OCR
RECOGNISING ACHIEVEMENT

Support for problem solving in GCSE Mathematics A J562 and Mathematics B J567

July 2010

Support for problem solving in GCSE Mathematics A J562 and Mathematics B J567

The School Mathematics Project

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Introduction

What these support materials aim to do

The GCSE Mathematics A and Mathematics B specifications for first teaching from September 2010 require candidates to apply the subject in a range of contexts ('functional mathematics') and to interpret, analyse and solve problems. Significant weight is given to these aspects in the mark schemes.

In response to requests from centres OCR has made this pack of materials available to help teachers prepare for these new elements. It contains, on pages 6–47, nineteen tasks on sheets that can be printed off and given to learners. Since the publications is downloadable in PDF format you can also project the sheets as an aid to class discussion.

The nature of the tasks

The tasks have been carefully developed and piloted by The School Mathematics Project in partnership with OCR, then refined in the light of feedback. They are not designed as practice exam questions but are intended to help learners develop the mental flexibility they will need for the new elements in the exam.

They are intended to be embedded into a scheme of work over a significant period of time, supporting – and being supported by – more conventional work.

The tasks vary in the amount of time they require.

While they all involve the ability to analyse and solve problems ('thinking skills'), they vary in structure and style: the functional tasks involve making sense of realistic data; other tasks require the learner to make sense of a spatial situation; some are essentially 'pure' investigations.

A resource graduated by difficulty

Tasks are presented in this pack roughly in order of difficulty: we have marked each task Foundation, Foudation/Higher or Higher to indicate the GCSE tier(s) for which we think it is suitable. However our order of difficulty and these markings are only for guidance, particularly as different parts of a task can vary significantly in demand. Change the order of tasks if you think that's right for your teaching groups. You can also modify the difficulty of a task by providing support (particularly in the early stages) or adding challenge.

It is not necessary to do all the tasks but it makes sense to choose a variety of types.

The teaching notes

We provide brief teaching notes on a sheet before each task and these indicate

- any resources that are needed
- in a few cases where is not obvious, the technical skills that will be needed
- teaching points that have arisen from piloting
- suggestions for ways you can develop the work

There is space around the teaching notes for you to record your own ideas.

Getting to know the tasks

The best way to prepare for one of these tasks is to work through it yourself, noting what you think might be particular points of difficulty for the group you teach and planning how to deal with them.

Because your main concern should be learners' methods and because some tasks can be approached in more than one way, we have not given 'answers' to all parts of the tasks. For convenience, however, we have given selected possible solutions after most teaching notes.

A resource for professional development

One of the teachers helping with the pilot ran a well-received in-service course in which teachers worked though and discussed some of the draft tasks. We hope this published version will prove useful for similar events.

Related support material

To support the stand-alone functional skills mathematics qualification, two resource packs, for Levels 1 and 2 respectively, have been developed by the SMP and are downloadable free from the OCR website. Like the present pack they contain reproducible tasks that have been carefully developed and piloted, together with teacher's notes. The Level 2 pack is particularly valuable as a source from which you can select additional support for the functional element in GCSE Mathematics: go to www.ocr.org.uk/functionalskills and follow the links.

Working with the problem solving tasks

The tasks and insights from piloting

The tasks in this pack are designed to support the learner in understanding the nature of a new problem and exploring different mathematical approaches to it, rather than imitating set routines without any depth of thought. Such independent thinking skills promote further study and are much prized by employers.

Some of the learners involved in the piloting were described to us as having limited educational aspirations and many had not encountered work of this type before. But learners frequently said the work 'made you think a lot' and often described it as fun. One wrote, 'It was good to work out how to read the question.' Another said, 'Doing it like this is challenging but when I started understanding it I found it a bit easier and it was good.'

Teachers were positive about taking part in the pilot. They often reported that classes were fully engaged with tasks and enjoyed doing something that was 'different from their ordinary classwork'. Describing 'Numbered discs' (pages 28–29), one wrote, 'I found this task was useful in getting pupils to have a go – to try out some ideas and if wrong try some more. This a common hurdle to get over in this school – fear of getting it wrong stopping them from having a go.'

Getting started with the tasks

When first given tasks of this kind many learners expect to be told what to do, rather than think about the problem themselves. That presents you with the quandary of how to offer support without doing the thinking for them. Here are some ideas.

- Before you give out the task sheet, ask the class some lead-in questions of your own to set the scene, to give yourself some idea of the knowledge your learners bring to the session and to stimulate discussion about the sorts of mathematical ideas that may be needed. But don't let this become a structured worked example of the unstructured problem that is to follow.
- When you give out the task have the class discuss in groups of two or three how they think they will approach it; say that you will ask a representative from one or more groups to give an explanation to the whole class of what they intend to do. Allow time for discussion to occur, for understanding to emerge and for learners to develop their explanations.
- Include 'open' questions that invite a range of responses and promote the higher level abilities of applying mathematics and putting mathematical ideas together. Don't just ask 'closed' questions that require a single fixed response derived from standard knowledge or procedures.
- By asking 'Why?', encourage learners to fill logical gaps in their explanations: they often find it hard to explain something they regard as obvious.
- Learners can then complete the work on their own if you wish, though if they continue in small groups it can give them (and you) much quicker and effective feedback on how well they are dealing with the problem.
- Develop a culture where 'mistakes' are seen as something to be learned from rather than to be ashamed of: bringing conflicting results into the open and encouraging learners to discuss them helps to identify and address underlying misconceptions, leading to more permanent learning than graded exercises designed to smooth away errors. You can give learners the reassurance of anonymity by having them respond to class questions on mini-whiteboards; you can then write some conflicting responses on the board for discussion. Do not immediately characterize any response as correct or incorrect: it is for learners to reach judgment by the end of the discussion.
- You'll find the new teaching approach becomes easier as you gain experience and learners come to realise what's expected. From time to time, perhaps at the end of a task, draw together the thoughts of the class concerning the new style of working. It is important that they gain self-awareness of the changes that have taken place and the reasons behind them.

Fitting the tasks into the curriculum

The difficulty of the tasks lies in applying mathematics in a new way rather than in the mathematical techniques themselves. You should therefore find it relatively easy to arrange your scheme of work so that learners are fluent with techniques that may be required for a task before you set the task.

However, avoid 'going through' a technique just before you give learners a task that needs it: that won't help foster the independent thinking that the tasks are intended to promote.

Occasionally the responses to your lead-in questions will tell you that learners are not ready for the task you are about to set them; if so, rather than leading them through the task 'by the nose', be prepared to set it aside, reflect on where the learning difficulties lie and return to it at a later date after you have organised appropriate learning experiences for them.

Some teachers feel that time spent on tasks of this kind is hard to justify, given the need to 'cover the syllabus'. However there is wide evidence that work based on learners' own discussion and reasoning leads to relatively solid understanding and is not at odds with the need to develop and maintain fluency with routine procedures.

Concern about the time required can be allayed if problem solving is started early in the curriculum and blended in with more familiar ways of teaching on a 'little and often' basis rather than being bolted on in the run up to GCSE.

Connecting, broadening and extending

Take opportunities to make connections with similar problems and related mathematics.

If learners suggest ways to broaden the scope of a task or modify it to suit their local situation or interests, encourage them to do so.

Some challenges are marked within the tasks and there is scope for you to provide your own.

When a class baulks at a significant jump in difficulty don't underestimate what can be achieved through classroom management that gets pupils working together. One teacher described what happened when a class 'who do not like to think' encountered the later, more difficult stages of 'Tournaments' (pages 25–27): '… at this point a majority of them switched off and were unwilling to attempt this part. I took a different tack and started to produce the required list on the board. I asked the class who would prefer to do it on the board, several volunteered and I selected a mixed group of students. Doing it this way they then completely focused for ten minutes on trying to solve the problem. After a few minutes some of the remainder of the class became involved and were both checking and correcting what the others were doing. I stepped back and let them continue, with only very occasional pointers about their strategy.'

1 Photo layout

Notes

Resources required

calculators

Optional resources

photos (or scissors and old magazines from which photos can be cut)
suitably sized rectangles of card
rulers
adhesive

Points to note

■ Piloting showed that many learners have difficulty with tasks that involve positioning shapes evenly: although the maths for a task like part (a) is simple (find the difference in height or width and divide by 2) many avoided it when dealing with the real situation, attempting instead to place the picture centrally by eye.

■ In parts (a) and (b) we have structured the problem slightly by starting with diagrams that highlight the equal border widths that need to be found. The important thing is for learners to reason logically from those diagrams to the mathematical steps they will use.

■ The piece of card in parts (a) and (b) is A4 size.

■ Either before doing part (a) or straight after it we recommend you give each learner a suitable card rectangle and photo, and ask them to calculate where to place the photo centrally on the card.

■ Some learners may feel more confident working in millimetres.

■ Some may use alternative approaches, for example calculating and marking the centre point of each edge and then aligning the centre points: such an approach is mathematically valid and should be given full credit. However it will not work with the trickier calculation of the vertical distances in part (b).

■ Another way to think about each problem is to turn the horizontal or vertical lengths into a simple equation and solve it. For example, the horizontal dimensions for Sue's picture give the equation

$$b + 17.8 + b = 29.6$$

or $$2b + 17.8 = 29.6$$

where b is the width of the left and right borders. This an elementary example of algebra having a use.

■ Here are examples of related questions you could ask.

