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ASSESSMENT and QUALIFICATIONS ALLIANCE

General Certificate of Secondary Education

Additional Mathematics 9306

Pilot Specification
2008

TEACHER'S GUIDE AND TEACHING RESOURCE

Further copies of this booklet are available from:

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Background Information

1 Introduction

1.1 Purpose

This resource has been written by Leeds University's Assessment and Evaluation Unit to support teachers in developing approaches to the type of problem-solving questions that will appear in the pilot GCSE in Additional Mathematics.

This Teachers' Guide has been provided to assist teachers in their preparation for the delivery of courses based on the new AQA GCSE specification 9306. The guide should be read in conjunction with the specification document and the specimen material that accompany them. As the subject content is the same as the present AQA GCSE's in mathematics, the Teachers' Guide produced in 2006 to support AQA's Specifications 4301 and 4302 will be useful to teachers of this pilot. The specifications and specimen assessment materials are also available from the GCSE Mathematics Department, AQA, Devas Street, Manchester, M15 6EX, Telephone: 0161 957 3852, Fax: 0161 957 3873

1.2 Introduction

This resource is designed to support the teaching and preparation for the problem solving element of the AQA Pilot for the new GCSE in Additional Mathematics.

The resource consists of:

- these introductory pages;
- 90 examples of problem solving questions on the CD-Rom;
- commentaries on the features of 30 of the problem solving questions;
- answers to the 90 questions;
- summary lists linking questions both to process skills and to content areas.

In the remainder of the introductory pages there are descriptions of:

- problem solving in the GCSE in Additional Mathematics;
- general strategies for problem solving in mathematics;
- five strategies that are helpful in solving the kinds of problems that are set in the examination papers for the GCSE in Additional Mathematics;
- teaching using the resource;
- a progression in student problem solving;
- the information given in the resource about all the problems;
- the elements of the commentaries on 30 of the questions.

2 Specification at a Glance
Additional Mathematics

- This is one of two pilot mathematics specifications offered by AQA. This specification is a linear specification leading to a GCSE in Additional Mathematics.
- There are two tiers of assessment, Foundation (C – G) and Higher (A* – D).

GCSE in Additional Mathematics Pilot (9306)	
Foundation Tier	
Written Paper (Calculator allowed)	100% of the total assessment 2 hours
Higher Tier	
Written Paper (Calculator allowed)	100% of the total assessment 2 hours

3 Assessment Issues

3.1 Problem solving in the GCSE in Additional Mathematics

Despite its title, the Pilot GCSE in Additional Mathematics does not contain any mathematics that is additional to that specified in the National Curriculum for Key Stage 4. The difference between GCSE Mathematics and GCSE Additional Mathematics is one of emphasis. This is enough to give the examination a different character, with a commensurate need for different preparatory teaching, but without requiring the teaching of new mathematics content.

Specifically, the GCSE in Additional Mathematics has proportionately more questions that call on mathematical processes that are not a matter of procedural or factual knowledge. There is:

- more reasoning;
- more justification of reasoning in explanations;
- more representation of a situation algebraically;
- more manipulation of algebra in previously unrehearsed ways;
- more visualisation;
- more problem solving – with more 'unstructured' questions, lacking the step-by-step build up to a solution that is found in many current GCSE questions.

Each of these kinds of mathematical process has been explicit in all versions of the Mathematics National Curriculum since 1989, and has always been taught as part of preparation for GCSE Mathematics. Nevertheless, the greater emphasis on these processes in the GCSE in Additional Mathematics means there is a need to have a more direct focus on each of them in preparatory teaching.

Mostly this will just be a matter of a little more practice, changing the balance in teaching to reflect the different balance in the examination – more practice in reasoning, in representing and manipulating, in visualising, in explaining and justifying – and this should be sufficient.

For problem solving, however, there is a likely need for focused teaching of strategies. Faced with unstructured problems without easy lead-in steps, many students do not know how to begin to find solutions.

The main aim of this resource is to offer the means to equip students with strategies to use on the problem solving questions on the examination paper.

- It is a source of problem solving questions similar to those in the Pilot GCSE in Additional Mathematics.
- It also describes how the problem solving strategies that are required for these kinds of questions might be taught.

4 Teaching Strategies

4.1 General strategies for problem solving in mathematics

In developing problem solving in mathematics throughout school, including in GCSE classes, teaching is likely to focus on general strategies that could be useful for any question, such as:

- thinking of the properties that the answer will have;
- formulating and testing hypotheses;
- eliminating options (paths or outcomes);
- representing the cases, relationships, features or examples symbolically, algebraically or diagrammatically.

There is also a 'meta' strategy of reviewing progress and going back and trying again when the chosen strategy is not working. Within this, 'trying again' has a number of different forms:

- trying the strategy again, only more carefully, presuming that the action has the capacity to succeed but was not being done well enough;
- trying the strategy again, but on a different basis, following an amended perception of the mathematics in the problem;
- trying a different strategy altogether.

Teaching of these general strategies has always been part of mathematics classes, including GCSE classes, and remains important.

However, the problem solving element of the GCSE in Additional Mathematics is a particular context for problem solving, in which the problems are constrained by the fact that they are in a timed examination subject to a mark scheme. A range of strategies can be identified that help with the kinds of problems that are set under such conditions, and which will be in the Pilot GCSE. Therefore, to prepare for the GCSE in Additional Mathematics, as well as a continuation of the teaching of general strategies, the classes can also be directed at the strategies that would be helpful for tackling the kinds of questions found in the examination. These are the five strategies focused on in this resource.

4.2 The five strategies of the resource

A student facing a new problem is initially likely to examine it to see if it is like a problem that they have done before - and if it is, will try to use the approach that had worked on the previous occasion. If the problem is not obviously like one they have done before, then they could consider the features of the problem to decide what might be done that could be helpful. They may ask themselves questions about the problem, such as: Are there examples or 'cases' that it might be helpful to set out systematically? Is there a procedural relationship between some elements that could be reversed to find others? Are there criteria to apply to possible solutions? Are there features with relationships between them that could be drawn out and expressed in some way (eg, linguistically, diagrammatically, symbolically and algebraically)? Is there some recognisable mathematics that could be followed through, extended or applied?

Progress on the problems in the GCSE Additional Mathematics are likely to be made through one or another of those possible approaches. As a result they represent the five strategies with particular relevance to the examination:

1. To set out cases systematically, and identify how many there are of relevant types.
2. To work backwards from a value given in the problem,
   (a) where the inverse is familiar, so just has to be applied but may have to be sustained over a number of steps.
   (b) where the inverse is unfamiliar, so has to be worked out 'from first principles'.
3. To find one or more examples that fit a condition for the answer, and see whether those examples fit with the other conditions in the situation, making adjustments until they do.
4. To look for and represent relationships between elements of the situation, and then act on them to see if any are useful.
5. To find features of the situation that can be acted on mathematically, and see where using them takes you (operating incrementally, yet speculatively).

For the purposes of this resource, these five possibilities for action, which can be developed as strategies for the individual, will be labelled as follows:

1. Set out cases.
2. Work back familiar; work back unfamiliar.
3. Find an example to fit.
4. Find key relationships.
5. Find mathematical features.

However, it should be noted that these labels are for the convenience of reference in the resource, and should not be used in the classroom context. To use labels like this in class is to invite students to misunderstand these possibilities for action as distinct sequences of activity that can be learned and applied as procedures. Problem solving is ***not*** a matter of identifying a problem as a 'type 3' and solving it by applying the 'find an example to fit' routine.

Even though the problems outlined in this resource are classified (section 9) in terms of which of these fives strategies the problem is most susceptible to, this is not the same as identifying problem types, since there are almost always different ways of tackling any problem. The purpose of the classification is to enable the resource to be used in a systematic way. The collection of five identified strategies are, between them, effective approaches to the kind of problem found in the GCSE in Additional Mathematics, and by associating each problem with a strategy, the resource offers a structure for rehearsing and developing strategies that will enable success in the exam.

* This resource has been developed specifically in the context of the questions written for the GCSE Additional Mathematics, and is not intended as a structure with applicability to all problem solving.

4.3 Teaching using the resource

It has long been recognised that problem solving strategies – whether the general strategies outlined previously, or the five of this resource – cannot be taught directly, that is by describing them and rehearsing them as procedures. Rather the teaching is indirect, by drawing attention to strategies as possibilities for action in the context of solving problems. This should be remembered in what follows: references to "teaching" should not be taken to imply "telling".

The resource contains a set of problems which offer opportunities for students to develop and use the five processes listed above – processes that should, if strategically deployed by them, improve their performance on the GCSE in Additional Mathematics. A classification is used in which process is matched to problem (section 9) but of course, being problems, there are always other ways to do any of them, and, being strategies, there are no guarantees that they will lead to a solution.

By working through the problems in an appropriate way, students will develop strategies and also have experience of problems that have been solved using them. Using the resource should therefore both enable the possibility of recognition of similarity between a new problem and an old one, and develop confidence in a set of actions that can lead to problem resolution.

Of course the resource could be seen and used as just a set of example questions for practice in problem solving, but it also offers the opportunity to use the examples in a more structured way. In teaching using the resource, therefore, even though answering the question is likely to be the first focus for the student, the main focus for teaching should be on the strategies – at least at first.

Organisation of the teaching

Experience with the problems in the resource can be organised in at least two different, and contrasting, ways.

(i) Lessons focusing on problem-solving
Problems are used that are all susceptible to the same approach, and then in later lessons, another set that suit a second approach, and so on.

In this approach each of the five strategies could be focused on in turn during a sequence of dedicated lessons, using an appropriate selection of problems for each strategy (following the mapping in Section 9). However, this does not mean labelling the lessons by the related strategy, because of the dangers associated with doing so. The context of students working on the identified particular problems increases the opportunity for that approach to be drawn attention to as a possibility for action – that is all. Of course, using problems that are classified against a particular strategy does not guarantee that that approach would work. Nor does it ensure that any particular student will recognise the value of the strategy that their attention is drawn to. However, the subtlety involved in learning to solve problems is such that an indirect approach is necessary, since the more direct approach of describing strategies and getting students to rehearse them has been shown to be ineffective.

(ii) Problem solving within lessons.
Problems are used that fit in with the content area, as part of teaching mathematics topics.

In this approach it will be important that the mathematics that is the subject of the teaching is 'used and applied' in the problems, and so the mapping in Section 10 will be a guide to which problems to use when. Since problems can often be approached in different ways, and this sometimes means using different mathematics, care would have to be taken to 'shape' the approach of the students so that the desired mathematics is being used. Whatever is done with the mathematics, however, the main focus of the teaching should still be on the problem solving strategies, and the progression in teaching outlined below should still be applied.

The relevance of mathematics learning

The main focus of the teaching using this resource is the development of problem-solving strategies. However, it should be recognised that working with problems in a social context, in which different ideas that students have are shared across the group, helps to develop awareness of inter-connectedness in mathematics, the knowledge of what is like what, and what fits with what, and what often goes with what, and this makes an important contribution to students' capacity to do problems.

4.4 A progression in student problem solving

First stage

Developing the 'strategies' as possible approaches

In the early phase of the work, as each problem is introduced to the students, and they make their first attempts at engaging with the challenge and devising an approach, the teacher draws attention to possibilities for action by making suggestions, by pointing out a feature of the problem, or by asking eliciting questions that bring the student to that awareness. The classification in section 9 is a guide to which approach each problem is most susceptible to, which indicates which possibilities the teacher might highlight. Through these means the students' range of possibilities for action – what they consider when looking at a new problem – is expanded to include those that correspond to the five strategies. They will have established that sometimes it is helpful to set out cases, sometimes to work back, sometimes to find examples that fit conditions and so on – but without thinking of them as techniques or identifying them through labels

Second stage

Developing awareness of the approaches as strategies

In the second stage, teaching as a new problem is introduced should begin with a discussion with students about what might be a helpful thing to do on that problem. The mapping in section 9 and / or the commentary on the question (Section 6) might be used as a guide about what would be a good approach. Over a series of problems it will be helpful for the teacher to ensure that the actions related to all five strategies are considered as possibilities, so that they remain in the students' minds as options.

Third Stage

Operating strategically

The last stage of the sequence of development of problem solving using these materials involves students deciding independently what to do - in effect practising problem solving using examples from the resource. However it should also be accepted that for some students there may be a need to remind them of possibilities as they consider each problem. At this stage the measure of success is not what they have learned or are aware of, but solving the problem in a reasonable length of time.

5 The CD-Rom of Questions

5.1 Introduction

The enclosed CD-Rom contains 90 problem-solving questions in PDF format. These are organised alphabetically. The CD also contains a PDF copy of this document.

All files on the CD may be copied for use in centres for the intended purpose only and must not be reproduced for any other reason, including commercially.

6 Commentaries

6.1 Introduction

For thirty of the problems in the resource a written commentary is given on the following pages. This provides detailed information about the problem, and how it can be approached. This includes a summary listing of the problem solving category, content area, tier and answer. The other elements of the commentaries, and what each is for, are as follows:

About the question

This section gives an overall perspective on the question - how it might be understood in mathematical terms.

Problem solving approaches

This section outlines how students might understand and then 'attack' the problem in relation to the strategy it is most susceptible to, including the questions they might ask of themselves, the possible triggers and prompts, etc.

Challenges / issues

This section mentions points of particular demand in the question which might trip students up, or common wrong approaches.

Finding an answer

This section sets out the mathematics that students might follow in moving towards a solution to the problem. Where these (or parts of them) are distinct, they are numbered, to aid clarity and ease of reference. Where there are too many of these to list (eg, variants on a theme), only the most likely alternatives are specified, with the others described generally.

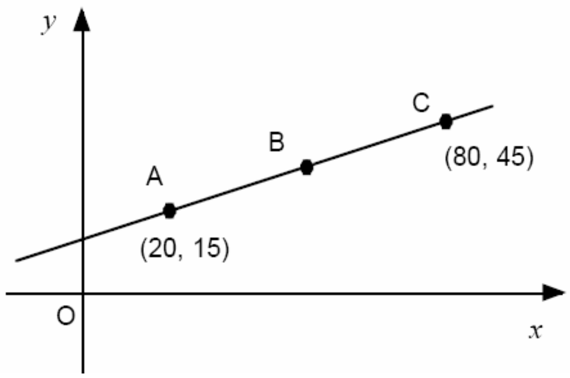
Follow up

In some questions, where appropriate, there is a description of a suitable further activity or two that represents further problem solving, a little different from that just done. This is not given for all problems, and in the other cases the heading is omitted.

