{\alpha}{\alpha-\beta}(A-\beta I)$ are true by (c)(i).	1	
Assume $X^k=\dfrac{\beta^k}{\beta-\alpha}(A-\alpha I)$ and $Y^k=\dfrac{\alpha^k}{\alpha-\beta}(A-\beta I)$, where k is a positive integer.		
$X^{k+1}=\dfrac{\beta^k}{\beta-\alpha}(A-\alpha I)\dfrac{\beta}{\beta-\alpha}(A-\alpha I)$ by the assumption		
$=\dfrac{\beta^{k+1}}{(\beta-\alpha)^2}(\beta-\alpha)(A-\alpha I)$ by (b)	1	Either one
$=\dfrac{\beta^{k+1}}{\beta-\alpha}(A-\alpha I)$		
$Y^{k+1}=\dfrac{\alpha^k}{\alpha-\beta}(A-\beta I)\dfrac{\alpha}{\alpha-\beta}(A-\beta I)$ by the assumption		
$=\dfrac{\alpha^{k+1}}{(\alpha-\beta)^2}(\alpha-\beta)(A-\beta I)$ by (b)		
$=\dfrac{\alpha^{k+1}}{\alpha-\beta}(A-\beta I)$	1	

Solution	Marks	Remarks
Hence the statement is true for $n = k+1$. By the principle of mathematical induction, the statement is true for all positive integers n.	1	Follow through
(iii) $XY = s(A-\alpha I)t(A-\beta I)$ $= st[A^2-(\alpha+\beta)A+\alpha\beta I]$ by (a) $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $YX = t(A-\beta I)s(A-\alpha I)$ $= st[A^2-(\alpha+\beta)A+\alpha\beta I]$ by (a) $= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	1	For both
$A^n = (X+Y)^n$ $= X^n + Y^n$ by the note given	1M	
$= \dfrac{\beta^n}{\beta-\alpha}(A-\alpha I) + \dfrac{\alpha^n}{\alpha-\beta}(A-\beta I)$ by (ii) $= \dfrac{\alpha^n-\beta^n}{\alpha-\beta}A + \dfrac{\alpha\beta^n-\alpha^n\beta}{\alpha-\beta}I$	1A	
	(9)	
12. (a) (i) $\overrightarrow{OM} = (1-a)\mathbf{i}+a\mathbf{j}$	1A	
$\overrightarrow{ON} = b(\mathbf{i}+\mathbf{j}+\mathbf{k})$ $\therefore \overrightarrow{MN} = b(\mathbf{i}+\mathbf{j}+\mathbf{k}) - [(1-a)\mathbf{i}+a\mathbf{j}]$ $= (a+b-1)\mathbf{i}+(b-a)\mathbf{j}+b\mathbf{k}$	1	
(ii) $\overrightarrow{AB} = \mathbf{j}-\mathbf{i}$ $\overrightarrow{MN}\cdot\overrightarrow{AB} = 0$ $[(a+b-1)\mathbf{i}+(b-a)\mathbf{j}+b\mathbf{k}]\cdot(\mathbf{j}-\mathbf{i}) = 0$	1M	
$-a-b+1+b-a=0$ $a = \dfrac{1}{2}$	1A	
$\overrightarrow{MN}\cdot\overrightarrow{OC} = 0$ $[(a+b-1)\mathbf{i}+(b-a)\mathbf{j}+b\mathbf{k}]\cdot(\mathbf{i}+\mathbf{j}+\mathbf{k}) = 0$	1M	
$a+b-1+b-a+b=0$ $b = \dfrac{1}{3}$	1A	
Alternative Solution $\overrightarrow{AB}\times\overrightarrow{OC} = (-\mathbf{i}+\mathbf{j})\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$ $= \mathbf{i}+\mathbf{j}-2\mathbf{k}$	1M	
$\overrightarrow{MN} // \left(\overrightarrow{AB}\times\overrightarrow{OC}\right)$ $\therefore \dfrac{a+b-1}{1} = \dfrac{b-a}{1} = \dfrac{b}{-2}$	1M	
Solving, we get $a = \dfrac{1}{2}$ and $b = \dfrac{1}{3}$.	1A+1A	