- My car is 1.80 m wide. My garage door is 2.10 m wide. How much clearance will I have on each side if I keep the car centred-up as I drive in?
- Design a layout for an A4 newsletter that will have three columns of text going across. (Learners will have to decide on the margins at the edges of the page and the gap between the text columns, then calculate carefully the width of the columns.)

Possible solutions

(a) Diagram showing borders 5.9 cm wide to the left and right of the photo and 4.2 cm above and below

(b) Diagram showing borders 3.0 cm wide to the left and right of the photos and three horizontal borders 3.2 cm wide

Foundation

1 Photo layout

Sheet 1 of 2

Sue wants to display a photo of herself and her friend on the piece of card.

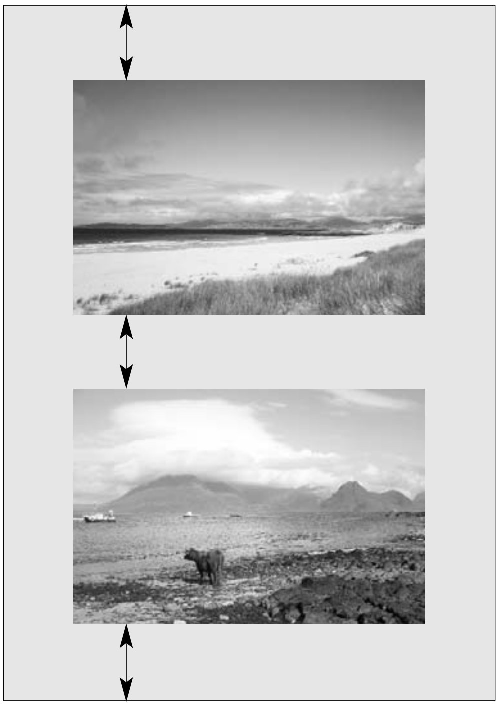
She wants these borders to be the same … … and she wants these borders to be the same.

Here are the dimensions of the photo and the piece of card.

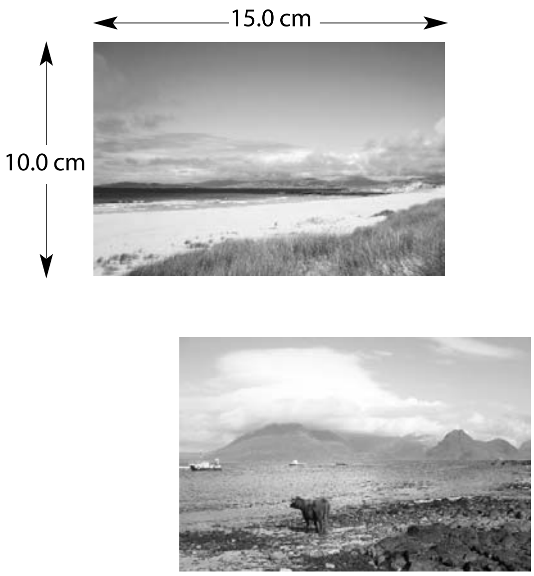
(a) Use the dimensions given to show where she should put the photo. Sketch a plan of the card, marking in enough measurements to show where the photo should go.

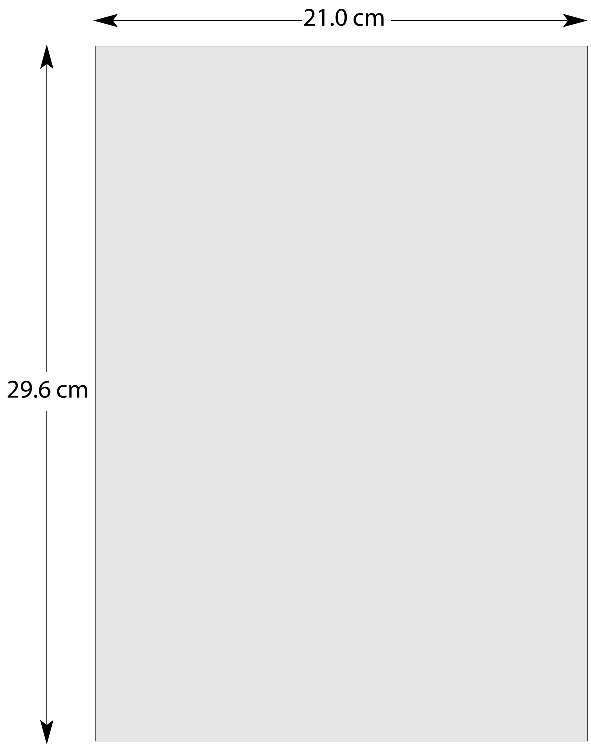
Billy wants to display two photos on a piece of card.

He wants these borders to be the same … … and he wants these three gaps to be the same.



Here are the dimensions of the piece of card and the photos (which are both the same size).





(b) Sketch a plan of the card, marking in enough measurements to show where the photos should go.

Foundation

2 Thorpe Park

Notes

Resources required

calculators

Optional resources

metre stick

Points to note

■ To start with you could just give out the first sheet and pose the following questions in class discussion to help learners become familiar with the price list and the definitions of adult, child and the two sizes of family. Then give out the second sheet for them to do the written questions.

- Jane is 15 years old.
  Does she count as an adult or a child?
- Jenny is two years old.
  Do you think she would get into Thorpe Park free?
- How much does it cost for an adult ticket bought on the day?
- How much does it cost to buy a Family of 5 ticket on-line?
- Jane and Mark have two children aged 14 and 5.
  Can they all get into Thorpe Park on a Family of 4 ticket?
- Ken has four children aged 11, 11, 8 and 6.
  Can they all get into Thorpe Park on a Family of 5 ticket?
- Paul is 61. Do you think he counts as a senior?

■ Learners may be unfamiliar with the height of a typical child at different ages. A metre stick should help them appreciate that a child who is less than a metre in height is likely to be quite young, certainly younger than 7 (the youngest child referred to here).

■ Learners will probably assume the parents of young children are not seniors. However someone might raise the point.

■ Learners could compare prices with another well-known adventure park such as Alton Towers or a more local venue.

Selected possible solutions

(a) £56

(b) £84

(d) We will just consider on-line tickets as they are cheaper, whatever the ticket type.
Separate tickets for 2 adults and 2 children cost £92.
A Family of 4 ticket costs £82 so that's the cheaper way.
If one of the adults is disabled separate tickets cost the same as a Family of 4 ticket.

2 Thorpe Park

Thorpe Park is an adventure park near London with lots of exciting rides.

Colossus

Samurai

You can book on-line in advance or pay on the day.
Here are the prices for the main season 2009.

	On-line	**On the day**
Adult *	£28.00	£35.00
Child **	£18.00	£21.00
Family of 4 ***	£82.00	£92.00
Family of 5 ****	£97.00	£115.00
Senior	£22.00	£24.00
Disabled/helper	£18.00	£20.00
Child under 1 metre	FREE	FREE

* Aged 12 or over

** Aged under 12 years

*** 2 adults (12 or over) and 2 children (under 12) or 1 adult (12 or over) and 3 children (under 12)

**** 2 adults (12 or over) and 3 children (under 12) or 1 adult (12 or over) and 4 children (under 12)

Slammer

Tickets booked in advance must be bought at least 24 hours before the date of visit.

(a) Joe and his daughter Ellie go to Thorpe Park and buy tickets on the day.
Ellie is 8.
How much do their tickets cost in total?

(b) Barry, Eva and and their son Donal plan to go to Thorpe Park and buy tickets on-line.
Donal is 14.
How much do their tickets cost in total?

(c) Helen and her sister Hazel decide to visit Thorpe Park with Hazel's two children Sasha and Roxy.
Sasha is 13 and Roxy is 10.

(i) Can they buy a 'Family of 4' ticket? Give a reason.

(ii) How much will they pay to buy their tickets on-line?

(d) Lorraine, Nick and their children Billy and Paula plan to go to Thorpe Park.
Billy is 8 and Paula is 10.
What is the cheapest way for them to buy their tickets?
Show how you decided.

(e) Babs has four children.
They are aged 15, 12, 10 and 7.
They wake up on a lovely day and all want to go to Thorpe Park.

They have £120 to spend.
Can Babs and her children afford to go to Thorpe Park?
Show clearly how you decided.

Vortex

3 Giving to charity

Notes

Resources required

1p and £1 coins
rulers
calculators

Points to note

■ Most schools engage in money raising activities for charity so the context is familiar and is usually found interesting.

■ Encourage learners to suggest their own methods for the estimate in (b) (i), rather than telling them an approach. The aim should be to get an estimate with the right order of magnitude; methods that achieve this could vary a good deal, for example:

- The coins are roughly the same size but a pound is worth 100 times the value of a penny, so the mile of pounds is worth 100 times the mile of pennies, which is …
- A pound is a bit bigger than a penny so there will be fewer pounds in the mile, so it will be worth a bit less than 100 times the pennies, so call it …

■ Some learners working in the Foundation tier will find difficulty with the pound's diameter of 2.3 cm. Get them to think back to what they did with the penny's diameter of 2.0 cm to deduce that there are 50 pennies to a metre.

■ The work can easily be broadened or adapted. After one class doing the task had found the value of a mile of pounds their teacher challenged them to calculate, on the basis of the number of students in their school, how much each one would need to contribute to make the mile. (This was met by protests that the staff should contribute as well and a discussion about what should happen if a family has, say, three children in the school.)