This is not to say, however, that follow up should not be done. Going on to solve some similar questions as consolidation of the new thinking can be merited on any question, as it raises self-awareness about the strategy and helps the feelings of confidence about problem solving. However, repeating a similar problem is no longer really 'problem solving', and repeated 'practice' of similar problems to establish a kind of procedure for each 'type' is counter productive, since there are not really problem types, and problem solving requires flexible rather than procedural thinking. For these reasons, how much to use similar problems as 'follow up' should be a decision that arises from the needs of the students in the group.

6.2 Abacus

The three points, A, B and C on this graph are equally spaced



What are the coordinates of point B?

_ _ _ _ _ _ _ _ _

Problem solving classification	*Find key relationships*
Content area classification	*Co-ordinates*
Tier	*Either*
Answer	*(50, 30)*

About the question

For this problem students need to get beyond seeing co-ordinates as simply a way of identifying location or as a way of representing linear equations graphically, and see them geometrically or numerically.

Problem-solving approaches

Essentially three pieces of information are given in this question - the co-ordinates of two of the points and the fact that all three points are equally spaced on the line. The solution comes from recognising the interrelationship between these pieces of information.

Challenges / issues

This could be a difficult question for students who are not used to using co-ordinates as a context for geometry or number. The main challenge is, perhaps, to see co-ordinates as measures (ie, vectors rather than scalars, though this language may not be helpful to students). Awareness of the x co-ordinate as a measure of how far the point is in the x direction from the y axis allows the problem to be seen either in geometric or numeric terms.

Finding the answer

Two approaches to this problem are the geometric and the numerical.

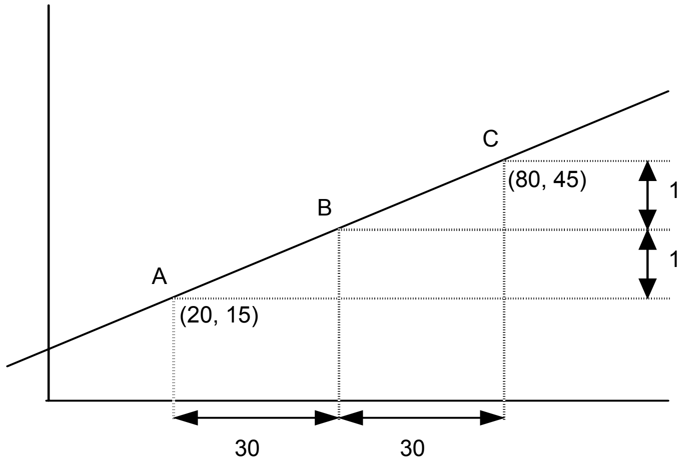
1. Geometrically, the problem can be seen separately in two dimensions.

Horizontally, A and C are 60 units apart and B is half way between the two. Vertically, A and C are 30 units apart and again B is half way between them. In this way it can be seen that the co-ordinates of B are:

(20 + 30, 15 + 15) = (50, 30)

2. Numerically, B can be seen as an 'average' of A and C ie, a representation of the middle.

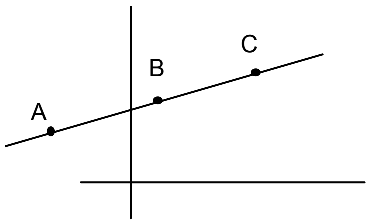
In this way of thinking, B's x co-ordinate is the mean (average) of the x co-ordinates of A and C, ie, (20 + 80) ÷ 2 = 50.
The y co-ordinate can be similarly calculated.

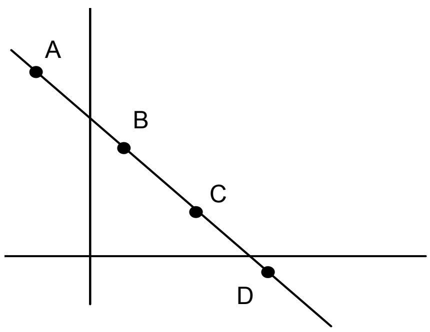


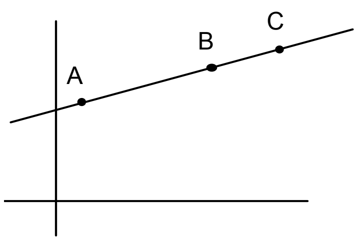
Follow up

Questions that require similar lines of thinking, but still retain a problem solving challenge could be ones where A, B and C are not all in the same quadrant, or where there are four equally spaced points instead of three.

Yet further challenge would come from situations where B is not the midpoint between A and C, but divides AC in a ratio, say 2 : 1

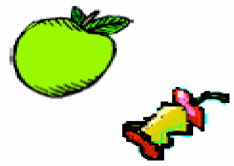






6.3 Apple Crumble

Lottie has a bag of apples.



She gives half of them to Fred.

Fred eats two and then has four left.

How many apples did Lottie have at the start? ------------------

Problem solving classification	*Work back – familiar*
Content area classification	*Number in correct context*
Tier	*Foundation (easiest)*
Answer	*12*

About the question

For many students, this is an 'obvious' question, which does not require any working out at all – they would 'just know'. For students who are problem solving at this level, however, it will be a challenge to create and retain a mental model of the situation that would enable a solution to be found, so will have to reason it out step by step.

Problem-solving approaches

The reverse operation of subtraction (from some being eaten) is adding, and the reverse of halving is doubling. Working back from the final situation to the starting point may need to be suggested, and possibly supported at each step.

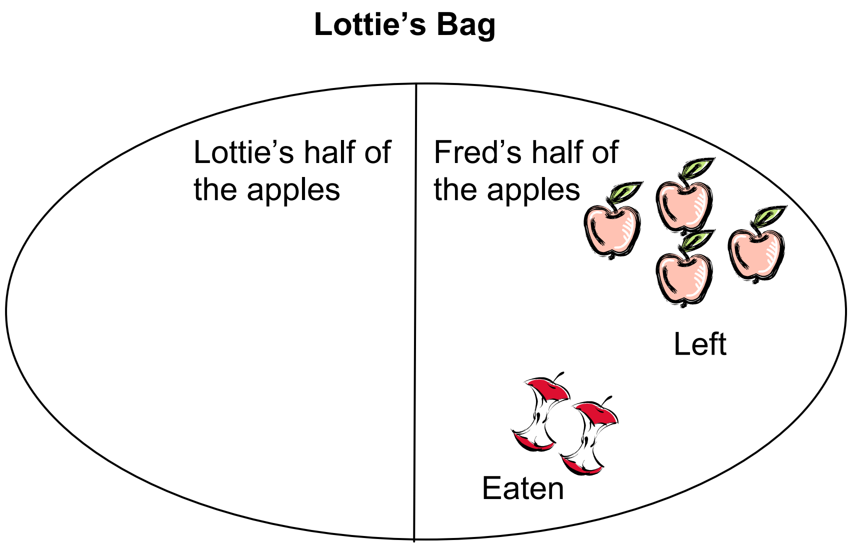
Challenges / issues

Some students may just try a starting number and see whether it 'works', but those who are problem solving at this level may not know how to strategically improve on their first attempt thereby making this method inefficient (at best). A more structured approach should be encouraged.

Finding the answer

1. Fred finished with four apples, and had eaten two, so working backwards he must have initially been given six. Therefore, Lottie must have had twelve (double six) apples to begin with.

2. An alternative is to 'model' the situation using labelled representations of the quantities, for example:



and work it out from there

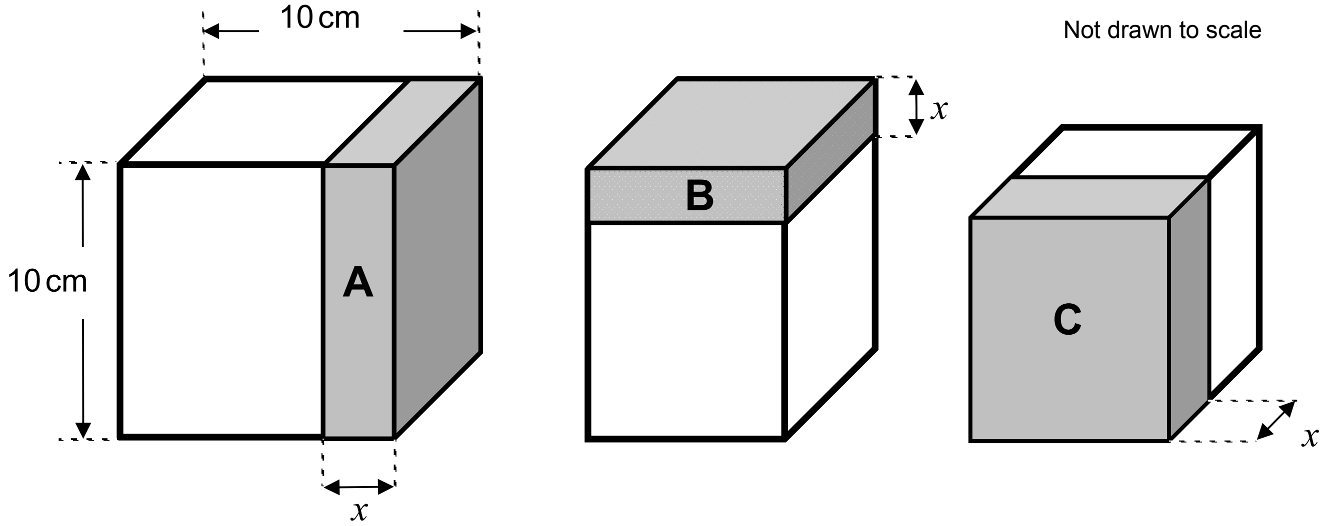
Follow up

Variants of the question which become increasingly (but gradually) more complicated are readily invented.

6.4 Boxclever

A cube has edges of 10 cm each.

Three slices each of thickness x cm, are cut off the cube.



Slice A is cut off the side, slice B is cut off the top and slice C is cut off the front.

What is the volume of each slice in terms of x ?

Problem solving classification	*Find key relationships*
Content area classification	*Volume and surface area; Basic algebra*
Tier	*Higher*
Answers	$A = 100x$ $B = 10x(10 - x) = 100x - 10x^2$ $C = x(10 - x)(10 - x) = 100x - 20x^2 + x^3$

About the question

This is an accessible geometric question - cutting slices off a cube. The complexity, which may not be initially apparent, is that each slice cut off affects the next slice to be cut. The expression of the volumes of these slices leads into algebra, which adds to the demand of the question, though the algebra is not where the problem solving lies.

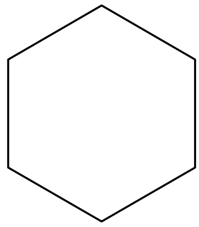
Problem-solving approaches

The key relationship in this question is the way that cutting a slice off the cube changes the dimensions of the cube. This can be achieved by a careful mental reconstruction of the process described in the question and noticing what features are changing as this happens. Out of this comes an awareness of how the next slice can be dealt with.

Challenges / issues	The challenge in this question is to recognise how cutting a slice off the cube affects the dimensions of the cube. Some students may not notice the clues in the changing size and shape in the diagram and instead imagine that, somehow, the second and third slicing are simple repetitions of the first. Careful thought and visualisation are needed - slicing a plasticine cube, or drawing lines on a wooden block may help. Once the problem is 'solved' in terms of understanding what is happening and being able to operationalise this, there may be a problem for some students in simplifying algebraic expressions.
Finding the answer	An important general strategic approach that should be employed for questions like this one (with relatively complex diagrams) is to add the given information to the diagram so that it becomes clearly labelled. Once this has been done the first step towards a solution is relatively unproblematic. The face of the cube is 10cm × 10cm and a thickness x cm is cut making a volume of $10 \times 10 \times x$ cm^3 (= $100x$ cm^3). This process has, however, changed the cube into a cuboid 10 by 10 by $(10 - x)$, so that the next slice measures 10 by $(10 - x)$ by x. This further alters the shape into a new cuboid 10 by $(10 - x)$ by $(10 - x)$, so the third slice measures x by $(10 - x)$ by $(10 - x)$.
Follow up	An extension for the students with the most confidence in algebra is to ask for the volume of the part of the cube that remains after the three slices has been removed. One approach to this is to subtract from 1000 the expressions for the three slices that have been removed. However, the remaining cube can also be seen as $(10 - x)^3$ The first approach is likely to have given a different expression for the answer. Can they show that the two are equivalent?

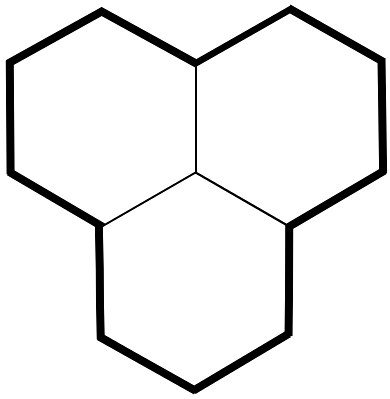
6.5 Bugeye

This hexagon has a perimeter of 24 cm



Not drawn to scale

Three of the hexagons are used to make this shape.



What is the perimeter of the shape? _ _ _ _ _ _ _ _ _ _ _ _

Problem solving classification	*Find key relationships*
Content area classification	*Perimeter*
Tier	*Foundation (easiest)*
Answer	*48 cm*

About the question

In terms of its content, this problem requires the student to know about perimeter and what it means and to have some understanding of regular polygons. What is unusual is that none of the lengths of the sides are given.

Problem-solving approaches

The important relationship in this question is between the lone hexagon and the larger shape. This relationship may be seen in more than one way - either that the larger shape is made up of three of the hexagons but with common sides, or that the length of each side is the same in both the lager shape and the lone hexagon. Asking students what information they require to answer the question, and what relationships they can see in the problem to help them deduce this may help to get them started.

Challenges / issues

Where students are likely to go wrong is in not fully understanding the relationships between the elements in the diagram, possibly because they believe the question is easier than it really is. This may lead some students to simply multiply 24 cm by 3, since there are three hexagons, not taking into account that some of the original sides no longer form part of the perimeter of the new shape. Other students operating at this level may give up on the problem, mistakenly believing that not enough information is given because they do not know any of the side lengths.

