Quadratics

The first thing to note about quadratics is the difference between an *expression*, an *equation* and a *curve*.

$$\begin{aligned} \textit{Expression} &: \quad 5x^2+15x+10 \\ \textit{Equation} &: \quad 0=5x^2+15x+10 \\ \textit{Curve} &: \quad y=5x^2+15x+10 \end{aligned}$$

When you have a quadratic *equation* you can happily divide both sides to get rid of common factors. For example here we can divide both sides by 5. This does not affect[1] the solutions.

$$0=5x^2+15x+10 \qquad \Rightarrow \qquad 0=x^2+3x+2.$$

However with an expression you can't get rid of common factors like that; you must retain them somehow.

$$5x^2+15x+10 \qquad \Rightarrow \qquad 5(x^2+3x+2).$$

The Formula

Given a general quadratic equation $ax^2+bx+c=0$ the two solutions (or roots) are given by[2]

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This *must* be committed to memory! You may find it useful to place a bracket round the $(4ac)$. All we need to do is plug the values of a, b and c into the formula and simplify. For example in the equation $2x^2+3x-2=0$ we can see that $a=2$, $b=3$ and $c=-2$.

$$\begin{aligned} x &= \frac{-3\pm\sqrt{3^2-4\times2\times(-2)}}{2\times2} \\ &= \frac{-3\pm\sqrt{9+16}}{4} \\ &= -\tfrac{3}{4}\pm\tfrac{5}{4} \\ &= \tfrac{1}{2} \text{ or } -2. \end{aligned}$$

Always, always, *always* remember that if you obtain something like $\sqrt{20}/2$ we can re-write this as $\sqrt{20}/\sqrt{4}$ which simplifies to $\sqrt{5}$. For practice show that the roots of x^2+6x+4 are $x=-3\pm\sqrt{5}$.

Note also that if the *discriminant* (b^2-4ac) is zero then there exists only one *repeated* root. If it is negative then there exist no roots.

Completing The Square

Given any quadratic expression we can complete the square. If we are dealing with an equation then we can also solve it. You must always "halve the number of x's" into the bracket[3]. For example

$$x^2+\boxed{6}x+3=(x+\boxed{3})^2-9+3=(x+3)^2-6$$

is an example of completing the square.

[1] Effect?

[2] The formula can be derived by completing the square on the general quadratic $ax^2+bx+c=0$.

[3] This is because $(x+a)^2=x^2+2ax+a^2$. Notice the a doubles to $2a$.

If the number of x's is not one then you have a choice of whether to factorise the coefficient out of the first two terms or the whole expression. For example

$$\begin{aligned} 3x^2 - 12x + 1 &= 3[x^2 - 4x] + 1 \\ &= 3[(x-2)^2 - 4] + 1 \\ &= 3(x-2)^2 - 12 + 1 \\ &= 3(x-2)^2 - 11, \end{aligned} \qquad \begin{aligned} 3x^2 - 12x + 1 &= 3[x^2 - 4x + \tfrac{1}{3}] \\ &= 3[(x-2)^2 - 4 + \tfrac{1}{3}] \\ &= 3[(x-2)^2 - \tfrac{11}{3}] \\ &= 3(x-2)^2 - 11. \end{aligned}$$

I don't know about you, but I prefer the left one because it usually avoids the fractions.

Once this is done the solution to the quadratic is simple by rearrangement. From above if we were asked to solve $3x^2 - 12x + 1 = 0$ then we would find

$$\begin{aligned} 3x^2 - 12x + 1 &= 0 \\ 3(x-2)^2 - 11 &= 0 \\ (x-2)^2 &= \tfrac{11}{3} \\ x &= 2 \pm \sqrt{\tfrac{11}{3}}. \end{aligned}$$

From completing the square we can also find the minimum (or maximum) of the curve $y = 3x^2 - 12x + 1$ because $y = 3(x-2)^2 - 11$. The smallest $(x-2)^2$ can be is zero, when $x = 2$, and the y-value will then be -11. So the coordinate of the minimum is $(2, -11)$.

Factorising

(Before any of the below look for is a simple factor of all three terms; for example $2x^2 - 4x + 6 = 2(x^2 - 2x + 3)$.)

If the x^2 coefficient is one then we just need to find two numbers that multiply to the constant and sum to the x coefficient. Then put these into brackets. For example $x^2 + 6x + 8$ the two numbers would be 2 and 4. So $x^2 + 6x + 8 = (x+2)(x+4)$. If we were solving $x^2 + 6x + 8 = (x+2)(x+4) = 0$ then we look for the values of x that make either bracket zero. Therefore $x = -2$ or $x = -4$.

When the x^2 coefficient is not zero then it becomes more complicated. We will use the example of $6x^2 + x - 2$. We need to find two numbers that multiply to -12 (obtained by $6 \times (-2)$) and sum to 1. Here they are clearly $+4$ and -3. We split the $+x$ term into $+4x - 3x$ (or $-3x + 4x$). We then factorise the first two terms and then the last two terms. You should find the resulting brackets are the same. The final factorisation should then become clear.

$$\begin{aligned} 6x^2 + x - 2 &= 6x^2 + 4x - 3x - 2 \\ &= 2x(3x+2) - 1(3x+2) \\ &= (3x+2)(2x-1), \end{aligned} \qquad \begin{aligned} 6x^2 + x - 2 &= 6x^2 - 3x + 4x - 2 \\ &= 3x(2x-1) + 2(2x-1) \\ &= (2x-1)(3x+2). \end{aligned}$$

Notice the result is the same both ways. You must be careful of negatives.

If we were being asked to solve $6x^2 + x - 2 = 0$ then $(2x-1)(3x+2) = 0$ so we look for the value of x when each of the brackets is zero to find the two solutions

$$\begin{aligned} (2x-1) = 0 \quad &\Rightarrow \quad x = \tfrac{1}{2}, \\ (3x+2) = 0 \quad &\Rightarrow \quad x = -\tfrac{2}{3}. \end{aligned}$$

(Finally if there is no x term then it is simple. For example $x^2 - 9 = (x+3)(x-3)$ and $4x^2 - 25 = (2x+5)(2x-5)$. This is from the difference to two squares; $x^2 - y^2 = (x+y)(x-y)$.)