■ For parts (c) and (d) measuring a single coin's thickness with an ordinary ruler isn't reliable; instead encourage learners to find out by measuring how many coin thicknesses make a centimetre (conveniently close to a whole number in both cases).

■ Learners can explore the mathematics of a charity event that they have participated in.

Selected possible solutions

(a) (iv) £805

(b) (ii) £70 000

(d) £483 000

3 Giving to charity

(a) Grit Lane Community School decide to raise money for charity by collecting enough 1p coins to make a mile when they are laid in a line edge to edge.

(i) Get a 1p coin and measure its diameter.

(ii) How many 1p coins make a metre, when laid in a line edge to edge?

(iii) There are nearly 1610 metres in a mile.
How many pennies will Grit Lane School need to collect?

(iv) How much will their 'mile of pennies' be worth?

(b) Leafy Grove Academy decide they will do the same thing but with pound coins.

(i) Based on your value of the mile of pennies, roughly what do you think the mile of pounds will be worth?

(ii) Now calculate a more accurate value for the mile of pounds.

(c) What would a pile of pennies a mile high be worth?

(d) What would a pile of pounds a mile high be worth?

(e) A certain breakfast radio show has about a million listeners at any given moment.
The show's host makes a disaster appeal.
He suggests that his listeners each donate 50p over their mobile phones.
In the past about 10% of his listeners responded to appeals of this kind.
He assumes a similar percentage will help this time.

How much money does he think his appeal will raise?

Think of some other interesting ways that a school or other large group of people could raise money for charity.
Work out how much you think would be raised, explaining what you have assumed in making your calculation.

4 How much does it cost a year?

Notes

Resources required

calculators

Points to note

■ Although all these problems seek the same conclusion – how much something that's used regularly costs a year – the information is given in different ways, as in real life. So the method of solution differs from problem to problem and the aim of the work is to get learners thinking logically about the steps they need to take in each case. Part (a) is **not** meant to produce a model solution to be followed in the rest of the work but instead should be used to practise showing enough working to be able to check back and see if any operation has been used invalidly (usually by confusing between multiplication and division or dividing the wrong way round); brief working that shows the operations and makes it clear where the data comes from is all that's needed, for example

(i) $80 \div 3 \approx 27$ days

(ii) $365 \div 27 \approx 14$ packets

(iii) $14 \times 2.20 \approx$ £31

Here each rounded result (rather than the calculator result) has been carried to the next stage . This is in the spirit of rough estimation, particularly where mental calculation is used, and it can help keep the work grounded in reality; but be on the watch for exam mark schemes where it may lose credit.

■ There is usually more than one valid way of solving each problem: if part (a) was presented in an unstructured way, a learner could calculate the number of teabags used in a year ($3 \times 365 = 1095$), the cost of a teabag ($220 \div 80 = 2.75$p) and then the year's cost ($1095 \times 2.75 \approx 3011$p). Usually each learner finds one of the valid approaches more natural than the others, so their confidence can be eroded if they are required to abandon it in favour of a method imposed on them: at the outset have the class working in groups to discuss methods, then share methods across the whole class; make it clear that you give your approval to all the valid methods (even any that seem to you a little awkward); if instead you impose your own fixed method on the class you are not teaching them anything about problem solving.

■ Each of these problems requires several steps and some learners may find that too much of a challenge to start with. Rather than getting bogged down in working through an example, start by asking questions that require fewer stages and have easy numbers that encourage mental arithmetic, but which are structurally varied, as in these examples.

- Dishwasher tablets come in boxes of 30 and I use one tablet every two days. How long will a box last me?
- Broadband costs me £20 per month. How much is that a year?
- A patient needs three tablets of a certain kind each day. The tablets come in boxes of 28. How long will a box last?
- A family uses three loaves of bread a week, costing £1.50 per loaf. How much is that a week?
- A carton of washing powder lasts me six weeks. How many cartons will I need to buy in a year?
- A bottle of shampoo costing £2.80 lasts me 20 days. How much is the shampoo costing me per day?
- I need three bags of sugar per month and each one costs 90p. How much is that per year?
- I buy instant coffee in jars that contain 200 g and cost £5. I use 2 g for every cup I make. How much is that per cup?

■ Some of the problems on the sheet contain values that are not needed in the calculation. Again this is how it is in real life, but look out for learners who are determined to use every bit of information given because 'this is what you do in maths lessons'.

■ The problem found most difficult during piloting was the toothpaste one (slightly amended in this published version). One teacher reported: 'They would come up with ridiculous answers and realise they were. This was a really good activity.'

Selected possible solutions

(b) Shower gel £14 per year
soap £16 per year
cat food £380 per year

4 How much does it cost a year?

(a) Cathy usually buys tea in packets of 80 tea bags.

(i) She drinks 3 cups of tea a day, using one bag for each cup.
How long will one packet last her?

(ii) How many packets does she need to buy each year?

(iii) The packets cost £2.20 each.
About how much does she spend on her tea in a year?

(b) This table shows some other things that Cathy uses up steadily.
Work out roughly what each thing costs her in a year.

Item	**Size of unit bought**	**Cost**	**Rate of use**
Milk	1 pint container	45p per pint	4 pints per week
Shower gel	500 ml bottle	£1.90 per bottle	10 ml every day
Soap	Pack of 4 bars	£1.20 per pack	1 bar per week
Toothpaste	100 ml tube	£2.30 per tube	3 ml twice a day
Cat food	390 g tin	70p per tin	$1\frac{1}{2}$ tins per day

(c) Find out about the size and cost of things that you or your family use regularly.

These could include bread, shampoo, baked beans, washing powder, frozen chips, DVD hire and bus fares.

Find out from home what the rate of use is in each case: the rate is likely to be an estimate.

Then work out roughly what you or your family spend on each thing in a year.

Foundation

5 Scale drawing 1

Notes

Resources required

rulers
pairs of compasses
angle measurers
calculators

Points to note

- (a), (b) and (c) do not have to be done together.
- The work was found to be well suited to a wide range of learners on Foundation tier courses and was generally enjoyed, though it required a lot of effort from some, particularly in grasping what was required from the written instructions.
- Although (a) covers standard topics, one teacher who assumed her fairly strong Foundation group would have no trouble with the angles discovered that the class drew them as bearings from a north line.
- Some failed to notice the scale requested for (b). The question gave rise to some realistic discussion about the driving skills that would be needed.
- Some learners failed to use a pair of compasses in (c) because the question didn't ask them to do so – even though they had previously used compasses to construct a triangle with three sides given.
- Even the most careful drawings will contain inaccuracies; you need to bear this in mind when checking solutions.
- 'Scale drawing 2' (pages 30–31) contains more demanding tasks of this kind.

Selected possible solutions

(a) 16 km (though slight inaccuracies in drawing the angles can result in a disparity of 1 km or more)

(b) (ii) Measured as 8.5 cm, so 4.25 m on the real bridge

(iii) Yes, but with only about 10 cm spare above the top 'corners' of the lorry, or 20 cm latitude on each side

(c) (ii) About 26 or 27 m^2

5 Scale drawing 1

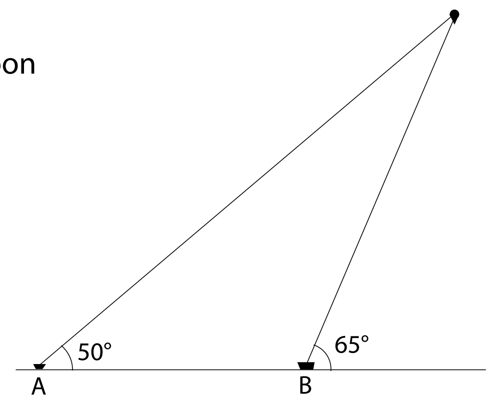
(a) Ship A sees a weather balloon due south at an angle of elevation of 50°.

At the same moment, ship B sees the weather balloon also due south at an angle of elevation of 65°.

Ships A and B are 6 km apart.

Draw the situation accurately using a scale where 1 centimetre represents 1 kilometre.

How high is the weather balloon above sea level?

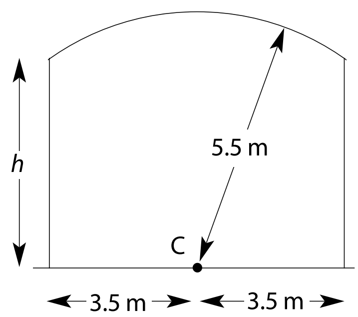


(b) This sketch shows an arch where a bridge goes over a road. The curve is an arc of a circle with its centre at point C.

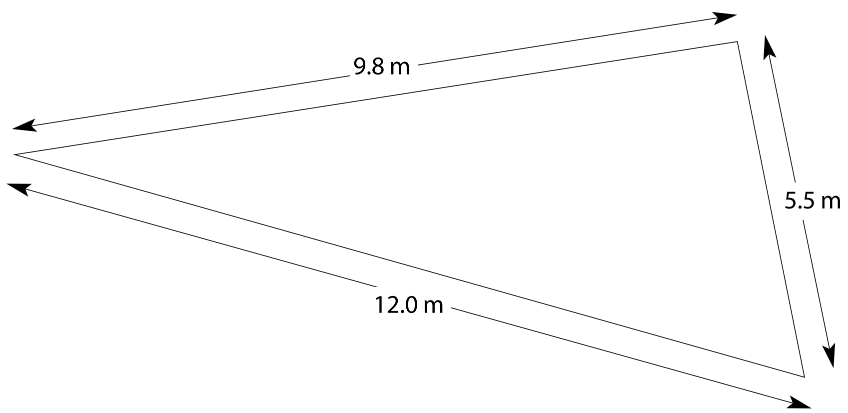
(i) Draw the arch accurately to a scale where 2 cm represents 1 metre.