Finding the answer

1. Dividing 24cm by 6 to find the length of one side as 4cm, and multiplying the number of sides in the new shape (12) by 4. For less able students it may be helpful for them to write or mark the diagram with useful information - for example writing 4 next to each side, and putting a line through each side as they count it to help keep track of the 12 sides.
2. Multiplying 24 cm by 3 to get 72 cm, and subtracting the lengths of the sides that are now in the 'middle' of the new shape, and so are not part of the new perimeter. This can be done by thinking that a third of the perimeter of each shape is now inside the new shape, or by working out that 6 sides (each of 4 cm) are now inside the new shape.

Some students taking this approach may see the new shape as having 3 sides 'inside' it, rather than 6, and give the (common wrong) answer of 60 cm.

6.6 Charterly

Here is part of a number chart.

Row			
1	6	8	10
2	12	14	16
3	18	20	22
4	24	26	28
5	30	32	34
6	36	38	40

The chart continues

(a) What number comes at the start of row 50?

(b) What is the number of the row that starts with 666?

(c) What is the number of the row that contains the number 248?

Problem solving classification	*Find key relationships*
Content area classification	*Sequences*
Tier	*Either*
Answer	*(a) 300, (b) 111, (c) 41*

About the question

Number grids should be relatively familiar to the students, though not necessarily as a recent part of their school mathematics. This particular example is a little out of the ordinary in its structure, and might require some exploration by students to enable them to see clearly how the grid 'works'.

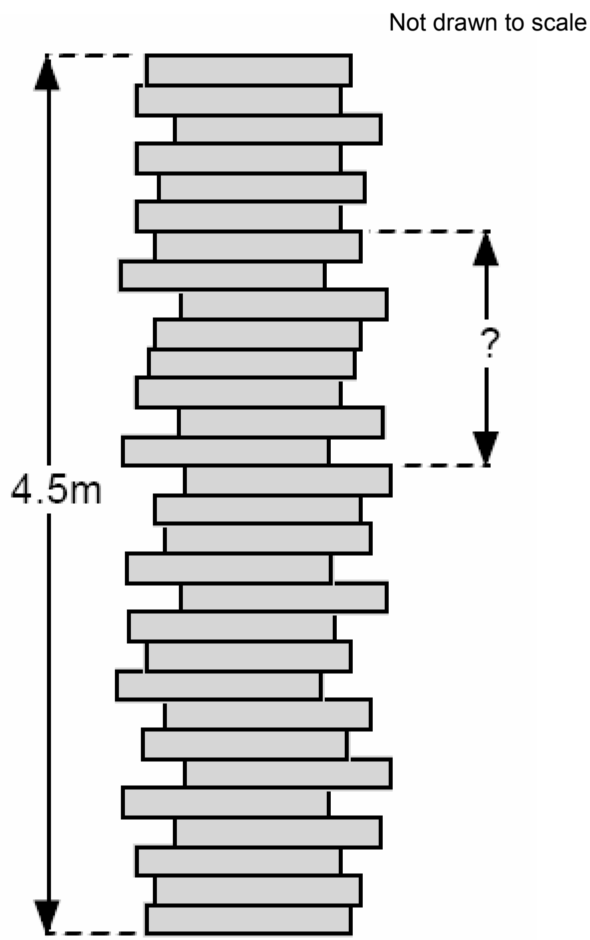
Problem-solving approaches	When presented with such a number grid, students need to be encouraged to make sense of the pattern(s) within the grid. The horizontal and vertical relationships between adjacent cells are the essence of the problem and could be explored to gain further insights into the mathematics in the table and to discover which aspects are the most useful. The problem is most easily understood through the multiples of six in the left-hand column.
Challenges / issues	For many students, it is obvious on being presented with the item that there is a pattern to it, but articulating this explicitly in mathematical terms may be demanding for some. A further difficulty lies in having sufficient confidence to be sure that the pattern thereby identified continues and allows the contents of rows not shown to be calculated.
Finding the answer	Students who can 'spot' that the first column is simply the six times table have essentially cracked the problem, provided that they are then able to use this knowledge to tackle the particular questions asked. Whilst this is the key relationship in the problem, there are other related patterns that can be used but none are as convenient and straightforward. For example, the numbers increase by 6 when moving vertically down a row, but this recognition is not alone sufficiently powerful to enable the solutions to be quickly worked out. Parts (a) and (b) follow straightforwardly based on multiples of 6, but part (c) is more difficult in that the given number (248) is not a multiple of 6. Students therefore need to realise that this number does not come at the beginning of a row and the solution requires them to find the nearest multiple of 6 just below 248. This will lead to the right answer for the row number but does require persistence across multiple steps for complete success.
Follow up	It is not difficult to make up grids of a similar nature to be used to consolidate the skills gained in tackling the original problem. Additional demands could be incorporated by including, for example, fractions, decimals or negative numbers.

6.7 Club Sandwich

A tower of 30 identical wooden blocks is 4.5 metres tall.

What is the distance from the top of the 16th block to the top of the 24th block?

_ _ _ _ _ _ _ _ _



Problem solving classification	*Find key relationships*
Content area classification	*Number in context; Ratio and proportion*
Tier	*Either*
Answer	*120 cm (or 1.2 m)*

About the question

This is a question about how a part or parts relate to the whole. The specified distance needs to be seen as a combination of eight blocks (from counting, or from interpreting the text) and then the height of the whole 30 blocks dealt with in an appropriate way.

Problem-solving approaches

The key relationship is between the part and the whole, but this can be thought of in different ways, either as combinations of ones, as a composite of factors, or in terms of proportion. Different students are likely to answer "How does this part relate to the whole?" in different ways, and each should be encouraged to develop how they 'see' it towards a solution.

Challenges / issues

Some ways of seeing the situation are easier to work with than others. For example, too close a focus on the position of the block to be found (noticing that there are 16 below and six above) might make finding the relationship (between 8 and 30) more awkward – but it is still feasible.

Another difficulty might be presented by units and conversion between them. Working in fractions of a metre rather than centimetres is relatively difficult, but ease in converting between them is not universal.

Finding the answer

An important step in all of the following approaches is to recognise (or count) that there are eight block-heights from the top of the 16th to the top of the 24th.

1. To find the height of one block, divide 4.5 m (or 450 cm) by 30 (15 cm). Then multiply this value by 8 to obtain the desired distance.

2. A more sophisticated approach is to find 8/30 of 4.5

3. There is also a method based on splitting the 8 up into a 5 and 3 (often arrived at after breaking up the whole tower into tens).

Since 5 is a sixth of 30, and 3 as a tenth of 30, the height is 1/6 of 450 (75) plus 1/10 of 450 (45).

6.8 Double Trouble

Use all the digits

0 1 5 0 1 5 0

To complete this multiplication

□□□ x 2 = □□□□

Problem solving classification	*Find an example to fit*
Content area classification	*Basic number*
Tier	*Foundation*
Answer	$550 \times 2 = 1100$ ***or*** $505 \times 2 = 1010$ ***or*** $055 \times 2 = 0110$

About the question

This problem has the appearance of a number puzzle. The mathematics at its heart is related to place value, and usually the solution is found only when that mathematics has been located.

Problem-solving approaches

At this level of ability, students may find it difficult to articulate a strategy. There may be only one thing to do: have a go and see what happens. In other words, try an example and see if it 'fits'. This is fair enough and, assuming the first attempt does not give a correct answer, it is at this stage that analysis is needed: why did the chosen arrangement not work? Features of the situation can then be identified and responded to. These are most likely to appear once a few unsuccessful attempts have been made.

Challenges / issues

This problem could be frustrating for someone with whose only strategy is pure trialling of combinations of digits, since this may take a long time to produce a solution that works. However, trials are probably necessary before the appropriate understandings that lead to a solution are arrived at. The challenge in this question is initially one of perseverance, therefore, and this may discourage some who feel that questions that cannot be done quickly cannot be done at all. Students may need help to see that further attempts can give clues.

Finding the answer

A few trials of numbers should show some significant, related features of the problem:

- a 3-digit number is being doubled to make a 4-digit number;
- the 3-digit number must be greater than 500;
- given the digits available, the first digit of the smaller number must be '5' and that of the larger number must be '1'

This will then allow some better structured attempts that should soon lead to a solution.

Follow up

One area of follow up is to find all the solutions. A discussion can be had over the acceptability of $055 \times 2 = 0110$

Similar problems that can be explored are ones such as

Use all the digits 0 0 1 2 2 5 5

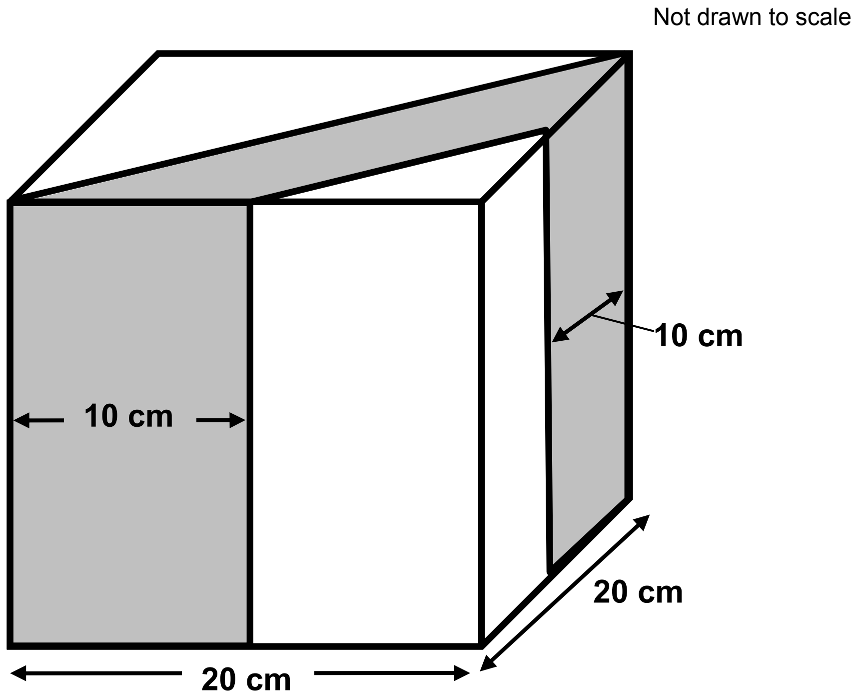
to complete:

$$\square\square\square \times 5 = \square\square\square\square$$

Again, the number of possible answers can be explored.

6.9 Ex-cube-me

A cube is cut into three parts by two vertical slices.



Find the area of the shaded part.

Problem solving classification	*Find mathematical features*
Content area classification	*Volume and surface area*
Tier	*Higher*
Answer	*3000 cm*^{3}

About the question

The question is concerned with areas and volumes. Students often find such problems difficult since they are not well-practiced in finding volumes of unusually-shaped objects. In addition, some of the lengths needed in the calculations are not given explicitly.

Problem-solving approaches

To make progress on such a problem, students must be encouraged to engage with the structure of the cube and its constituent parts. They must be able to locate aspects of their mathematical knowledge that can be usefully brought to bear in solving it. To this end, students might be encouraged to ask themselves questions such as: "What is the volume of the cube?", "Is it necessary to calculate any lengths?", and "What is the nature of the shaded shape?"

Challenges / issues

Students often find volume and area problems such as this one difficult because of insufficient capability in 3D visualisation. For example, the slice along the diagonal of the cube cuts the cube into two equal portions, and a potential error is to assume that the second slice cuts the half portion of the cube into half again giving an answer of $2000\,\text{cm}^3$. Not being able to 'see' precisely the nature of the shaded object may make this problem very difficult for a number of students.

Finding the answer

1. Only the most confident students are likely to immediately calculate the shaded strip on the top face of the cube, most likely as the difference in the area of two triangles, giving

$$\frac{1}{2} \times 20 \times 20 - \frac{1}{2} \times 10 \times 10 = 150\,\text{cm}^2$$

The shaded shape is then a right prism and its volume is its cross-sectional area times its height ($150 \times 20 = 3000\,\text{cm}^3$).

2. An alternative approach is to calculate the volume of the cube, $8000\,\text{cm}^3$, take a half of it, $4000\,\text{cm}^3$. This then is the volume of the triangular faced prism containing the piece whose volume is sought.

The other slice to be removed to give the shaded shape is also triangular prism, with a cross-sectional area $\frac{1}{2} \times 10 \times 10 = 50\,\text{cm}^2$. Thus the volume of the triangular prism to be removed has volume $50 \times 20 = 1000\,\text{cm}^3$. The desired volume is then $4000 - 1000 = 3000\,\text{cm}^3$.

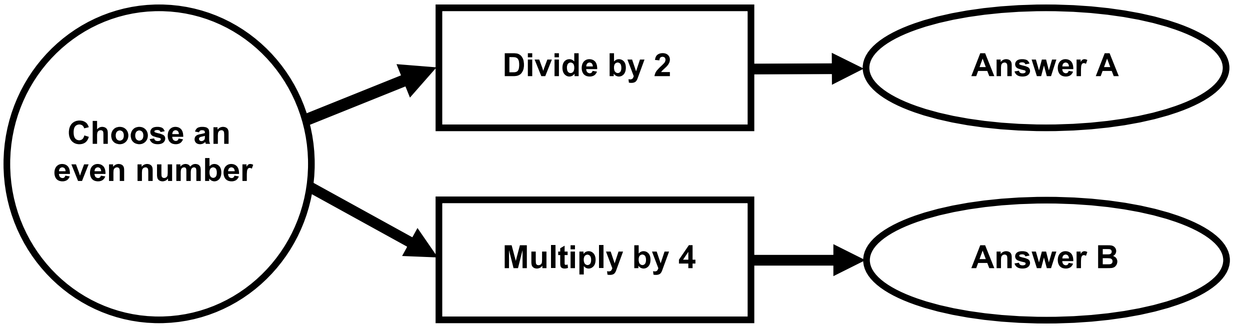
3. A third method is to argue that the shaded strip on the upper face of the cube is three times the area of the small triangle on the same face. Hence the volume of the shaded object is three times the volume of the triangular prism. This volume is $\frac{1}{2} \times 10 \times 10 \times 20$ and multiplication by three gives the answer.

Follow up

Extensions to other cuboids can be constructed. Alternatively, slicing in perpendicular directions provides challenging problems, though the corresponding diagrams can be difficult to draw clearly.

6.10 Explain 7

Here is a flow chart.