Solution	Marks
(iii) $\overrightarrow{MN}=\frac{-1}{6}\mathbf{i}-\frac{1}{6}\mathbf{j}+\frac{1}{3}\mathbf{k}$	
The shortest distance between the lines AB and OC	
$=\left\lvert\overrightarrow{MN}\right\rvert$	
$=\sqrt{\left(\frac{-1}{6}\right)^2+\left(\frac{-1}{6}\right)^2+\left(\frac{1}{3}\right)^2}$	1M
$=\frac{\sqrt{6}}{6}$	1A
	(8)
(b) (i) $\overrightarrow{AB}\times\overrightarrow{AC}=(-\mathbf{i}+\mathbf{j})\times(\mathbf{j}+\mathbf{k})$	
$=\mathbf{i}+\mathbf{j}-\mathbf{k}$	1A
(ii) Let the intersecting point of the two lines OG and MN be P.	
Since P lies on MN, let $\overrightarrow{MP}=\lambda\overrightarrow{MN}$.	1M
$\overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MP}$	
$=\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\lambda\left(\frac{-1}{6}\mathbf{i}-\frac{1}{6}\mathbf{j}+\frac{1}{3}\mathbf{k}\right)$	
$=\frac{3-\lambda}{6}\mathbf{i}+\frac{3-\lambda}{6}\mathbf{j}+\frac{\lambda}{3}\mathbf{k}$	1A
Since P lies on OG, $\overrightarrow{OP}\ //\ \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$.	
$\therefore\ \frac{3-\lambda}{6}=-\frac{\lambda}{3}$	1M
$\lambda=-3$	
<u>Alternative Solution</u>	
Since P lies on OG, $\overrightarrow{OP}\ //\ \left(\overrightarrow{AB}\times\overrightarrow{AC}\right)$.	
Let $\overrightarrow{OP}=t(\mathbf{i}+\mathbf{j}-\mathbf{k})$	1M
$\therefore\ \overrightarrow{MP}=t(\mathbf{i}+\mathbf{j}-\mathbf{k})-\left(\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)$	
$=\frac{2t-1}{2}\mathbf{i}+\frac{2t-1}{2}\mathbf{j}-t\mathbf{k}$	1A
Since P lies on MN, $MP\ //\ MN$.	
$\therefore\ \frac{\frac{2t-1}{2}}{\frac{-1}{6}}=\frac{-t}{\frac{1}{3}}$	1M
$t=1$	
Hence the coordinates of P are $(1,1,-1)$.	1A
	(5)



Solution	Marks
13. (a) Let $u = x - p$.	1M
$\therefore \quad \mathrm{d}u = \mathrm{d}x$	
When $x = 0$, $u = -p$; when $x = 2p$, $u = p$.	
$\therefore \quad \int_0^{2p} \mathrm{f}(x-p)\,\mathrm{d}x = \int_{-p}^{p} \mathrm{f}(u)\,\mathrm{d}u$	1A
$= 0$ since f is an odd function	1
$\therefore \quad \int_0^{2p} [\mathrm{f}(x-p)+q]\,\mathrm{d}x = 0 + [qx]_0^{2p}$	
$= 2pq$	1A
	(4)
(b) $\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} = \dfrac{\sqrt{3}+\dfrac{\tan x - \frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}\tan x}}{\sqrt{3}-\dfrac{\tan x - \frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}\tan x}}$	1M
$= \dfrac{3+\sqrt{3}\tan x+\sqrt{3}\tan x-1}{3+\sqrt{3}\tan x-\sqrt{3}\tan x+1}$	
$= \dfrac{1+\sqrt{3}\tan x}{2}$	1
	(2)
(c) $\int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,\mathrm{d}x = \int_0^{\frac{\pi}{3}} \ln\left[\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)}\cdot 2\right]\mathrm{d}x$ by (b)	
$= \int_0^{\frac{\pi}{3}} \left[\ln\dfrac{\sqrt{3}+\tan\left(x-\frac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\frac{\pi}{6}\right)} + \ln 2\right]\mathrm{d}x$	1M
Consider $\mathrm{f}(x) = \ln\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}$.	
$\mathrm{f}(-x) = \ln\dfrac{\sqrt{3}+\tan(-x)}{\sqrt{3}-\tan(-x)}$ $= \ln\dfrac{\sqrt{3}-\tan x}{\sqrt{3}+\tan x}$ $= \ln\left(\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right)^{-1}$ $= -\ln\dfrac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}$ $= -\mathrm{f}(x)$	1M
$\therefore \quad \mathrm{f}(x)$ is an odd function	1A
$\therefore \quad \int_0^{\frac{\pi}{3}} \ln(1+\sqrt{3}\tan x)\,\mathrm{d}x = \int_0^{2\times\frac{\pi}{6}} \left[\mathrm{f}\left(x-\frac{\pi}{6}\right)+\ln 2\right]\mathrm{d}x$	
$= \dfrac{\pi \ln 2}{3}$ by (a)	1A
	(4)