(ii) Measure *h*, the height of the side wall, on your drawing. Write down what it would be on the real bridge.

(iii) Would a lorry 5.2 m high and 3.0 m wide be able to get through the arch? Show on your drawing how you decided.



(c) This is a sketch plan of a triangular plot of land.



(i) Draw the plan accurately to a scale where 1 cm represents 1 metre.

(ii) After measuring where you need to, calculate the area of the plot of land.

6 Thought puzzles

Notes

Points to note

■ These are short puzzles designed to encourage use of mental arithmetic in a flexible way, often with an element of thinking backwards.

■ After a little thought, many learners will see the answers to these questions 'in a flash'. So it can be hard to help someone who is struggling with a question without giving the game away. Perhaps the best thing is to ask them to read you the question, wait for a while then enquire what the question is asking. If that does not get them started you can perhaps ask them to try some possible values systematically and see whether they give the right result (or are too big or too small): that can sometimes start them thinking more logically.

■ You can make up more questions of these kinds if you think they are needed; or working in pairs, one learner can make up such questions and try them on the other learner.

■ Harder questions requiring backwards thinking are included in 'Spanish treasure puzzles' on pages 34–35.

Selected solutions

(a) Octavia is 16; her brother is 23.

(c) 12 m by 3 m

(e) 7, 9, 17 (He could have achieved them in any order.)

Foundation

6 Thought puzzles

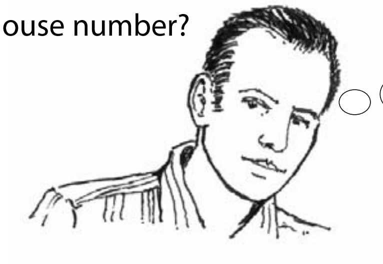
(a) Octavia is thinking about her age.

My brother is 7 years older than I am.
Our ages add up to 39.

How old are Octavia and her brother?

(b) Toby is thinking of his house number.

What is his house number?



It's an even number.
It has two digits. The digits add up to 11.
The digits multiplied together give 24.

(c) Bethany is thinking of the vegetable garden that her girl guide troop have created.

What are the dimensions of the vegetable garden?

It's rectangular.
Its area is 36 square metres.
Its perimeter is 30 metres.

(d) Kamala is thinking about chocolates.

How many chocolates were there in the box to start with?

I bought a box of chocolates.
I ate two-thirds of them.
I gave my sister half of what I hadn't eaten.
Now there are 3 chocolates left.

(e) Ted is thinking of his last three scores when playing bowls.

Their median was 9.
Their mean was exactly 11.
Their range was 10.

What were the three scores?

7 Socks

Notes

Optional resources

coloured counters, cubes or socks in a bag or box

Points to note

- During piloting, some teachers described this task as a 'long starter'.
- If learners can't get started with the task it can be tricky to give help without giving the game away. One approach is to try a related activity with coloured counters (or you could use cubes or cut-out pictures of coloured socks – or you could use real socks!). Fill a bag with blue, red and yellow counters (or whatever you've chosen). Now pick one out at a time, until a pair is found. You could record how often a pair is found after 2 counters, 3 counters and so on. Then you could discuss why a pair is found after 2, 3 or 4 counters and why you never need to pick a fifth counter. Now ask the class to try and apply this to the first socks question.
- Part (e) has a Higher tier level of difficulty.
- During piloting some learners extended the task to a third colour. You could, for example, add 8 yellow socks to the drawer and then consider p red socks, q blue socks and r yellow socks. You could also consider the case of a three-footed space creature who needs three matching socks!

Selected solutions

(a) 3

(c) 12

7 Socks

In my sock drawer I have 6 identical loose red socks and 10 identical loose blue socks.

I share a room with my little sister.
I get up early and don't want to put the light on and disturb her.
I reach into the drawer without being able to see the socks.

(a) How many socks must I take out of the drawer to be sure of getting a matching pair?

(b) How many socks must I take to be sure of a blue pair?

(c) How many socks must I take to be sure of a red pair?

(d) What if I have 4 red socks and 3 blue socks?

(e) What if I have p red socks and q blue socks?

8 Moving house

Notes

Optional resources

scissors

Points to note

- During piloting, learners worked on part (a) with the minimum of explanation from the teacher. Typically, discussion centred around how much time, if any, they should allow for lunch.
- A branch of applied mathematics called operational research treats problems like these in a more formal way. Within it, a planning problem like part (b) would be dealt with by a technique called critical path analysis, which helps managers schedule the stages of a project at times that will result in the quickest completion of the whole project. Critical path analysis may reveal ways in which a whole project can be speeded up by applying more resources just to certain stages (as with the question whether and when Linda should help).
- For part (b) learners could cut out and label rectangles (using a standard size for a person's work for a day) and move them around until they form a plan that meets all the conditions.
- There could be discussion about those jobs that could be speeded up when done by more than one person and other jobs that could not because of the need for drying time.

8 Moving house

Josh and Katy are moving house.
Their friend Linda has promised to help them.

They have to pack all their things into boxes and clean each room in the house.

Katy estimates how long it will take one person to pack up the things in each room and then clean it.

Room	**Approximate time**
Kitchen	4 hours
Dining room	2 hours
Lounge	3 hours
Main bedroom	3 hours
Spare bedroom	2 hours
Study	3 hours
Bathroom	2 hours
Shower room	1 hour
Shed	5 hours
Garage	3 hours

They want to do all the packing and cleaning in one day, starting at 8:30 a.m.

(a) Make a list of jobs for each person, showing roughly when they should aim to start and finish in a room.

Before they move into their new house, Josh and Katy want to do some decorating.

The table shows the decorating jobs they need to do and the time it will take for **one person** to complete each job.

The 'time needed' includes time for any paint, wallpaper or plaster to dry.

Job	Time needed	What needs to be finished before this job is started
Painting woodwork		
Sand and prime	1 day	Wallpaper stripped
Undercoat	1 day	Woodwork sanded and primed
Top coat	1 day	Undercoat on woodwork
Papering walls		
Strip off wallpaper	1 day	
Repair wall plaster	1 day	Wallpaper stripped
Paper walls	2 days	Top coat of paint on woodwork **and** wall plaster repaired **and** second coat of paint on ceilings
Painting ceilings		
Repair ceilings	1 day	
First coat	1 day	Ceilings repaired
Second coat	1 day	First coat of paint on ceilings

Josh and Katy start the decorating on a Monday.

(b) **(i)** Plan the jobs that need to be done day by day.

(ii) They plan to move in when the decorating is finished.
When can they move in?

(iii) Could the job be speeded up if Linda helped on some days?
Show how on a revised plan.

9 Tournaments

Notes

Points to note

■ Make sure learners understand the knock-out tournament diagram before they start the questions. To simulate the tournament round-by-round you could spin a coin for each match (for example, heads Adele to win, tails Barker to win) and the class could write the winner in the relevant rectangle.

■ There are many valid responses for part (d). For any number of players from 9 to 16, a tournament can be designed with four rounds.

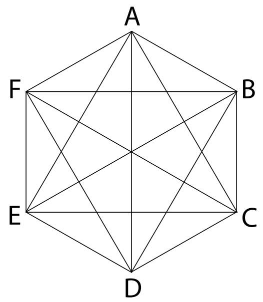
■ A round robin tournament is sometimes called an American tournament.

■ A tricky thing in part (e) is to ensure that every possible pair of different letters has been used once and once only. You may be able to nudge learners towards drawing an incidence matrix like the one below and ticking off each cell that has been used; you can discuss why only the cells in this triangle need to be considered.

	A	B	C	D	E	F
A		✓	✓	✓	✓	✓
B			✓	✓	✓	✓
C				✓	✓	✓
D					✓	✓
E						✓
F						

The matrix representation shows that, whatever the number of players, the number of matches will be a triangle number. Subtracting the number of cells in the leading diagonal from the total number cells in the matrix and then dividing by two because we are only using one of the remaining triangles gives $(n^2 - n)/2$ cells, which can be factorised to $n(n-1)/2$.

■ The complete set of pairs can be represented as the lines on a mystic rose, such as the one shown here for six items.



■ In part (f) there must be at least 9 rounds so each player gets a chance to play each of the other 9 players once. With 2 players involved in each match there can be no more than 5 matches played simultaneously in a round. But can there be an efficient tournament of exactly 9 rounds with 5 matches in a round? A way to explore this is to number the rounds 1–9 and put these numbers in the incidence matrix instead of ticks, seeking to ensure, for example, that the C row and C column together contain only the numbers 1–9 (a sudoku-like procedure).

	A	B	C	D	E	F	G	H	I	J
A		1	2	3	4	5	6	7	8	9
B			3	4	5	6	7	8	9	2
C				5	6	7	8	9	1	4
D					7	8	9	1	2	6
E						9	1	2	3	8
F							2	3	4	1
G								4	5	3
H									6	5
I										7
J										

A systematic approach has worked in the matrix above indicating that for any even number of players n a tournament of $n - 1$ rounds, each with $n/2$ matches can be organised. This gives a total of $(n/2)(n - 1)$ matches, which is equivalent to the triangle number formula derived differently above.