Explain why (B – A) is always a multiple of 7

Problem solving classification	*Find mathematical features*
Content area classification	*Basic algebra; Symbols and formulae*
Tier	*Either*
Answer	*An explanation that suggests that the difference between half a number and four times the number is three-and-a half times the number. Since the original number is even, 3.5 times the number will be a multiple of (3.5 × 2) = 7* ***or*** *Algebraically, (B – A) = 4n – 0.5n = 3.5n and n is even, so it is already a multiple of 2; thus (B – A) is a multiple of (2 × 3.5) = 7*

About the question

Initially the problem offers a mystery – '7' does not obviously result from division or multiplication by 2 or 4. Although the question lends itself to an explicit algebraic analysis ("let us call the number *x*..."), it can be tackled in a less formal way for those students who struggle with explicit algebra.

Problem-solving approaches

Probably the best initial approach to this question for most students is to try numbers in the flow chart and see what happens. From this it should be possible to discern more general features of the mathematics, which can be explored further, perhaps by further trialling.

Challenges / issues

The challenge for most students will be to generalise. It will become clear that putting an even number into the flow chart will always generate (B – A) as a multiple of 7; but no matter how many specific examples are tried, this will not show that it is *always* the case. The step needed is to start thinking about what happens to 'a number' (unspecified) that could be called '*n*' or it could be left as 'a number'.

Finding the answer

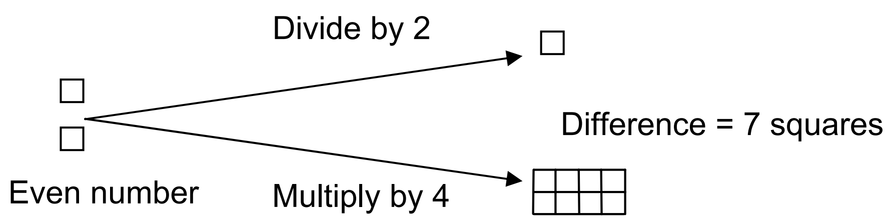
1. The most efficient solution comes from using algebra. The use of a symbol to represent a number will help considerable in expressing a general result.

An even number can be represented by $2n$ (since evenness implies a multiple of 2). Dividing by 2 and multiplying by 4 gives n and $8n$ respectively. Their difference is $7n$ which is by definition, a multiple of 7.

Students who do not see that $2n$ is an even number, can of course begin with a single letter to represent the starting number and deal with the evenness at a later stage.

2. For those students whose formal algebra is not so strong, method 1 can be carried out in a less formal, pre-algebraic way. So, an even number is twice a particular number and then four times the even number is eight times the particular number and their difference must be seven times that number, which is a multiple of seven.

3. The processes can also be represented pictorially:



Follow up

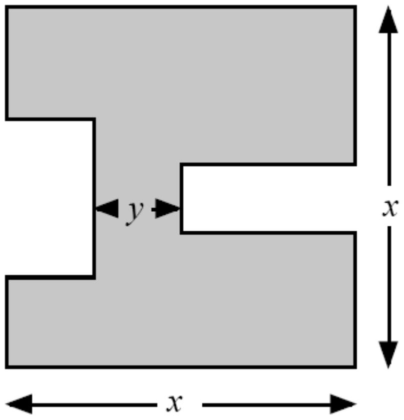
Similar questions based upon starting with an even number, then dividing by 2 and multiplying by, say, 5 and explaining why the difference between the two answers is always a multiple of 9 (or why the sum is a multiple of 11) can be readily constructed.

Another would be to start with a multiple of 3, then divide by 3 and multiply by 3 and explain why the difference between the answers is always a multiple of 8.

It should be noted however, that these are more a consolidation of understanding than opportunities for fresh thinking.

6.11 Eye Test

The diagram shows a square of side length x with two rectangles cut out of it.



Find the perimeter of the shaded shape in terms of x and y.

Problem solving classification	*Find key relationships*
Content area classification	*Perimeter; Basic algebra*
Tier	*Higher*
Answer	$6x - 2y$

About the question

This is an apparently simple situation, but quite a challenging question, requiring a geometrical perception that will evade many and leave them supposing that not enough information is given. Some patience and careful thinking are needed to see that all that is needed for a solution is there in the question.

Problem-solving approaches

The information apparently missing concerns the dimensions of the cut out rectangles. These must have a relationship with the shaded shape and its given dimensions: if not the whole shape, what about parts of the shaded shape? The edges? Can these crucial relationships be expressed in mathematical form?

Challenges / issues

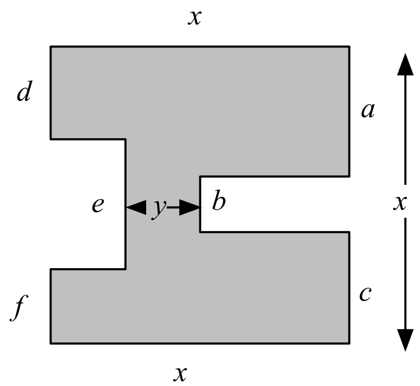
The challenge of this problem is to see a way in, given the minimal amount of information. The diagram is deliberately drawn to discourage any tendency to assume that unknown lengths are simple proportions of any given lengths. It is always possible to make such assumptions as a way of getting started and then realising (it is to be hoped) that they are unnecessary.

Finding the answer

The key understanding needed is to realise that, since the white spaces are rectangular all the edges in the diagram are parallel or perpendicular. Going down the right-hand side of the shape, the sum of the vertical lines must be x (this includes one side of the white rectangle); the same is true going down the left-hand side.

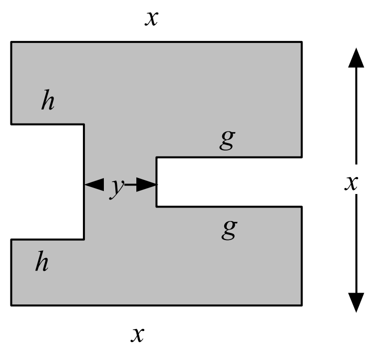
Going horizontally across the shape, the upper edges of the two white rectangles, added to the gap of y must be x. The same is true of the lower edges of the white rectangles and the gap of y.

In this way all the edges that form the perimeter of the shape are accounted for in terms of x and y.



$a + b + c = x$

$d + e + f = x$



$g + h + y = x$, and so
$g + h = x - y$

Hence, the complete perimeter
$= x + a + g + b + g + c + x + f + h + e + h + d \; = 6x - 2y$

6.12 Half Take

Marcus thinks of a number between 25 and 35
He divides the number by 2 and then subtracts 0.5

He takes his answer, divides it by 2 and then subtracts 0.5

He repeats this process a number of times and gets zero.

What number did he start with?

– – – – – – – – – – –

Problem solving classification	*Work back – familiar*
Content area classification	*Fractions and decimals*
Tier	*Either*
Answer	*31*

About the question

The problem has the appearance of more familiar inverse problems: 'I think of a number, do something, this is the answer, what number did I first think of?' As such, it is deceptively accessible - until it is realised that the number of steps in the process is not stated.

Problem-solving approaches

The most constructive approach to a solution is one based on inverting the series of operations. Some may be tempted into a seemingly easier option of trying each number from 25 to 35 in turn but this will mostly lead to a lengthening string of decimals that overshoots the zero target and may well cause many to give up.

Challenges / issues

The challenges for the students may be two-fold. One is recognising that there is a better way of solving the problem than trying numbers until one works. The other is in implementing the better method (inverting) ie, understanding how to invert the operations, and invert the sequence of their application.

Finding the answer

The most direct solution strategy is to repeatedly undo what has been done starting from zero. It may be helpful to represent the mathematics diagrammatically:

number → divide by 2 → subtract 0.5 → next number

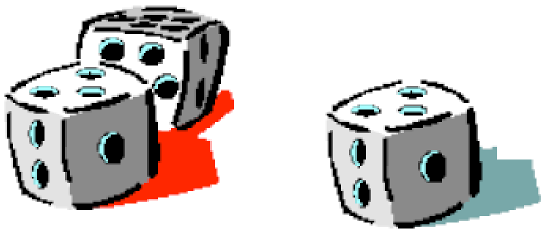
and from that work out the reverse process:

previous number ← $\boxed{?}$ ← $\boxed{?}$ ← number

This should help students construct an inverse sequence from zero, repeatedly adding 0.5 and doubling the answer, giving the sequence 0, 1, 3, 7, 15, 31, 63 etc.

Reference to the question, and the range of Marcus' starting number, gives the answer.

6.13 Highroller



Three dice are numbered 1 to 6

Two of them are red and one is blue.

All three dice are rolled.

What is the probability that the total on the two red dice will equal the score on the blue dice?

_ _ _ _ _ _ _ _ _ _ _

Problem solving classification	*Set out cases*
Content area classification	*Probability*
Tier	*Higher*
Answer	$\frac{5}{72}$

About the question

This problem is about the ability to handle cases carefully – dividing up the sample space in an appropriate way to enable the question to be answered effectively. It appears deceptively simple but without bringing additional structure to the problem it is very difficult to tackle efficiently.

Problem-solving approaches

The students must ask themselves how this problem can be broken down into parts that are amenable to the sorts of techniques that they are familiar with in more straightforward examples of probability. Will tree diagrams help? Is a 3-way sample space do-able? How many actual equally likely outcomes are there? Amongst these, can the desired outcomes simply be counted?

Challenges / issues

The difficulty lies in the added layer of complexity beyond that reached in more standard questions, in that there are three dice, rather than just two. How the techniques that work for two dice can be extended to this more difficult situation is the key to finding the solution.

Finding the answer

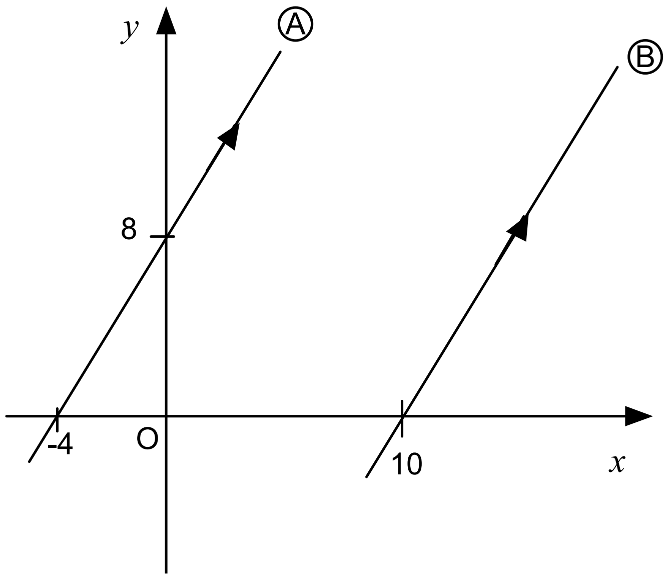
1. This item can be tackled in a very probabilistic way using, for example, a tree diagram starting with the single blue die and this will involve lots of fractional calculations. Such a method, carefully constructed with irrelevant parts of the tree diagram properly omitted, can bring success, though does require great attention to detail to cover all the cases correctly.
2. An alternative, more counting-based approach would be to consider the two red dice as a whole and then to match the scores on this pair to the single score on the blue die. This method has the effect of bringing the complexity of the problem down a level by making it into a problem requiring the calculation of scores on a pair of dice matching the scores on a single die. There is still work to be down in determining the correct sample space for the pair of red dice and then accounting for the matching on the blue but this is within student's experience of more standard problems.

Follow up

A natural extension that might encourage similar problem-solving strategies could include matching scores on two pairs of six-sided dice, or using three die, but not all six sided.

6.14 Javelin B

The lines A and B are parallel



Not drawn to scale

What is the equation of line B?

_ _ _ _ _ _ _ _ _ _ _

Problem solving classification	*Work back – unfamiliar*
Content area classification	*Linear graphs*
Tier	*Higher (hardest)*
Answer	$y = 2x - 20$

About the question

This problem is about equations of straight lines and the important mathematical qualities that are shared between parallel lines, and those properties that are different. Cleary, students need a reasonable understanding of slope, intercept and of how to form an equation from them.

Problem-solving approaches

The problem relies on students being able to construct the equations of straight lines from diagrams of the lines. Thus, the question is unusual in being the reverse of what is normally asked of students – sketching a line given its equation. To tackle the problem students need to be aware of what information determines the equation of a line, and of how they can go about finding such data from that presented on the graph.

Challenges / issues

Students confident with the $y = mx + c$ formulation for the equation of a straight line still have some work to do since m is not given explicitly in the question for either line. In addition to this, the intercept of line A is shown but that for B is not.

Finding the answer

The two pieces of information that determine the equation of line B (slope and intercept) need to be found. The fact that parallel lines have the same slope needs to be understood so that the gradient of line B can be inferred from that of line A. Then the increase in y over increase in x formulation for gradient can be applied to the triangle containing the intercepts for line A. This gives 8/4 = 2 and so $m = 2$ for line B.

The intercept for line B can be found in (at least) two distinct ways:

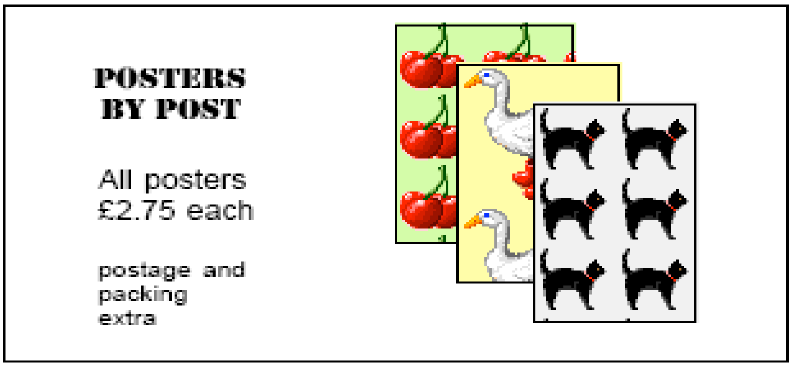
1. Solving an equation for $y = 2x+c$ through the point (10, 0).

2. An alternative method, involving proportionate thinking based on the known gradient (2), would be to note that on line B at (10, 0) moving 10 units left must be equivalent to moving 20 units down so that the intercept of B must be at (0, –20).

Follow up

Other related problems using parallel lines can be constructed, with negative slope an additional complication that could be included.

6.15 Last Poster



Posters cost £2.75 each.

You have to pay postage and packing charges as well.

These are:

postage and packing	
1 to 10 posters	£3.25
11 to 20 posters	£6.00
21 to 30 posters	£8.75
Over 30 posters	£11.50

Zeke has £50 to spend.

How many posters can he get by post if he spends £50?