Solution	Marks

14. (a) The volume of the solid of revolution

$= \pi \int_0^h (25 - y^2)\, dy$ — 1M

$= \pi \left[25y - \frac{y^3}{3} \right]_0^h$

$= \left(25h - \frac{h^3}{3} \right) \pi$ — 1

(2)

(b) (i) By (a), $V = \left(25h - \frac{h^3}{3} \right) \pi$ for $0 \le h \le 4$

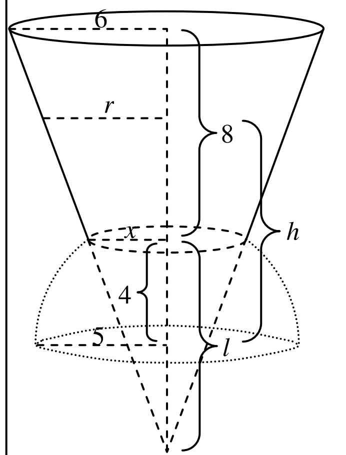
$\frac{dV}{dt} = \left(25 \frac{dh}{dt} - h^2 \frac{dh}{dt} \right) \pi$ — 1A

When $h = 3$, $8 = (25 - 3^2) \pi \frac{dh}{dt}$

$\frac{dh}{dt} = \frac{1}{2\pi}$

i.e. the rate of increase of the depth of coffee is $\frac{1}{2\pi}$ cm s^{-1}. — 1A

(ii) Let x, l, r and h be the lengths as shown in the figure.



$x^2 + 4^2 = 25$

$x = 3$ — 1A

By similar triangles, $\frac{x}{l} = \frac{6}{8 + l}$ — 1M

$24 + 3l = 6l$

$l = 8$

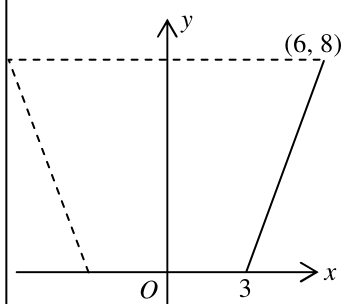
By similar triangles, $\frac{r}{h - 4 + l} = \frac{6}{8 + l}$

$r = \frac{3(h + 4)}{8}$ — 1A

$\therefore\ V = \left[25(4) - \frac{(4)^3}{3} \right] \pi + \frac{\pi}{3} \left[\frac{3(h + 4)}{8} \right]^2 (h + 4) - \frac{\pi}{3} (3)^2 (8)$ — 1M

<u>Alternative Solution</u>

Locating the origin at the centre of the base and the *x*-axis along the base of the frustum, the equation of a slang edge of the frustum is



$\frac{y - 0}{x - 3} = \frac{8 - 0}{6 - 3}$ — 1M

$x = \frac{3}{8}(y + 8)$ — 1A

$\therefore\ V = \left[25(4) - \frac{(4)^3}{3} \right] \pi + \pi \int_0^{h-4} \frac{9}{64} (y + 8)^2\, dy$ — 1M

$= \frac{236\pi}{3} + \frac{9\pi}{64} \left[\frac{(y + 8)^3}{3} \right]_0^{h-4}$

i.e. $V = \frac{164\pi}{3} + \frac{3\pi}{64} (h + 4)^3$ — 1

Solution	Marks
(iii) After 15 seconds, $\frac{164\pi}{3}+\frac{3\pi}{64}(12+4)^3-2\times 15=\frac{164\pi}{3}+\frac{3\pi}{64}(h+4)^3$	1M
$\frac{3\pi}{64}(h+4)^3=192\pi-30$	
$h+4=4\left(\frac{64\pi-10}{\pi}\right)^{\frac{1}{3}}$	1A
$h\approx 11.73>4$	
$V=\frac{164\pi}{3}+\frac{3\pi}{64}(h+4)^3$	
$\frac{\mathrm{d}V}{\mathrm{d}t}=\frac{9\pi}{64}(h+4)^2\frac{\mathrm{d}h}{\mathrm{d}t}$	1A
After 15 seconds, $-2=\frac{9\pi}{64}\left[4\left(\frac{64\pi-10}{\pi}\right)^{\frac{1}{3}}\right]^2\frac{\mathrm{d}h}{\mathrm{d}t}$	
$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{-8}{9\pi^{\frac{1}{3}}(64\pi-10)^{\frac{2}{3}}}$	
≈ -0.0183	
i.e. the rate of decrease of the depth of coffee is 0.0183 cm s^{-1} .	1A
	(11)