■ Finding the number of ways of combining items belongs to a branch of mathematics called combinatorics. It's relevant to this kind of planning task and important in more advanced work on probability.

■ In parts (i) and (j) devices such as seeding and divisions may arise. All the sports leagues are in fact tournaments and you may find some of your learners know a lot about their complexities.

Selected possible solutions

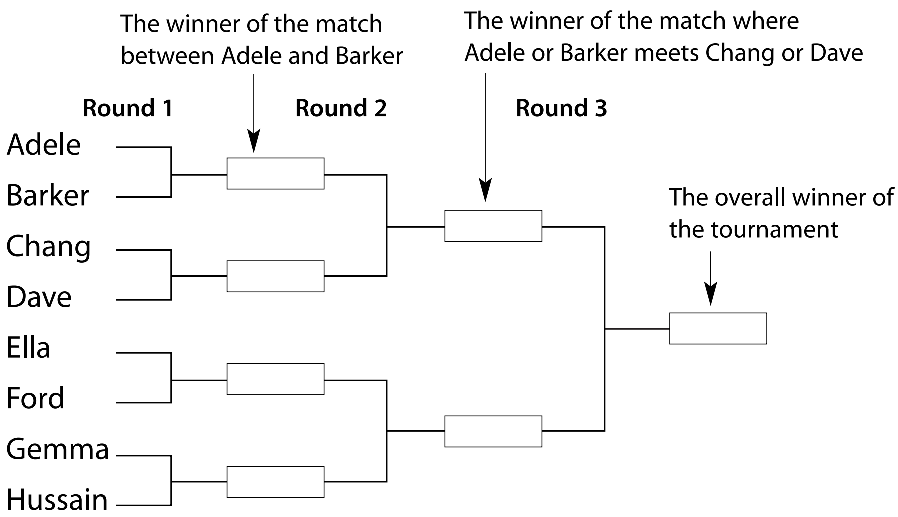
(a) No, if they meet at all it will be in round 3 and there one will beat the other and go on to the final playing some other player.

(b) Yes, in round 3 if each of them wins their matches in the first two rounds.

9 Tournaments

Eight school students are playing in a tennis tournament.

It is a **knock-out tournament**: if you lose a match you do not play any more matches.
The organisers make this plan.



(a) Adele and Dave are the school's star players.
They hope to meet in the final.
Could this happen?

(b) Ella is dreading having to play against Gemma.
Could this happen? If so, in which round?

(c) In the tournament shown above there are 3 rounds and 7 matches.
If the number of players is doubled how many rounds and matches will be needed?
Draw a diagram like the one above to show how the tournament would be organised (you can use A, B … instead of names).

(d) Show on a diagram how you would organise a tournament for an 'awkward' number of players like 12 or 13.

Another way to arrange matches between players is a **round robin** tournament.
Each player in it meets every other player once.
At the end the overall winner is decided by adding up points.
These could be the scores from matches or (say) 2 for a win, 1 for a draw and 0 for losing.

Here is how a round robin tournament could be arranged for 4 players – A, B, C and D.

Round 1	**Round 2**	**Round 3**
A plays B	A plays C	A plays D
C plays D	B plays D	B plays C

Check that every player has played every other player once.

(e) Show how a round robin could be arranged for 6 players – A, B, C, D, E and F.
It could start like this (but it doesn't have to) …

Round 1	**Round 2**
A plays B	A plays C
C plays D	…
E plays F	

(f) For a round robin tournament with 10 players,

(i) how many rounds would be needed?

(ii) how many matches would there be in each round?

(iii) how many matches would there be altogether?

(g) How many matches are there altogether in a round robin with n players?

A tournament can be between teams, rather than individual players.
Sometimes 'awkward' numbers of teams or players have to be fitted in.

(h) Five basketball teams – P, Q, R, S and T – decide to have a tournament.
Each team has to play every other team once.
In each round, 2 matches are played and the fifth team rests.
Show how the tournament could be arranged.

Tournament structures can be a hot topic among sports fans.

(i) What are the advantages and disadvantages of the knock-out tournament and the round robin from the point of view of players and spectators?

(j) If you know about some other tournament structure, describe how it works (with the help of a diagram) and explain its advantages and disadvantages.

Foundation/Higher

10 Numbered discs

Notes

Optional resources

Small scraps of paper to substitute for discs

Points to note

■ One way to record results is by a two-way table with the totals in the body of the table (see below).

■ Parts (c), (d) and (f), are easier to deal with if the numbers are restricted to positive integers. Learners may also use zero if they want to, or indeed negative integers; doing so adds to the possible correct solutions. Make sure that in giving their answers to these parts learners clearly state a pair of numbers for each disc.

■ For the experimental approach using scraps of paper in part (c) you could have learners working in pairs; after some haphazard choices of numbers you may find they naturally apply some logic. A more abstract approach, which may appeal to some learners, is to set up a two-way table with the smallest total placed top left and the largest placed bottom right. The task then is to find the row and column headings, which are the numbers on the discs. Here is how it would work for part (c).

		on grey	
		?	?
on white	?	2	
	?		11

4 and 9 to go here

Selected possible solutions

(b) $2 + 6 = 8$
$2 + 9 = 11$
$5 + 6 = 11$
$5 + 9 = 14$

(d)

	2	6
1	3	7
5	7	11

(e) $2 + 2 + 3 = 7$
$2 + 2 + 6 = 10$
$2 + 7 + 3 = 12$
$2 + 7 + 6 = 15$
$4 + 2 + 3 = 9$
$4 + 2 + 6 = 12$
$4 + 7 + 3 = 14$
$4 + 7 + 6 = 17$

10 Numbered discs

(a) Sathya has two discs, one white and one grey.

The white disc has 3 on one side and 4 on the other.

The grey disc has 1 on one side and 7 on the other.

If Sathya throws the two discs in the air they could land like this 3 7,
giving a total score of 10.

Or they could land like this 4 7, giving a total score of 11.

What other ways could they land, and what would the total score be in each case?

(b) Austin has some discs like Sathya's but they have different numbers.

The white disc has 2 on one side and 5 on the other.

The grey disc has 6 on one side and 9 on the other.

Write down all the possible ways Austin's discs could land, giving the total score in each case.

(c) Karen has two discs of different colours, each with a number on both sides.
When her discs land the different possible total scores are 2, 4, 9 and 11.
What numbers are on Karen's discs?
(If you have trouble getting started, use two small scraps of paper as 'discs' and write a number on each of the four sides. Experiment with rubbing out and replacing these numbers until your discs give you the four totals you need.)

(d) Afzal has two discs, each with a number on both sides.
When his discs land the only possible total scores are 3, 7 and 11.
What numbers are on Afzal's discs?

(e) Chris has three discs, one white, one grey and one black.

The white disc has 2 on one side and 4 on the other.

The grey disc has 2 on one side and 7 on the other.

The black disc has 3 on one side and 6 on the other.

If Chris throws the three discs in the air they could land like this 2 2 6,
giving a total score of 10.

What other ways could they land, and what would the total score be in each case?

(f) Challenge

You have three discs.
You want to get these possible total scores when the discs land: 3 4 5 6 7 8 9 10
How would you number the discs?

11 Scale drawing 2

Notes

Resources required

rulers
pairs of compasses
angle measurers
calculators

Points to note

- 'Scale drawing 1' (pages 16–17) contains simpler tasks of this kind.
- (a), (b) and (c) do not have to be done together.
- (b) and (c) focus on standard GCSE topics but (c) is unstructured and requires a significant amount of independent thought.

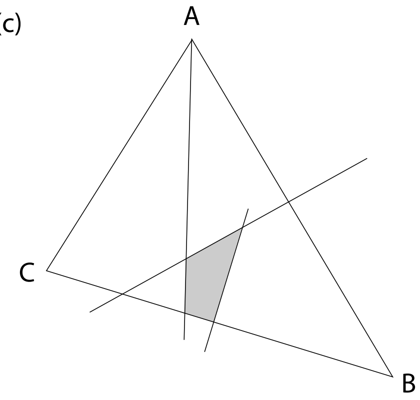
Selected possible solutions

(a) (ii) 9.5 cm

(b) (i) 26 km

(ii) 290°

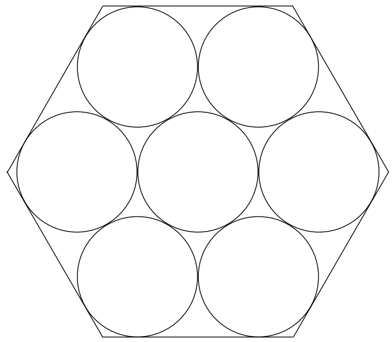
(c)



The cost of digging is about £90.

11 Scale drawing 2

(a) A hexagonal box is being designed to hold seven large chocolates as shown.
Each chocolate will be a circular disc 6 cm in diameter.



(i) Draw the arrangement of chocolates full size.
(Position the centres of the circles carefully so the circles will touch as shown.)
Now add the hexagonal box, touching the outer circles.

(ii) By measuring, find the length of one side of the box.

(b) Make a scale drawing of the following situation, using a scale where 1 cm represents 2 km.

Ship A is 17 km from a lighthouse, on a bearing of 053° from the lighthouse.