____

Problem solving classification	*Work back – unfamiliar*
Content area classification	*Number in context*
Tier	*Either*
Answer	*16*

About the question

This question requires students to have a good grasp of number and money. It also involves careful reading of a large amount of information, and correct interpretation of the information in the tables in order to recognise that the postage costs cannot be ignored.

Problem-solving approaches

This question is the reverse of what is usually asked in typical money problems, usually finding the cost given the quantity. Here the cost is given and number of posters has to be calculated. A useful starting point might be to encourage working forward with a given number of posters in order to gain a fuller understanding of the nature of the question.

Challenges / issues

The two challenges here are that the number of posters is not given, and that the postage costs must be accounted for. Further, the postage costs themselves may be misunderstood as the price of the posters, for example, thinking that 11 or 20 posters can be bought for £6.00. Some students ignore the postage altogether since this makes for a far easier, though incorrect, solution.

Finding the answer

1. One way to work through this problem is to estimate how many posters could be bought and work forward through the problem adjusting the number of posters until the correct numbers are found. Students could begin by diving £50 by £2.75 to provide an approximate starting point, or recognise by mental calculation that 10 posters are too few and 20 posters are too many, and working inwards from there.
2. Alternatively, once it is known that 10 posters is too few, and 20 is too many, the postage must be £6, so 44 divided by 2.75 gives the number of posters.
3. The formal version of this is to represent the problem using symbols, for example, with p = number of posters and n = postage and packing costs

   $(2.75)p + n = 50$

   There must be more than 10 but fewer than 20 posters, so n = £6.00

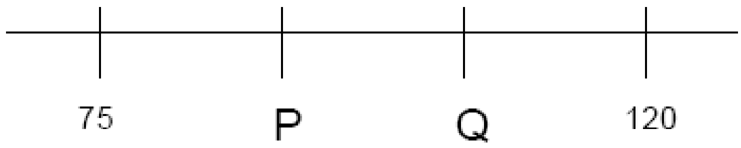
   $(2.75)p + 6 = 50$

   $p = (50-6)/2.75$

   $p = 16$

6.16 Lineup

Four numbers are equally spaced on a number line.



Find the numbers represented by P and Q

P _ _ _ _ _

Q _ _ _ _ _

Problem solving classification	*Find an example to fit*
Content area classification	*Number in context; Sequences*
Tier	*Foundation*
Answer	*P = 90, Q = 105*

About the question

This problem requires students to be able to translate the information that the numbers are 'equally spaced' into more tangible mathematics, thereby allowing P and Q to be found. This is quite a challenge for some students since they must be able to cope with the subtly of the difference between the numbers on the line and the equal spaces between them.

Problem-solving approaches

For students solving problems at this level, the most appropriate approach is to try out values for P and Q and to see if they work or can be made to work. At this level, finding a pair for P and Q that 'works' (through careful adjustment if necessary) is genuine problem solving.

Challenges / issues

A common error might be to think that since there are four numbers, a division by four of the difference between 75 and 120 is necessary.

Finding the answer

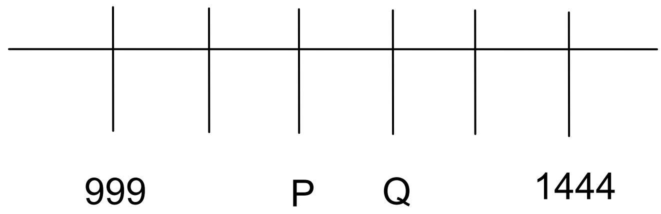
1. Many students will find a way into this problem by exploration – for example, by trying out values for the 'gaps' to see if they work. An obvious starting point for such an approach is 10, giving P and Q as 85 and 95 and the right-end point as (incorrectly) 105. Suitable adjustments can then be made.
2. A constructive approach, which is likely to be employed only by some students, is to calculate the overall difference (45) and to divide this by the number of gaps (3) to give the distance between each of the numbers (15).

Follow up

The second method is worth discussing in class to make the point that it will work for all such problems, even when the numbers make the first, more exploratory approach, too difficult.

An example of a more "difficult" problem, one not particularly amenable to the approach given in method 1 above can be introduced, for example, the following:

Six numbers are equally spaced on a number line.



Find the numbers represented by P and Q

6.17 Meanstreet

Three numbers have a mean of 23

Two of the numbers have a mean of 12

Two of the numbers have a mean of 30

What are the three numbers?

_ _ _ _ _ and _ _ _ _ _ and _ _ _ _ _

Problem solving classification	*Find mathematical features*
Content area classification	*Averages and range; Simultaneous equations*
Tier	*Higher*
Answer	*9, 15, 45 (in any order)*

About the question

Although this question is concerned with mean values, the procedures usually associated with finding the mean (and so on) are not directly useful in solving it – it is more about a broader understanding of means, such as knowing that the total of n numbers with a mean of m is nm.

Problem-solving approaches

This is another of those questions where the best strategy is to consider what can be done with the mathematical features of the situation, to try things and see where it leads. Some students might start with the totals of the numbers, others with lists of possibilities for combinations of the numbers. Some might give labels to the unknowns and create equations, others might create grids and populate them. There is not a 'right way' or a necessary insight here.

Challenges / issues

It is possible to see a way forward, but not be able to see it through. For example, some of the students who feel compelled to create equations might not be able to handle those equations effectively to find solutions. Equally, some of the students who populate the page with possible numbers might not be able to connect them up appropriately to find the answer.

The dilemma in that situation is whether to persevere or to 'change tack' - given that the student is not in a position to know whether the approach that they have adopted has the potential to be successful, or whether they personally have the knowledge and skills to make it work.

Finding the answer

1. Since three numbers have a mean of 23, their sum must be 69. Similarly, the sum of two of the numbers must be 24, and the sum of (a different) pair of the numbers must be 60.

   Taking the first pair of these relationships implies that one of the numbers must be the difference between 69 and 24 ie, 45. Then the last piece of information gives that another of the numbers must be 15, and hence third number must be 9.

2. Method 1 can, of course, be formalised into equations:

   If $x + y + z = 69$, and $x + y = 24$, then z must be 45;
   and as $x + z$ is 60, y is 15,
   so x must be 9

3. There is a method based on working directly with possible means:

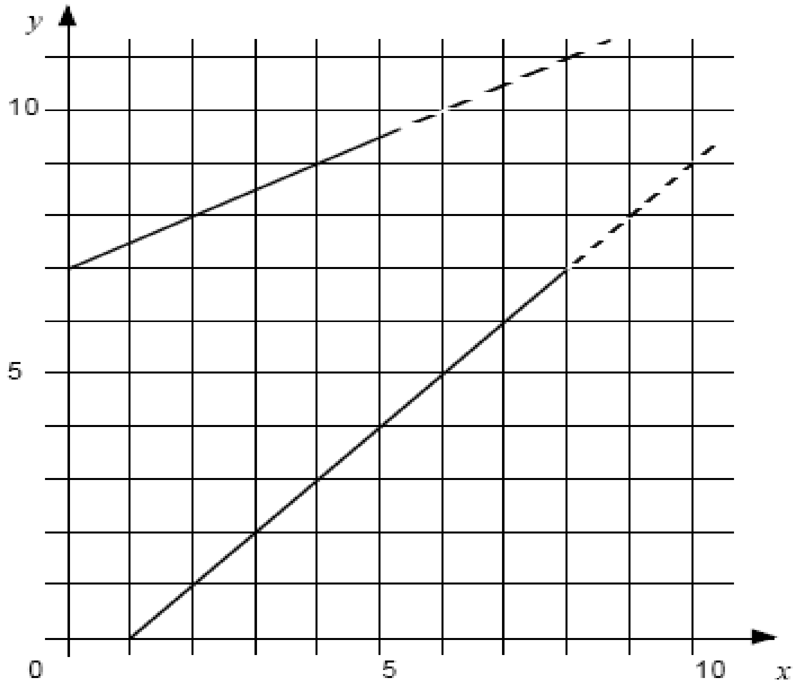
   The numbers with a mean of 12 might be 6 and 18, 7 and 17, 8 and 16 and so on.

   Those with a mean of 30 might be 18 and 42, 17 and 43, 16 and 44 and so on.

   The mean of 6, 18 and 42 is 22, that of 7, 17 and 43 is $22\frac{1}{3}$ and so on.

Considering such combinations one can home in on the correct answers, though a discussion as to why this method works, but only for whole number solution examples, might prove illuminating.

6.18 Meet



Find the co-ordinates of the point where these two lines meet if they are extended.

____

Problem solving classification	*Find mathematical features*
Content area classification	*Linear graphs, ratio and proportion*
Tier	*Higher*
Answer	*(16, 15)*

About the question

Superficially the problem appears to be about straight lines and/or the solution to a pair of, as yet unknown, simultaneous equations. However there are other patterns in the problem that can provide a way in to it and to the solution that will be more mathematically straightforward for many students.

Problem-solving approaches

The problem is tantalising in that it is clear that the lines do meet but little information is explicitly given. Hence, from a problem-solving point of view, students need to be encouraged to look for any mathematical features that they can work with, to see if any help towards a solution. The most obvious 'standard' techniques are not always the 'best' in that, for example, an equation-based approach, whilst guaranteeing success, is difficult to follow through to completion.

Challenges / issues

The key challenge in this problem is in turning the given information into useful mathematics. The students might notice how the slopes differ, thereby guaranteeing that the lines do indeed meet at some point, and then be encouraged to describe patterns in the vertical (or indeed horizontal) changes in the distance between the two lines.

Finding the answer

1. For students confident in algebra and the formulation of straight-line equations, a standard approach of finding the formulae of the two equations and solving them simultaneously will guarantee an answer. However, for many students the degree of algebraic confidence required by this method will be beyond them, as it requires sustained engagement with slope, intercept and then manipulative algebra.

2. An alternative method based on sequences can be to spot that the vertical distance between the two lines decreases by 1 unit with every increase of 2 units in the x direction, which allows a sequence to be constructed:

eg, for the lower line: (2,1), distance 7; (4,3) distance 6;
(6,5) distance 5 and so on.

This ends in distance zero at the point where the lines meet. An equivalent pattern can be found using the horizontal separation of the two lines.

3. A simpler method also based on sequences is to write out sequences for the points that each line passes through:

eg, (0,7), (2,8), (4,9) for the upper line
(2,1), (4,3), (6,5) for the lower line

and then continue the patterns to find the point that the two sequences have in common.

6.19 Pair de deux

The rule for a sequence of numbers is:

(first number, last number) ⟶

(first number + last number, first number – last number)

eg, **(5, 3)** **(8, 2)**

5 + 3   5 – 3

Here is part of the sequence that follows this rule.

Write in the missing number pairs

(___, ___) (___, ___) (1, 2) (3, –1) (2, 4) (___, ___)

Problem solving classification	*Work back – unfamiliar*
Content area classification	*Sequences*
Tier	*Higher*
Answer	$(\frac{1}{2}, 1)$ $(1\frac{1}{2}, -\frac{1}{2})$ $(6, -2)$

About the question

Going back in sequences is often done by using the inverse of the rule for the sequence. However, going back in this sequence is made problematic by the nature of the rule, which does not have an inverse. The way the rule affects the numbers has to be worked out, and then applied in reverse. This is made even more tricky by the involvement of negative numbers and non-whole numbers.

Problem-solving approaches

Working backwards with unfamiliar operations is usually best approached by practising the forward operation to improve understanding of its effects. There is one opportunity to apply the rule forwards to find the last pair of coordinates, but mostly the understanding is achieved by trying examples to get to (1, 2).

A different approach to working backwards is to look for a pattern and then extend it.

Challenges / issues

Working backwards with unfamiliar operations is usually best approached by practising the forward operation to improve understanding of its effects. There is one opportunity to apply the rule forwards to find the last pair of coordinates, but mostly the understanding is achieved by trying examples to get to (1, 2).

A different approach to working backwards is to look for a pattern and then extend it.

Finding the answer

1. One method involves trying different combinations towards findings pairs whose sum and difference are the required outcomes. This leads to understanding of the relationships between the characteristics of the inputs and outputs (eg, that if 'x' is smaller than 'y', the 'y' output is negative) that can be used to find the actual solutions.

2. An alternative approach begins by noticing a doubling relationship between (1, 2) and (2, 4), which, so long as the final set of coordinates has been worked out correctly, would be reinforced by the relationship between (3, –1) and (6, –2). Applying this relationship backwards by halving, then gives $(\frac{1}{2}, 1)$ and $(1\frac{1}{2}, -\frac{1}{2})$, and these can be tried out forwards as a check. Care must be taken to emphasise this last point – that the apparent pattern is based on very limited information and must be checked by employing the rule forwards to confirm its validity.

3. A formal algebraic approach will construct solutions as follows:

$$(x, y) \longrightarrow (x + y, x - y)$$

and so simultaneous equations can be formed, firstly for the number pair (1,2):

$$x + y = 1$$
$$x - y = 2$$

Once solved, a second pair can be formed from this solution.

Follow up

The power of the algebraic approach can be exemplified by using more complex examples, for example, rules of the form:

$$(x, y) \longrightarrow x + ay, ax - y)$$

with a as some fixed constant, and so on. Sequences can be thus generated and missing terms discovered regardless of the relative complexity of the rule generating them.

6.20 Pecuniary

P and Q are whole numbers.

P is greater than 10 and less than 20

Q is greater than 100 and less then 200

(a) What is the largest value that (P + Q) could have?

_ _ _ _ _ _ _ _ _

(b) What is the smallest difference there could be between P and Q?

_ _ _ _ _ _ _ _ _

Problem solving classification	*Find an example to fit*
Content area classification	*Symbols and formulae*
Tier	*Either*
Answer	*218, 82*

About the question

This question is about numerical reasoning. By being set in words rather than symbolic inequality statements, it encourages students to consider what the words actually mean mathematically. The first part is relatively straightforward, though the second part needs a more careful weighing up of the options.

Problem-solving approaches

Students need to identify the information that is required for the solutions and try the possible combinations that will meet the conditions. A clear statement of the possible values of P and Q that may be required is a valuable aid to making the actual choice. In other words, can examples of P and Q be found that are known with certainty to have the required characteristics?

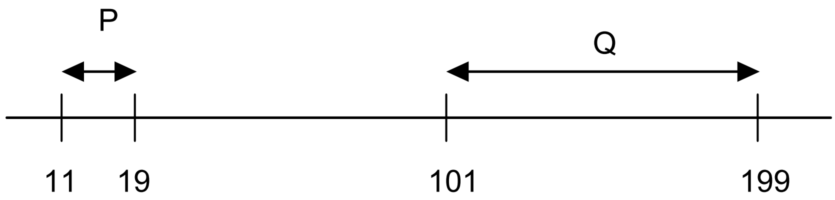
Challenges / issues

For many students the first part might seem almost obvious: the largest value of (P + Q) will be when P and Q are each at their maximum values. But it does not follow that the smallest difference is when each is at its minimum value - this is careless thinking. The challenge then, is in avoiding such mistaken logic and in engaging properly with the constraints of the problem, especially in the second part. The key question is "How do you know that what you have done is correct?" in reference to the smallest difference between P and Q.

Finding the answer

1. The ranges of P and Q do not overlap so largest and smallest values of P and Q can be considered as the candidates for pairing up to give the largest combined value and the smallest difference. This can be done purely numerically, choosing one number from each of 11 or 19 and 101 or 199

2. Given that the second part is the harder to reason out, a helpful approach may be to visualise P and Q on a number line:

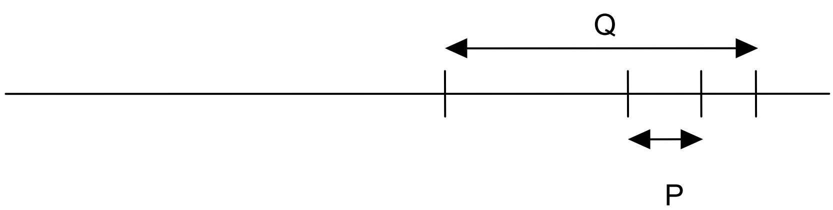


The least difference between P and Q will be represented by the gap between the two ranges, ie, when P is maximum and Q is minimum.

Follow up

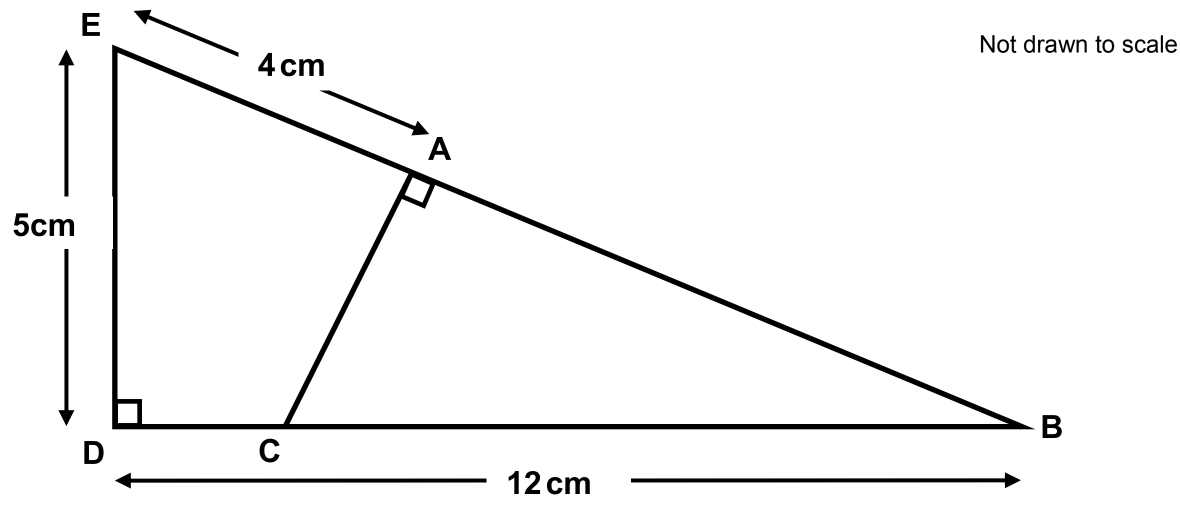
Simply changing the numerical values can produce other questions, though this will offer only a repetitious exercise, once the lines of reasoning are established. More thought will be needed if the ranges of P and Q overlap or the range of P is entirely within the range of Q. In this case it is worth asking what the greatest difference between P and Q is.

greatest difference between P and Q?



6.21 Perp perp

The diagram shows two right-angled triangles ABC and DEB



Find the length of line AC

– – – – – – –

Problem solving classification	*Find mathematical features*
Content area classification	*Pythagoras' theorem, Transformations and similarity*
Tier	*Higher (hardest)*
Answer	*3.75 cm*

About the question

This question requires strong visualisation skills in order for students to be able to 'see' the two key aspects of the problem - firstly, that the side AB is calculable as a section of a side in a right-angled triangle, and secondly, that the triangles ABC and DBE are similar. Even with this knowledge, the demand level of the problem, involving non-standard applications of both Pythagoras and similar triangles, remains high.

Problem-solving approaches

Across a group of students there are likely to be a variety of different initial approaches to the problem, and they may not know for certain that any progress made will result in the desired solution. Some might opt for trigonometry first, some may try a method based on similar triangles, and others go immediately for Pythagoras. Even though the last of these probably provides the 'best' first step, the other approaches, properly employed, will lead to a deeper understanding of the problem, and thereby to eventual success.

Challenges / issues	The desired length AC forms part of a triangle (ABC) for which no lengths are given. This immediately marks the problem out as unusual and therefore requires strategic thinking of how to get started on finding AC. Once Pythagoras has been used to find AB there still remains the difficulty of recognising (whether consciously or not) the similarity in the problem and then of managing this correctly to move towards an answer for AC.

Finding the answer

As part of the solution, the side AB (9 cm) has to found using Pythagoras in triangle DEB, although this does not have to be the first step in the solution.

There are then two main strategies for finding AC.

1. The most straightforward in terms of calculation is to use similar triangles to see that:

$$\frac{AC}{AB} = \frac{ED}{DB}$$

from which AC can then quite easily be found. However, this method does require significant levels of confidence in dealing with similarity in this context.

2. A more trigonometric, though equivalent, approach would involve actually calculating angle EBD as arctan (5/12) and then using this angle and side AB to find AC in triangle ABC. This might lead to a non-exact answer as the results of calculator use and rounding of intermediate answers. However, working with fractions and avoiding calculators (ie, stating that the angle is acrtan (5/12) rather than actually working it out in degrees) will not lead to such problems, and requires the similarity aspects necessary in method 1 above.

Follow up

Class discussion could involve students evaluating which of the approaches they think is 'best' and why. A good follow-up exercise would be to find DC (2.25 cm) or BC (9.75 cm) using similar techniques to those described above.

6.22 Repeater

556, 484 and 333 are examples of numbers with repeated digits.

How many whole numbers from 1 to 201 have repeated digits?

_ _ _ _ _ _ _

Problem solving classification	*Set out cases*
Content area classification	*Organising data*
Tier	*Higher*
Answer	*38*

About the question

This problem is straightforward to state but requires very careful attention to detail in ensuring that all the cases have been correctly accounted for. Can students correctly manage to detail all the repeating digit numbers in the desired range, and be sure that they have done so?

Problem-solving approaches

The problem as it stands has insufficient structure to be tackled unless students are to consider all the numbers from 1 to 201 individually, which is something to be discouraged. The natural questions to ask are "What different types of numbers with repeated digits are there?", or more broadly, "What is a useful classification system for the numbers 1 to 201 that will help solve this problem?" Answering such questions should help in making the solution more manageable.

Challenges / issues

Many students are likely to get an answer to this problem but the challenge for them is in ensuring that the answer is correct. Common errors include double counting of certain numbers (eg, 111) and of missing out individual numbers (eg, 100) or whole classes of numbers (eg, 112 to 119). The key again is in having a coherent system that covers all cases.

Another common error is to miscount, for example incorrectly deducing that the set 110, 111, 112,…, 119 contains 9 numbers, not 10.

An additional question that might arise is whether leading zeros should be allowed, so that, for example, 001, 002,… are included in the set. They should not be.

Finding the answer

Any efficient solution is likely to begin by dividing the numbers from 1 to 201 into those with 1, 2 and 3 digits respectively. The first of these can then be ignored and the second of these only has nine members with repeated digits (11, 22,…, 99). It is dealing correctly with the 3-digit numbers that is the difficult part.

There are a variety of slightly different approaches but all must deal with the following types:

- The numbers with the first two digits repeated (eg, 110, 111, 112,…, 119). There are 10 of these in total.
- The numbers with last two digits repeated (eg, 100, 111, 122,…, 199, 200). There are 11 of these in total.
- The numbers with a 1 at the beginning and at the end (eg, 101, 111, 121,…, 191). There are 10 of these in total.

However, 111 appears in all three of these groups and so has been trebled counted. The final total therefore is 9 (2-digits) + 29 (3-digits) making 38 in all.

Follow up

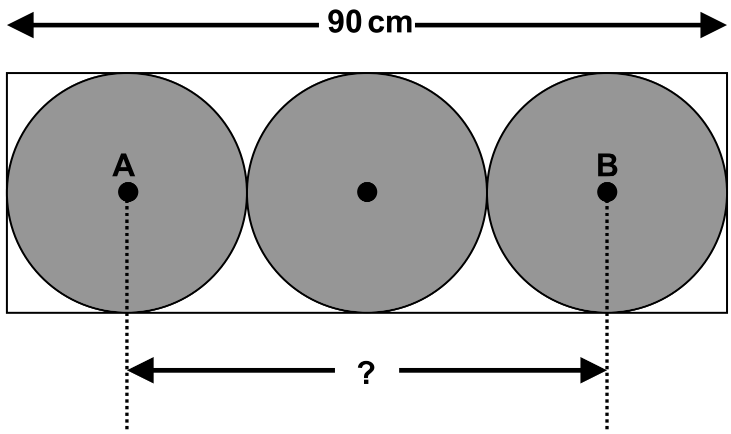
Giving a different range (eg, 101 to 301) provides a suitable follow-up but the problems can become quite difficult to manage, especially beyond 3 digits.

6.23 Roller

Three identical circles fit inside a rectangle.

The length of the rectangle is 90cm.

Not drawn to scale



Find the distance between the two centres, A and B.

_ _ _ _ _ _ _ cm

Problem solving classification	*Find key relationships*
Content area classification	*Two-dimensional shapes, Number in context*
Tier	*Foundation (easiest)*
Answer	*60 cm*

About the question

Although this question appears to be about circles it involves only some basic knowledge about circles ie, that the diameter of the circle goes through its centre and that the radius of the circle is half the diameter. This may not have been recently covered by students and is often implicit rather than explicit. Basic number calculations are also required, but their demand is minimal.

Problem-solving approaches

The key relationship in this question is between the shapes in the diagram, that is, that the three identical circles side by side are the same combined 'length' as that of the rectangle. This fact then provides key information about the circles themselves.

Challenges / issues

The challenge with this question is in seeing the relationship between the three circles and the length of the rectangle. Once this relationship has been seen the mathematics is straightforward.

Finding the answer

Distance AB can be calculated directly by finding and using the radius (90 cm/6), diameter (90 cm/3) or a combination of these.

Follow up	Alternatively, once the radius is found, AB is 90 cm minus two of these, or, equivalently, 90 cm minus one diameter.

6.24 Rollover

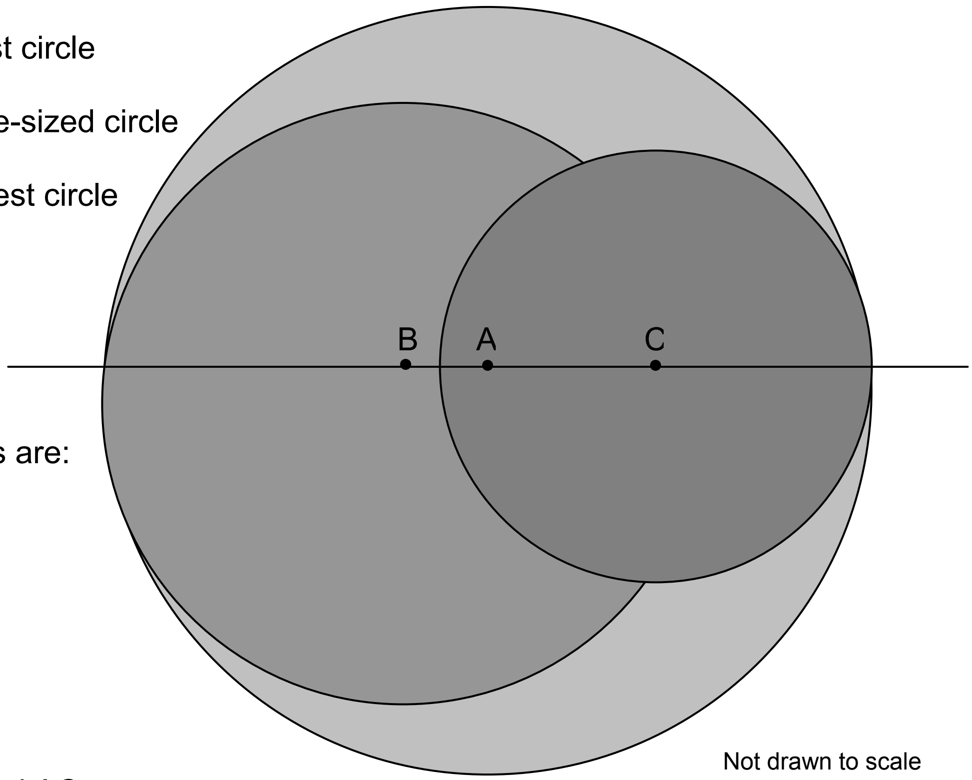
Three circles overlap each other as shown in the diagram

The centres of the circles are all on the same straight line

A is the centre of the largest circle

B is the centre of the middle-sized circle

C is the centre of the smallest circle



The diameters of the circles are:
22 cm, 16 cm and 13 cm

Calculate the lengths BA and AC

BA _ _ _ _ _ _ _ _ _ cm

AC _ _ _ _ _ _ _ _ _ cm

Problem solving classification	*Find mathematical features*
Content area classification	*Two-dimensional shapes*
Tier	*Either*
Answer	*BA = 3 cm, AC = 4.5 cm*

About the question

While apparently about circles, the question uses only straightforward aspects of circles – the radius and diameter. It is necessary to recognise how the circle diameters can be related to distances along the line.

Problem-solving approaches

This is an example of a question which responds best to an approach which locates features of the situation that can be acted on mathematically, and use them speculatively. Students should be encouraged to ask themselves what they notice about the diagram, and what information is provided that they can use to decide distances on the diagram.