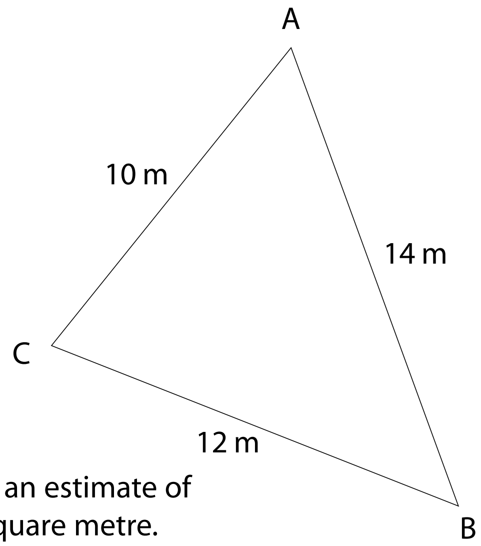
Ship B is 22 km from the same lighthouse, on a bearing of 330° from the lighthouse.

(i) How far apart are ships A and B?

(ii) Draw an accurate north line going through ship A.
What is the bearing of ship B from ship A?

(c) These are the dimensions of a triangular field.

Some treasure has been buried in the field and these are the only clues to its whereabouts.



The treasure is closer to corner B than to corner A, closer to corner C than to corner B and closer to side AB than to side AC.

Draw the field to a scale of your own choice.

By accurate construction find the region of the field in which the treasure must lie.
Shade this region and, from measurements, calculate an estimate of the cost of digging up this area at a rate of £20 per square metre.

12 Consecutive numbers

Notes

Points to note

■ This work assumes the ability to multiply out two linear expressions to obtain a quadratic expression. Simple factorisation of expressions is also needed.

■ During the piloting, putting algebra to use like this was unfamiliar to some learners, even though the algebra itself is straightforward. In part (a) even confident learners needed help to see how letting the first of the three numbers be n, so that $n + 1$ and $n + 2$ become expressions for the next two, is the key to producing a proof; but after that they could produce their own proofs for parts (b) to (d).

■ The work highlights the important distinction between identifying a general truth by trying several cases and proving it by algebra.

Selected possible solutions

(a) (i) $1 + 2 + 3 = 6$
$5 + 6 + 7 = 18$
These and other cases are multiples of 3.
Let n be the first of any three consecutive integers.
Then the next two integers are $n + 1$ and $n + 2$.
The total is $n + (n + 1) + (n + 2)$,
which simplifies to $3n + 3$,
which factorises to $3(n + 1)$.
Since 3 is a factor, this expression will represent a number that is a multiple of 3, whatever the value of the integer n.

(c) (i) 4, 5, 6, 7
$4 \times 7 = 28$
$5 \times 6 = 30$
The difference is 2.

(ii) This also works for 7, 8, 9, 10 (products 70 and 72) and for 20, 21, 22, 23 (products 460 and 462).
Let n be the first of any four consecutive numbers.
Then the next three numbers are $n + 1$, $n + 2$ and $n + 3$.
The product of the first and the last is $n(n + 3)$, which simplifies to $n^2 + 3n$.
The product of the middle two is $(n + 1)(n + 2)$, which simplifies to $n^2 + 3n + 2$.
The difference between these expressions is 2.

12 Consecutive numbers

Whole numbers that follow on from one another are called consecutive numbers, for example 5, 6, 7.

(a) **(i)** Choose three consecutive numbers and add them together.
Do you always get a multiple of 3?
Try to **prove** what you have found using algebra.

(ii) What happens when you add any four consecutive numbers?
What happens when you add any five?
Use algebra to confirm your results.

(b) **(i)** Choose three consecutive numbers.
Multiply the first number by the last (in other words, find their product).
Find the square of the middle number.
Now calculate the difference between these two results.

(ii) Follow the same steps with other groups of three consecutive numbers.
What do you find?
Try to prove the result using algebra.

(c) **(i)** Now choose four consecutive numbers.
Find the product of the first number and the last number.
Find the product of the middle two numbers.
Now calculate the difference between these two products.

(ii) Do the same with other groups of four consecutive numbers.
What do you find?
Try to prove the result.

(d) Consider a situation where you have five consecutive numbers.
Use algebra to predict what the difference will be between the square of the middle number and the product of the first and last numbers.
Try this out with some sets of five consecutive numbers to check whether your prediction is correct.

13 Spanish treasure puzzles

Notes

Points to note

■ These are mixed questions that roughly increase in difficulty. (a) is suitable for Foundation and some of the others may be also. There is easier work of a similar kind in 'Thought puzzles' on pages 18–19.

■ In part (d) some use of algebra is better than unfocused guesswork but it is not necessary to write and formally solve four simultaneous equations.

Selected solutions

In each case there should be evidence of a logical method.

(b) 16 old Spanish ounces

(c) 59

13 Spanish treasure puzzles

(a) Each of these bags contains some gold doubloons.

The mean number of doubloons in a bag is 5.
The median is 6.
The mode is 2.

What numbers of doubloons are in the bags?

(b) Benito has a hoard of 5 gold ingots.
These ingots have a mean weight of exactly 12 old Spanish ounces.

Someone steals one of Benito's ingots. (They will pay for it!)
The mean weight of the remaining ingots is exactly 11 old Spanish ounces.

What was the weight of the ingot that was stolen?

(c) A band of 5 pirates find a hoard of doubloons.
They try to share them out equally but there are 4 doubloons left over.
So one pirate is made to walk the plank.

When the 4 remaining pirates try to share out the doubloons,
they find there are 3 doubloons left over.
So another pirate has to walk the plank.

When the 3 remaining pirates try an equal share-out, 2 doubloons are left over.
So …

When the 2 remaining pirates try a share-out, 1 doubloon is left over.

What is the smallest number of doubloons there could be in the hoard?

(d) Four Spanish princesses are given some jewels.

Beatriz gets twice as many jewels as Alicia.
Dolores gets 3 times as many jewels as Cecilia.
Alicia gets 7 more jewels than Cecilia.
Beatriz gets 3 more jewels than Dolores.

How many jewels does each princess get?

Higher

14 Shapes and algebra 1

Notes

Points to note

■ The algebra involves carefully collecting terms in a linear expression and solving a linear equation.

■ Like 'Consecutive numbers' on pages 32–33, this is work where algebra is put to use, something that many classes were unfamiliar with. After giving it to a class working at about levels 6 and 7 one teacher said, 'This was a good task with all pupils able to access it, and it provided an excellent reason for using algebra and an excellent opportunity for lots of discussion and reasoning.' One learner said, 'I thought it was harder than it was and I overcomplicated it.'

■ Some learners will need to be coaxed away from trial and improvement. The key step is writing one or more expressions from the information given; you need to check that this is happening in part (a) (i) and provide guidance if it's not. One teacher commented, 'What I liked was that you could give support without giving away all of the answers.' In part (b) (i) learners may fail to see that 'what is the length of QR?' is asking for an expression and not a particular value.

■ It was common for learners to be unsure whether two more than y should be written as $y + 2$ or as $2y$. This may just reveal a momentary lack of focus or it may indicate that they do their regular algebra moving symbols around on the page by rote, without ever thinking about the underlying arithmetic represented by the symbols.

■ Some learners realised that a T-shape has the same perimeter as a rectangle fitted around it, thus simplifying the work – and the same for an L-shape.

Selected possible solutions

(b) (i) $(y + 2)$ cm

(ii) $8(y + 2) + 4y = 52$

$12y + 16 = 52$

$12y = 36$

$y = 3$

So PQ = 3 cm, QR = 5 cm

(d) Let QR = x cm

PQ = $(x + 3)$ cm

$$\overset{\text{KL}}{(2x+2)} + \overset{\text{LM}}{2} + \overset{\text{MN}}{x} + \overset{\text{NO}}{(x+3)} + \overset{\text{OP}}{2} + \overset{\text{PQ}}{(x+3)} + \overset{\text{QR}}{x} + \overset{\text{RK}}{2} = 32$$

$$6x + 14 = 32$$

$$6x = 18$$

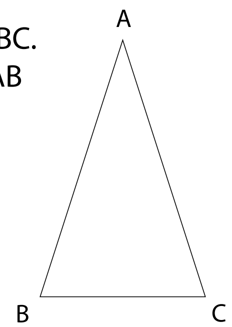
$$x = 3$$

QR = 3 cm, RK = 2 cm, KL = 8 cm, MN = 3 cm, NO = 6 cm, PQ = 6 cm

14 Shapes and algebra 1

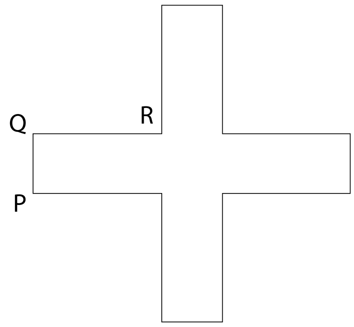
(a) **(i)** In this isosceles triangle side AB is three times the length of side BC. Given that BC is x cm long, write an expression for the length of AB and an expression for AC.

(ii) The perimeter of the triangle is 24.5 cm. Form an equation in x and use it to find the lengths of the three sides.