Challenges / issues	There are a number of distracting features of the diagram that can get in the way of making steps towards a solution. Students might not, for example, see the value of identifying the distances from B and C to the edge of the large circle. Progress is helped considerably by clear marking onto the diagram of the information given in the text.
Finding the answer	**1.** Write given values onto the diagram, which suggests other values that may be calculated and entered. **2.** Reasoning that AB is a 'B' radius subtracted from an 'A' radius, and that AC is a 'C' radius subtracted from an 'A' radius. Both methods can be checked since the total length of the line in the diagram is the large diameter (22 cm), whilst BA plus AC plus the other two radii should add to this total.

6.25 Rooting range

$\sqrt{20\,000}$ = 141.4 (correct to 1 decimal place)

What is the smallest whole number that has a square root equal to 141.4 (correct to 1 decimal place)?

– – – – – – – –

Problem solving classification	*Find an example to fit*
Content area classification	*Properties of number including place value*
Tier	*Higher*
Answer	*19,980*

About the question

This is a demanding test of ability to deal with the number system and a precise understanding of decimal rounding. It also requires a flexibility of thought to be able to invert the usual procedure of taking a square root and rounding the answer. Not all students will find the question immediately accessible.

Problem-solving approaches

The criteria to be fulfilled in this question include the unusual condition of a limiting case, the smallest number from a range of possible answers. It may be helpful to first find an example or two without that condition – a whole number that has a square root equal to 141.4 (correct to 1 decimal place) – and then consider how to approach finding the smallest.

Challenges / issues

Clear thinking is needed to recognise exactly what the problem is asking. Some students may not recognise at first that 141.4 (correct to 1 decimal place) represents a range of values, since it is a rounding. The question has been restricted to whole numbers to avoid the further demand of dealing with unending decimal expressions.

Finding the answer

The implications of '141.4 (to 1 decimal place)' need to be explored. The least value is 141.35 (assuming the 5 and upwards rounding convention).

Squaring 141.35 gives 19,979.8225

However, a whole number is required, so the choice is between 19,979 and 19,980.

It may be obvious that the lower value is too small and that 19,980 is the value needed, but there is no harm in checking the square roots of both of them, just to be sure:

$\sqrt{19,979} = 141.347...$, which does not round to 141.4

$\sqrt{19,980} = 141.3506...$, which does round to 141.4.

Follow up

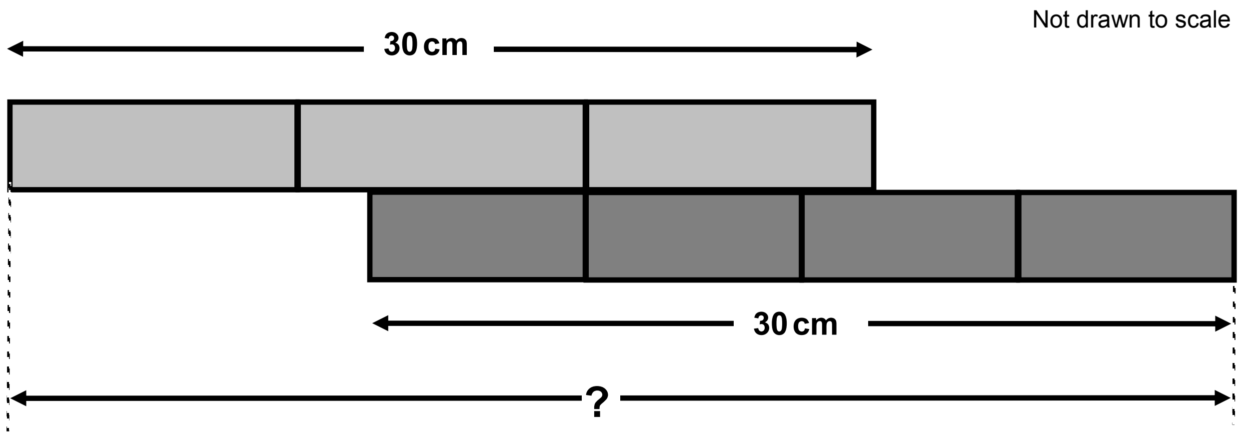
The natural follow up is to ask what the *largest* whole number is such that its square root is 141.4 (to 1 decimal place).

Further consolidation of the understandings may be achieved by changing the starting number, or by asking for the range of whole numbers with square roots of 141.42 (correct to *2* decimal places).

6.26 Side by side

Here are two 30 cm strips of card.

One is divided into thirds and the other is divided into quarters



What is the total length of this arrangement?

_ _ _ _ _ _ _ _ cm

Problem solving classification	*Find key relationships*
Content area classification	*Number in context; Fractions and decimals*
Tier	*Either*
Answer	*42.5 cm*

About the question

The accessible problem of the total length of two overlapping lengths is given a twist in this question by how the overlap is indicated, a consequence of how each is divided. As a result, fractions are brought into play, but exactly what fractions of what should be calculated is not initially obvious.

Problem-solving approaches

The key to this question is recognising the significance of the vertical division that is in the same place in both strips. Focusing on what can be directly deduced from what is given on the diagram (ie, the lengths of the parts of each strip) can help to direct attention to this.

Once this realisation has been made explicit, other important and useful relationships can come into play, for example, the difference in lengths between the strip divisions.

Challenges / issues

Many students will approach this question by trying to estimate the length of one of the overlaps, for example, the part of the lower card strip which extends beyond the end of the upper card strip, and then may become bogged down in the fractions (about one and three quarters of seven and a half) or make an insufficiently supported guess.

Some students see that the answer is two thirds plus three quarters of 30, but try to do the addition of the fractions first, and struggle. (It is far easier to work out each length first.)

Finding the answer

1. Write on the diagram the lengths of the parts of each strip, and see which five lengths can be combined to give the overall length.
2. Calculate the lengths to the left and right of the common line, two thirds of 30 and three quarters of 30.

Follow up

Inverse problems could be asked, for example, "Arrange the two strips so that the total length is 35 cm".

Note that if the two strips were 12 inches long (and hence, divisible by three and four), the total length could be made to be anything from 12 up to 21 inches, or 24 (but not 22 or 23).

6.27 Skywalker

Luke has £3.20 and Lottie has £4.50

How much money will they have if they share their money?

£ _ _ _ _ _ _ _ _

Problem solving classification	*Find an example to fit*
Content area classification	*Number in context*
Tier	*Foundation (easiest)*
Answer	*£3.85*

About the question

This may not be a problem as such for many students. It is about the distribution of a fixed amount in two quantities. The fact that the quantities are to be equal enables a procedure to be used - if it is known.

Problem-solving approaches

Although this could be seen by many students as 'just' a question about the mean of the two values, or 'just' a matter of adding the two amounts together and halving them, students who are problem solving at this level are more likely to approach it as a change situation - where Lottie gives Luke some money until they have the same. As such it will be about trying examples and applying the criterion of equal amounts.

Challenges / issues

For students who are problem solving at this level, there may be a challenge in keeping track - with the risk that Lottie's money will have been decreased by a different amount than Luke's money has increased.

Finding the answer

1. Add the two values (£7.70) and halve it.

2. If Lottie gives Luke 50 p, how much will each have, and is it the same?

What about other amounts?

Follow up	Small variants could be made, such as "How much would they each have if Lottie ended up with exactly 50 p more than Luke?" This would demand genuine problem solving from more students, but would also be at a suitable level for those who found the original question a problem (provided they have succeeded on the original).

6.28 Spinalot

Spinner A has 6 equal sections and Spinner B has 8 equal sections

Each section of the spinners contains the number 1, 2 or 3

All three numbers appear on each spinner.

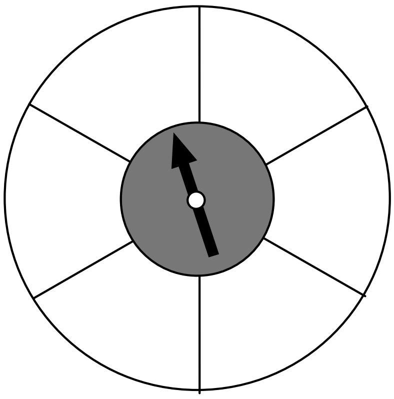
Write numbers in the spinner sections so that:

A score of 1 is more likely on Spinner A than Spinner B,

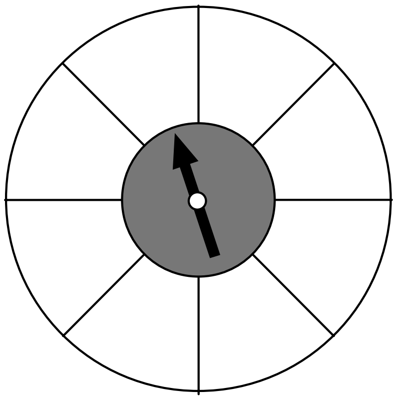
A score of 2 is more likely on Spinner B than Spinner A,

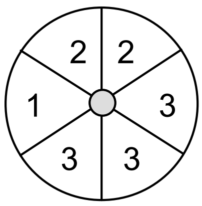
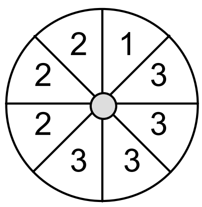
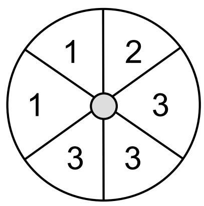
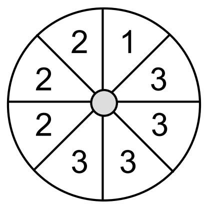
A score of 3 is more likely on either spinner

Spinner A



Spinner B



Problem solving classification	*Find an example to fit*
Content area classification	*Probability*
Tier	*Either*
Answer   or   The layout of numbers on any individual spinner can, of course, be re-arranged.	

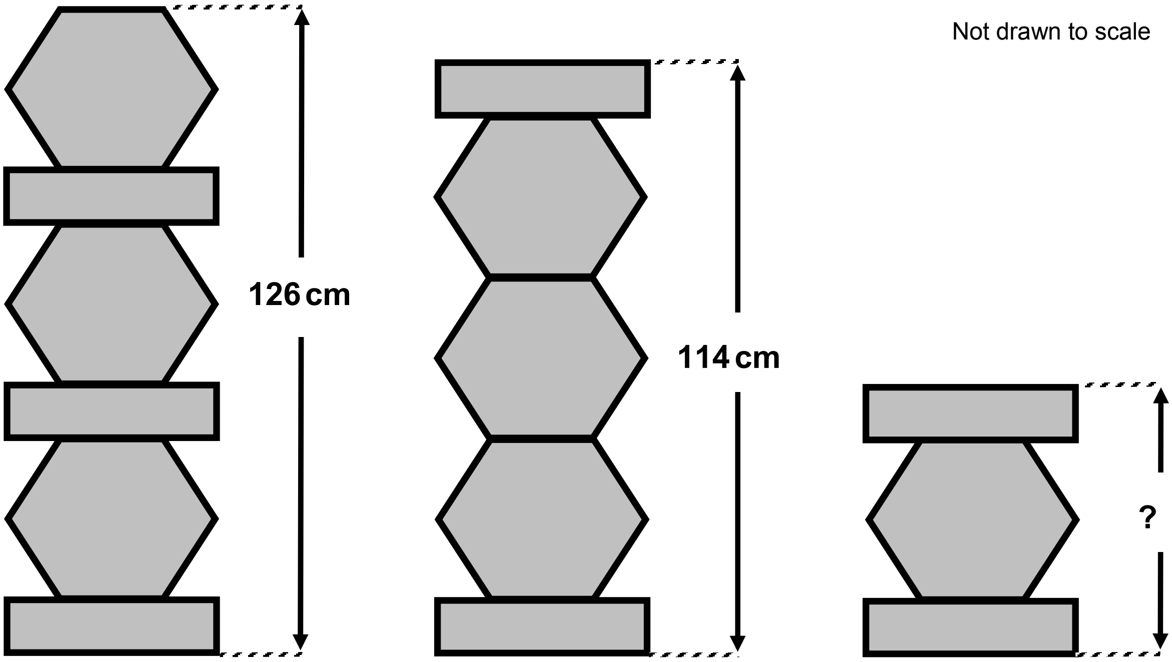
About the question

This is an accessible question, because to get started you only need to understand probability at the elementary level of dealing with terms such as 'more likely' and 'less likely'. The difficulty of the question is in the detail, which sets constraints that students may not initially notice. For '3' to be equally likely on either spinner, its sections will need to cover, in total, the same fraction of the circle (or the same angle subtended at the centre) on either spinner. This illustrates how the problem that appears to be about low level probability actually requires higher level geometric or fraction knowledge for its solution.

Problem-solving approaches	Many students are likely to experiment with the question by working their way through the conditions in the order given until they reach a point of contradiction - such as when the sections left for the '3's do not match the requirement that '3' is equally likely on either spinner. Having found an example that does not yet fit, the response may be to go back and check for mistakes. Assuming there are none, the next step must be to reconsider what has been done so far and recognise that adjustments can be made ie, that there were other possibilities in meeting the earlier conditions.
Challenges / issues	The tendency to look at the diagram rather than read the text means that some students will take for granted that each spinner has equal sections, but not engage with the fact that one has six sections and the other has eight sections. This may lead them simply to count sections on either spinner and come up with an apparent solution. Once it is realised that one spinner has larger sections than the other, it may be worth exploring what students think are the determining factors that make one number more likely than another number to come up on a spinner. They are likely to talk vaguely in terms of area (which is only true if a lot of assumptions are made). Fractions or angles are a better approach.
Finding the answer	**1.** Examine the constraints in the question first and decide which is the most significant. Making '1' or '2' more or less likely on either spinner appears to offer a lot of room for manoeuvre; making '3' equally likely on both spinners turns out only to have one arrangement (where '3' occupies half of each spinner). That makes it the best place to start and then the other two constraints can be fitted around it. **2.** Take what comes in the order it comes and make adjustments later, as necessary. Making '1' more likely on A than B has several possibilities, though it is probably safer to go for the minimalist one and choose one section on each spinner. Making '2' more likely on B again has several options. It is unlikely that these first two steps will leave just the right number of sections on each spinner for '3' to be correctly arranged. Once it is worked out what spaces on each spinner are needed for '3', the tentative allocations for '1' and '2' can be adjusted.

6.29 Towerism

These towers are made of identical hexagons and identical rectangles.