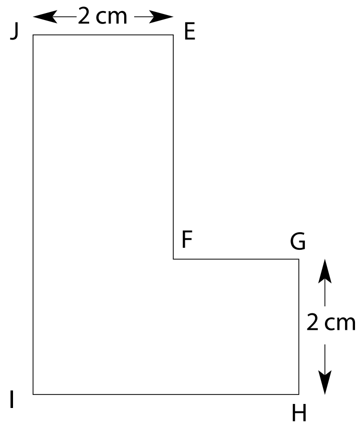
(b) **(i)** This cross has four lines of reflection symmetry. Side QR is 2 cm longer than side PQ. Given that PQ is y cm long, what is the length of QR?

(ii) The perimeter of the cross is 52 cm. Form an equation in y and use it to find the lengths of PQ and QR.



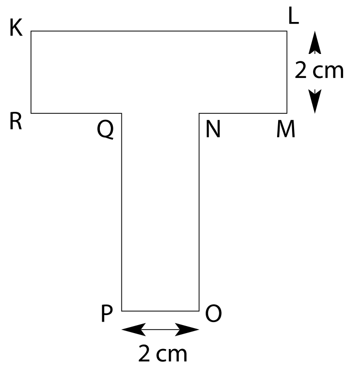
(c) **(i)** In this L-shape side EF is twice as long as side FG. Given that FG is k cm long, what are the lengths of EF, IH and JI?

(ii) The perimeter of the L-shape is 29 cm. Form an equation in k and use it to find the lengths of EF and FG.



(d) This T-shape has one line of reflection symmetry. Side PQ is 3 cm longer than side QR. The perimeter of the T-shape is 32 cm.

Find the lengths of the sides that are not given.



Higher

15 Clock puzzles

Notes

Points to note

■ The questions are roughly graded in difficulty: during piloting, most learners found enough to engage with and ended with something that challenged them. One teacher wrote, 'This type of task did engage the students and kept them motivated throughout the lesson.'

■ In part (a) (iii) some learners will assume that at 1:30 the hour hand is still pointing to 1 and will calculate the angle 150°; however the hour hand is now halfway between 1 and 2. Similarly in part (a) (iv) at 3:15 the hour hand is not pointing to the same place as the minute hand, so the angle is not 0°. The idea that the hour hand's slow movement has to be taken into account is fundamental to parts (a) and (b); one teacher remarked, 'It was good because it showed students that a problem isn't always as easy as it seems.'

■ It's easy to make up more questions like those in part (a) if you think that's necessary before doing (iv) and (v) or moving on to part (b); or learners can work in pairs making up such questions and giving them to their partner (or in threes: A sets the question, B answers and C is the referee; then roles change round).

■ In part (b) there are 22 times in the twelve hour cycle at which the two hands are separated by the given angle. Learners are only expected to find the more obvious ones and even so 80° is challenging.

■ Mathematically talented learners could explore why there are as many as 22 times at which a given angle occurs (though only 11 times at which the hands are 180° apart). Some might even develop a method for finding all the times; however such a project would be well beyond the level of demand of problem solving in GCSE.

Selected possible solutions

(a) (iv) $7\frac{1}{2}°$ (v) 70°

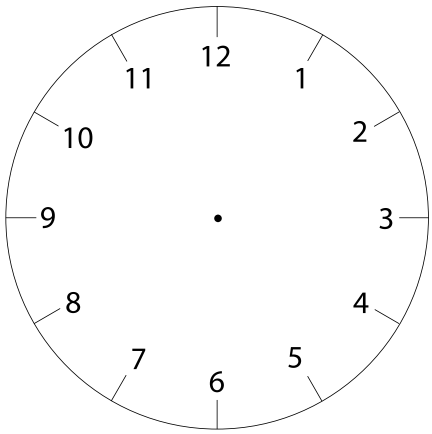
(b) (iii) 4:30 (iv) 1:20 (v) 9:45

(c) At 4 o'clock that afternoon

(d) (i) The watches will coincide 144 days later ($24 \times 60 \div 10$), namely at noon on 25 May (or 24 May if it's a leap year).

(ii) Ali's watch coincides with Pete's every 96 days ($24 \times 60 \div 10$).
To find the number of days after which all three watches show the right time we need the LCM of 144 and 96.
$144 = 2^4 \times 3^2$
$96 = 2^5 \times 3$
So their LCM is $2^5 \times 3^2 = 288$
So all three watches coincide 288 days later on 16 October (or 15 October if it's a leap year).

15 Clock puzzles



(a) Calculate the angle between the minute hand and the hour hand of a clock at each of these times.
Explain, using a sketch if necessary, how you worked each one out.

(i) 9:00 **(ii)** 7:00 **(iii)** 1:30 **(iv)** 3:15 **(v)** 5:40

(b) Give a time at which the angle between the hands is the following.

(i) 60° **(ii)** 120° **(iii)** 45° **(iv)** 80° **(v)** 22.5°

(c) Terri has a clock with hands and Parvinder has one too.
To start with, both clocks show the right time.

Then, at 10 o'clock one Monday morning, Terri's clock starts to go backwards (but at the right speed).

When will both clocks show the same time again?

(d) Pete and Elsita both have 24-hour digital watches.

Pete's watch runs at the right speed but Elsita's loses 10 minutes every day.
At noon on 1 January they both show the right time.

(i) When will they both show the right time again?

(ii) Challenge
Ali's 24-hour digital watch also shows the right time at noon on 1 January, but it loses 15 minutes every day.
When will all three watches show the right time again?

Higher

16 Queen power

Notes

Resources required

squared paper

Optional resources

scissors

Points to note

■ This is a rich task that can be approached in a variety of ways. Encourage learners to try to explain the statements they make and to prove any generalisations.

■ If they find it difficult move forward at any point you could pose relevant questions along the lines of the following.

- Do you have to think about every square on the board? How does the symmetry of the board help you?
- In how many directions can the queen move from a corner? How many squares can she move on to in each of these directions?
- What happens if you move the queen from one edge square to another? Do you add to or subtract from her power? What happens to the horizontal power and vertical power? What happens to the diagonal power? Can you explain this?
- What happens if you move the queen towards the centre of the board? Do you add to or subtract from her power? How much do you add or subtract? Can you explain this?
- Are there any areas of the board where the power of the queen remains the same? Can you explain this?
- Where on an n by n board does the queen have the most power? What happens when n is even, and when n is odd?

■ Later in their investigation, some learners may want to draw a queen with her power radiating out in eight directions, which they view though square 'windows' of different sizes and in different positions. You could have scissors and paper ready for this.

■ Other investigations can be based on the power of the queen, for example, 'What is the minimum number of queens that are needed to command every square on a standard chess board?' Invite learners to come up with their own investigations, based on the queen or other chess pieces.

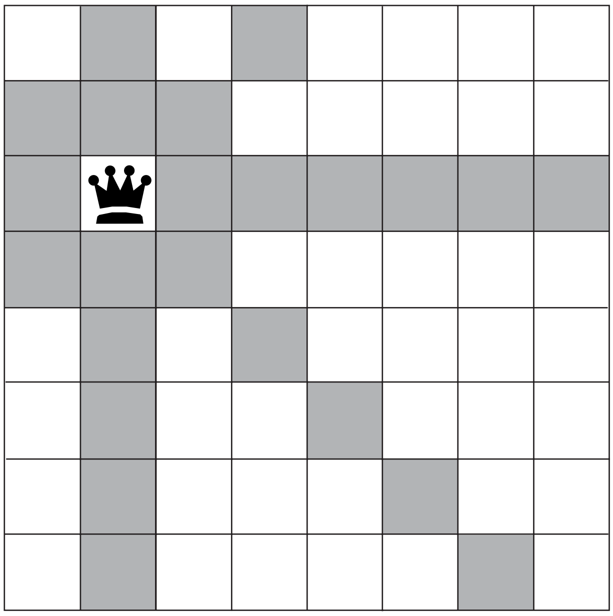
16 Queen power

Chess is a game where pieces move on an 8 by 8 board of squares.
The pieces move in different ways.

The queen is a very powerful piece.
She can move any number of squares in any direction – up or down, left or right, or diagonally.

From this position on the board, the queen can move to 23 different squares.
These squares are coloured grey.

We can measure the 'power' of a queen on a particular square as the number of squares she can move to.
So the queen here has a power of 23.



(a) Investigate the power of the queen on different squares on the 8 by 8 board.
Where does she have the most power?
Where does she have the least power?

(b) Investigate the power of the queen on smaller and larger square boards.
Where does the queen have the most power on each board?
Where does she have the least power?

(c) What is the greatest and least power of a queen on an *n* by *n* board?

17 Shapes and algebra 2

Notes

Points to note

- The algebra involves collecting linear terms and dealing with expressions that are sums of products of two variables like *ab* and f^2 .
- Some learners are puzzled by the terms 'square units' and 'units' in part (a) (and later in (f) and (g)). If we treat *a* as a pure, dimensionless number, which for example we can multiply by *b* to get *ab*, *a* itself cannot be a length but *a* centimetres can be. In non-practical problems like these we have no need for particular measures like metres and square metres; instead we talk about *a* units and *ab* square units, sometimes called 'arbitrary units'. Using this convention strictly, the edge lengths on the diagrams should be labelled '*a* units' and so on. Note that in the sciences a different convention is often used where a variable such as *a* is treated as a length or other dimensioned quantity, not a pure number.
- During piloting, some learners were unsure what to do in part (a), thinking that the expressions were obvious so something difficult must be required (see the selected possible solutions). This part should help clarify the later phrase 'an algebraic expression for …'
- In part (c) (i) the shape can be dissected in different ways and the areas added; or it can be seen as a square with a square subtracted from it, an approach that requires more care with the algebra. There are similar choices for the subsequent area questions.
- As in 'Shapes and algebra 1' (pages 36–37) some learners realised that a T-shape has the same perimeter as a rectangle fitted around it, thus simplifying the work – and the same with an L-shape.
- If learners are completely stuck with part (g) (ii) you can give the hint that they consider what it is about the area expression $2f^2 + 6fg$ that differs from the area expression they obtained in part (e) (i).

Selected possible solutions

(a) (i) Area of a rectangle = breadth × length
$= a \times b$
$= ab$ square units

(ii) Perimeter $= a + b + a + b$
$= 2a + 2b$ units

(c) (i) $r^2 + 2rs$ square units

(ii) $4r + 4s$ units

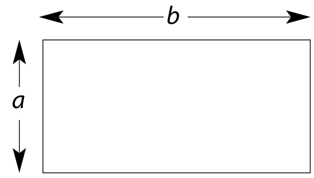
(e) (i) $10wv$ square units

(ii) $8w + 8v$ units

17 Shapes and algebra 2

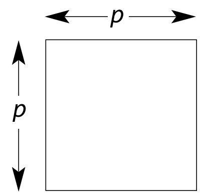
(a) **(i)** Explain why the area of this rectangle is ab square units.

(ii) Explain why the rectangle's perimeter is $2a + 2b$ units.



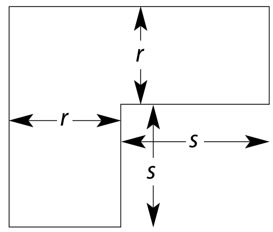
(b) **(i)** Write an algebraic expression for the area of this square.

(ii) Write an expression for the square's perimeter.



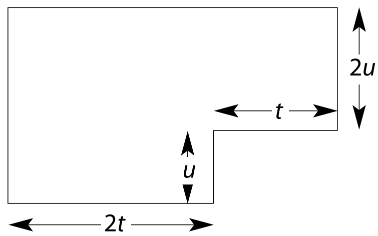
(c) **(i)** Write an expression for the total area of this L-shape.

(ii) Write an expression for the L-shape's perimeter.



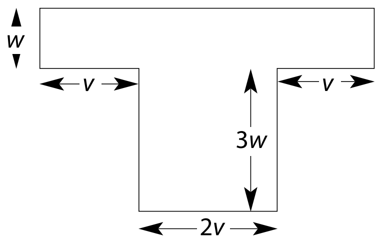
(d) **(i)** Write an expression for the total area of this L-shape.

(ii) Write an expression for the L-shape's perimeter.



(e) **(i)** Write an expression for the total area of this T-shape.

(ii) Write an expression for the T-shape's perimeter.



(f) Sketch each of these shapes, using algebra to label sides.

(i) A rectangle with area $3a^2$ square units and perimeter $8a$ units

(ii) An L-shape with area $4b^2$ square units and perimeter $10b$ units

(iii) A T-shape with area $13c^2$ square units and perimeter $20c$ units

(g) **Challenges**

Sketch each of these shapes, using algebra to label sides.

(i) An L-shape with area $4de$ square units and perimeter $6d + 4e$ units

(ii) A T-shape with area $2f^2 + 6fg$ square units and perimeter $6f + 8g$ units

18 Friday the 13th

Notes

Resources required

a calendar for the current year

Optional resources

calendars for other years

Points to note

■ Learners could start by finding the number of Friday the 13ths in the current year.

■ If the class then find it difficult to move forward with the task you could ask some of these questions.

- Do you think the number of Friday the 13ths in the current year is unusually low or unusually high?
- Do you have any idea what the maximum number will be for any year? Do you think there will be a year with, say, ten Friday the 13ths?
- Do you have any idea what the minimum number will be for any year? Do you think there will be a year with, say, no Friday the 13ths?
- Is there any kind of repeating pattern in the number of Friday the 13ths in each year?
- Can you find a rule to give you the number of Friday the 13ths in a year if you know the day of the week that New Year's Day falls on?

■ During piloting some learners took this exploratory task to a challenging level, getting as far as establishing the number of years in the repeating cycle of years, taking account of leap years. On the other hand one teacher was prompted by this task to do simpler work on the calendar with a Foundation group: they became very interested in what day of the week their birthday would fall on over a cycle of years.

18 Friday the 13th

Some people think it is unlucky when the 13th of the month falls on a Friday.

Fear of Friday the 13th has been given the name 'paraskevidekatriaphobia'.*

How many Friday the 13ths can you expect in a year?

* Names given to conditions in medicine and psychology are made up of parts that come from suitable Latin or ancient Greek words.

In this case the words are ancient Greek …

paraskevi = Friday

dekatreis = thirteen

phobos = fear

19 Tile patterns

Notes

Resources required

calculators

Points to note

■ The necessary technical skills include finding circle-related areas, using Pythagoras and calculating proportions as decimals or percentages. Exact expressions (in terms of π and surd notation) may be used for areas and proportions (see below).

■ The problem solving element centres on dissecting shapes in ways that allow the required areas to be found efficiently.

■ Because of its high technical and problem solving demand, this last task in the pack is a very demanding one, much of it being more difficult than anything that will appear in GCSE. It stretched even the most able learners during piloting. It is well suited to working in pairs or threes, so that learners can build on one another's knowledge, debate about the direction the work should take and check one another's work.

■ Errors often involve losing sight of the need to halve, quarter or double an area. It is also easy to forget whether an expression (or value) is an area or a proportion.

■ Learners may be inclined to base their calculations on the measured edge length of the squares as they appear on the page. However this is not a convenient value and since the proportion shaded is what's ultimately required – not actual areas – learners may come to realise that squares of any size could be considered. So using an edge length of 1 unit or 2 units would simplify the work.

■ Answers for the proportions may be given as a decimal or as a percentage. Learners who are confident about obtaining exact expressions (in terms of π and surd notation) should be encouraged to do so. Using this approach with design R amounts to a proof of the elegant fact that the proportion is exactly a half; that's different from a mere calculation producing some value that happens to round to a half.

Selected possible solutions

(a) (i) In Q, considering a square with vertices at the mid-points of the sides of the given square leads to the conclusion that the shaded area is significantly less than a half of the area of the given square. No similar simple argument seems to be available for P or R, and visual estimation may not be conclusive.

(ii) There needs to be a clear record of how the dissections have been used, perhaps including a 'cartoon strip' or labelling of the regions with letters.

P: $\frac{1}{2}\pi - 1$ ($\approx 0.57 = 57\%$)

Q: $\frac{1}{4}\pi - \frac{1}{2}$ ($\approx 0.29 = 29\%$)

R: Shaded area =
area of circle – shaded area in Q
leading to proportion = $\frac{1}{2}$ ($= 50\%$)

(iii) P

(b) Shaded area = shaded area in P – area of circle
leading to proportion = $\pi(\sqrt{2} - 1) - 1$
($\approx 0.30 = 30\%$)

19 Tile patterns

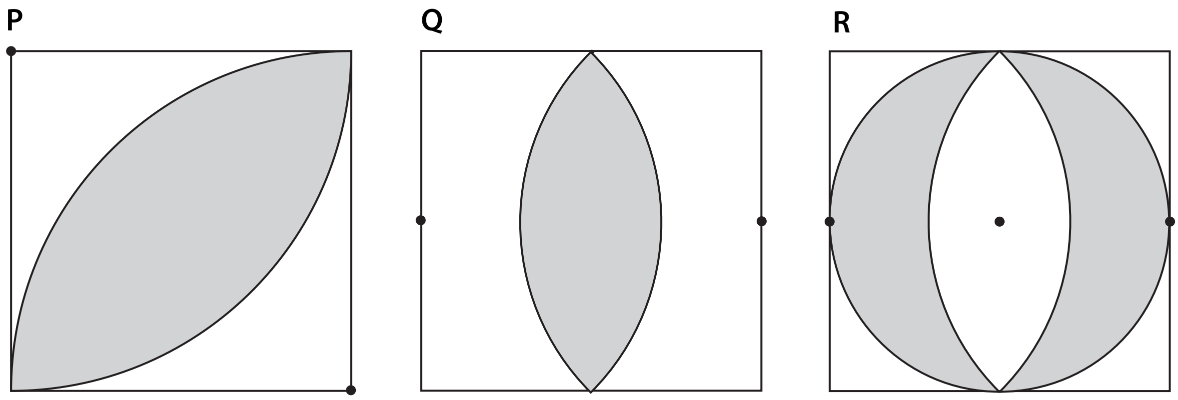
Each tile design on this sheet is drawn in a square.
Each design has two lines of symmetry.

You can draw each one using a ruler and a pair of compasses.
Each curved line is part of a circle that has one of the dots as its centre.

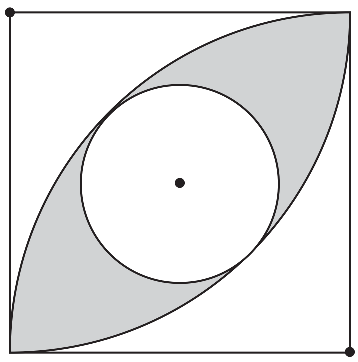
(a) **(i)** In which of the designs P, Q and R do you think more than half of the tile is shaded?

(ii) For each design, work out the proportion of the tile that is shaded.

(iii) In which design is the highest proportion of the tile shaded?



(b) **Challenge**
What proportion of this design is shaded?



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