Calculate the height of the smallest tower

_ _ _ _ _ _ _ _ _ cm

Problem solving classification	*Find key relationships*
Content area classification	*Simultaneous equations, symbols and formula*
Tier	*Either*
Answer	*54 cm*

About the question

This item involves comparison of cases in a way that is an informal equivalent to the processes of simultaneous equations.

Problem-solving approaches

The problem involves finding the relationship between the height of the towers and the elements within them. The two towers where the height is given need to be compared to be able to do this. Considering what elements are the same and different in the two towers and how their heights differ will help students tease out the relevant information.

Challenges / issues

Some students may base their answer on the visual impression that the second tower appears to be twice the height of the third. This has to be seen only as an approximation, with mathematics needed to work out the exact answer.

Finding the answer

1. One solution strategy is to express the information given as a pair of simultaneous equations and to then solve them:

$$3h + 3r = 126$$
$$3h + 2r = 114$$
$$r = 12\,\text{cm}$$

$$3h + 3(12) = 126$$
$$3h = 90$$
$$h = 30\,\text{cm}$$

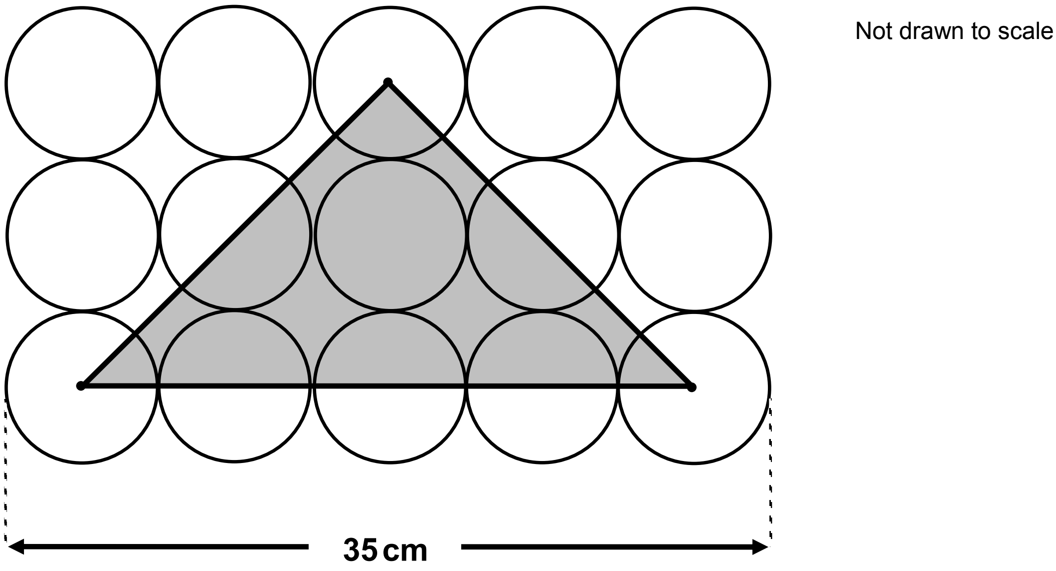
These measurements can then be used to calculate the height of the smallest tower.

2. A less formal approach which involves the same type of thinking is to compare the 126 cm and 114 cm towers, crossing out the shapes that correspond in both towers to reveal that the 12 cm taller tower is different only because it has one extra rectangle. The appropriate number of rectangles can then be subtracted from the height of one of the towers to give the height of three hexagons.

6.30 Tribubble

The diagram shows 15 identical circles, arranged as a rectangle and shaded triangle

The vertices of the triangle are at the centres of circles.



Calculate the area of the shaded triangle

_ _ _ _ _ _ _ _ _ cm^2

Problem solving classification	*Find mathematical features*
Content area classification	*Area*
Tier	*Either*
Answer	*196 cm^2*

About the question

This problem mixes two contexts of mathematics - the area of a triangle and the properties of circles. Information is given in one context and has to be related to the other to solve the problem.

Problem-solving approaches

The information given, the length of a rectangle, in terms of circles, has to be explored to see what further information it can provide. The hope is that something that relates to the triangle can be discerned and that this can then be used to find the answer to the problem. An awareness of what is needed in order to find the area of the triangle will help in the recognition of useful information that can be deduced from the circles.

Challenges / issues

Faced with the need to find the area of a triangle, there are two broad approaches that students usually try to employ. One is to look for suitable measurements that allow the use of 'the formula'; however, such measurements are not immediately apparent in this example. The other is to see if a method related to 'counting the squares inside' is available. The triangle does contain circles and part-circles that can be accurately quantified, once it is realised it is a 45°, 45°, 90° triangle and that the sides of the triangle pass through the centres of circles. However, it also contains gaps between circles that are unknown and are very difficult to calculate, certainly at this level.

Hence the familiar techniques are not sufficient in solving this problem and the challenge is to bolster these using the additional information given in the problem.

Finding the answer

Some thought about the given 35 cm and the number of circle diameters that make up this length will reveal that the diameter of a circle is 7 cm. The base of the triangle is equivalent to four circles, making it 28 cm. The height of the triangle is equivalent to two circles, making it 14 cm. From this the area can be calculated using the $\frac{1}{2}$ base × height formula.

Slight variations in the method are possible in terms of seeing the triangle, not in direct base and height terms, but as half of an encompassing rectangle, 2 circle diameters by 4 circle diameters, though this still relies on the key feature of the circle diameter.

There are other ways of "seeing" the problem. For example, the triangle can be viewed as forming half of a square, and this might appear to provide an easy route to a solution. Unfortunately, the side lengths of the square are not easily found, requiring a far from simple application of Pythagoras.

Follow up

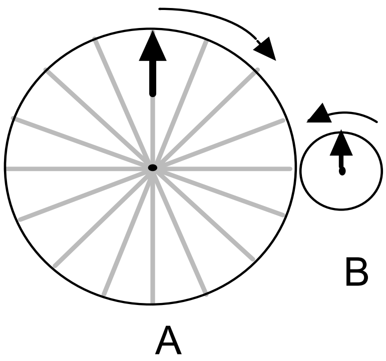
Using different shaped triangles on the same grid (but retaining the property that their vertices are at the centres of circles) will give some consolidation to students' understanding of the method. A further challenge would be to invert the process, by giving the area of the triangle and asking for the diameter of a circle. For example, on the diagram above, the area of the triangle could be given as 400 cm^2, and the challenge would be to find the diameter of a grid circle that this implies.

6.31 Wheelie bin

The two wheels A and B turn together, in opposite directions

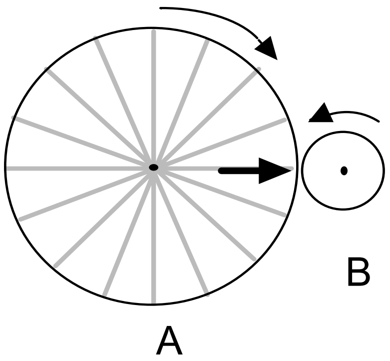
As wheel A makes one complete turn clockwise,
wheel B makes four complete turns anticlockwise.

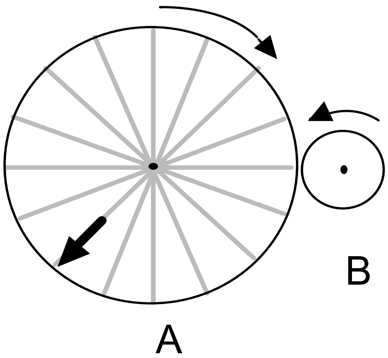
This diagram shows how the wheels look at the start



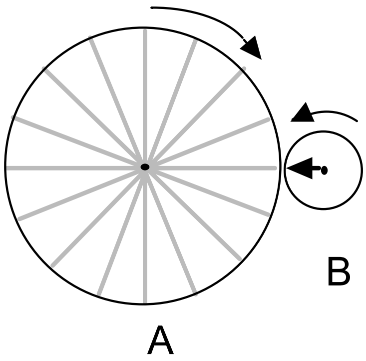
The diagrams below show new positions after turning.

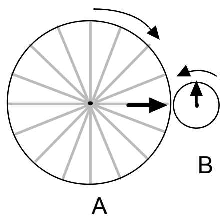
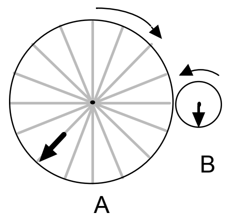
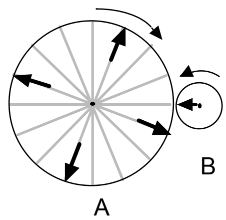
In each case, draw in the missing arrow on wheel B.





In this diagram, draw **all** the possible positions for the arrow on wheel A



Problem solving classification	*Find key relationships*
Content area classification	*Ratio and proportion*
Tier	*Higher*
Answers: First and second parts  	Third part 

About the question

Although this question is about ratio and proportion, the geometric context is important, since without a clear understanding of the dynamics, students are unlikely to see how to use their knowledge of ratio and proportion. The spokes on the larger wheel are given to clarify exactly what question is being asked and to allow precision in answering.

Problem-solving approaches

A way to understanding the key relationship between the wheels as they turn is to envisage the actual process, step by step. Students need to understand precisely how far the small circle turns as the large one turns, and be able to see explicitly that the angles turned through are always in the ratio 4:1. This knowledge can then be applied to the questions.

Challenges / issues

Although a broad statement of the relationship is given, its implications are not immediately apparent - exactly what happens still has to be thought through. After complete turns of the larger wheel, the starting arrangement reappears. Many students jump to conclusions instead of engaging with the painstaking accumulation of information needed to answer the questions.

Finding the answer

Whilst it is possible to reason that, since the smaller wheel turns four times as the larger wheel turns once, a quarter turn on the larger wheel implies a single complete turn on the smaller wheel, some students may struggle to reason that out and it may be easier to start in a more cautious way: a complete turn on the larger wheel implies four complete turns on the smaller, half a turn on the larger implies... and a quarter turn implies...

Alternatively, one can start from the other end, so to speak. For every amount of turn on the large wheel there is four times the amount of turn on the smaller. Thus a turn through one section (1/16) on the larger wheel means 4/16 or a $\frac{1}{4}$ turn on the smaller wheel.

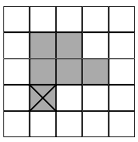
The third part of the question is the inverse of this: inferring the amount of turning on the larger wheel to produce a given amount of turn on the smaller. This leads to four possible answers as a quarter turn on the large wheel returns the smaller wheel to its starting position.

7 Answers

7.1 introduction

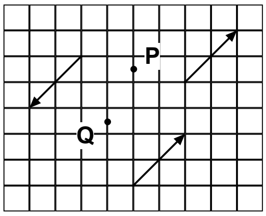
The answers to all questions in the resource are given below. For some of the questions, the answers are not clear cut, for example the answers that involve explanations or justifications. Here the resource gives a guide to what might be expected, and the kind of thing that is 'on the right lines'. In these cases judgement will need to be exercised for some student responses – or a discussion started. The answers are organised alphabetically reflecting the order of the problems on the CD-Rom.

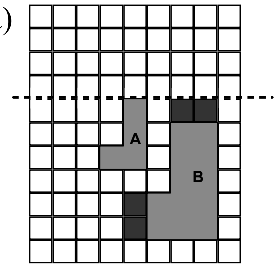
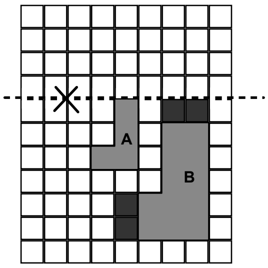
Item	Answer
3Rex	(a) Length = 4, Height = 2 (b) C is (9, 8)
Abacus	(50, 30)
Apple crumble	12
April 1st	The number of days in April, May and June total 91, which is a multiple of 7
Arwick 40	£6
Bouncy-bouncy	5
Boxclever	Slice A = $100x$ Slice B = $10x(10 - x) = 100x - 10x^2$ Slice C = $x(10 - x)(10 - x) = 100x - 20x^2 + x^3$
Bugeye	48 cm
Bunch of pens	8

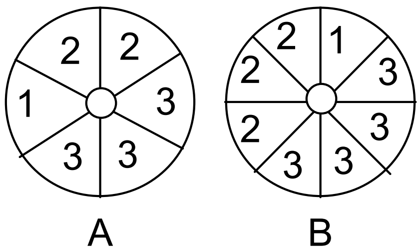
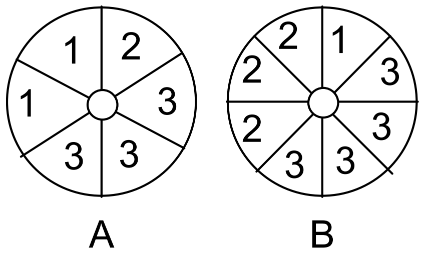
Item	Answer
Charterly	(a) 300 (b) 111 (c) 41
Club sandwich	120 cm (or 1.2 m)
Coin double	Janice 50 p 20 p 5 p, Jeremy 50 p 50 p 50 p
Crate-ivity	(a) 2 (b) 6
Cubical	First part: 14 Second part: 
Cuboid ratio	Length = 15 cm, height = 30 cm, depth = 45 cm
Cupid	p = 7, q = 8
Digitification	841
Double trouble	$550 \times 2 = 1100$ or $505 \times 2 = 1010$ or $055 \times 2 = 0110$
Ex-cube-me	$3000\,\text{cm}^3$
Expand	a = 6, b = – 3 and a = – 6, b = 3

Item	**Answer**
Explain 7	An explanation that suggests that the difference between half a number and four times the number is three-and-a half times the number. Since the original number is even, 3.5 times the number will be a multiple of $(3.5 \times 2) = 7$ **or** Algebraically, $(B - A) = 4n - 0.5n = 3.5n$ and n is even, so it is already a multiple of 2; thus $(B - A)$ is a multiple of $(2 \times 3.5) = 7$
Eye test	$6x - 2y$
Factory square	(a) Any one of 15, 21, 35, 105 (b) The square of any number from 15 to 31 eg, 400, 900
Fire rescue	1 litre
Five times	6
Fivegrand	4765
Flight cost	£33
Form	(a) $x^2 - 5x = 0$ (b) $x^2 - 14x + 49 = 0$
Gang of four	72 cm
Graphy	$y = 2x(x - 3) = 2x^2 - 6x$
Half take	31
Happylappy	19 cm
Highroller	$\frac{5}{72}$
Hotel	7

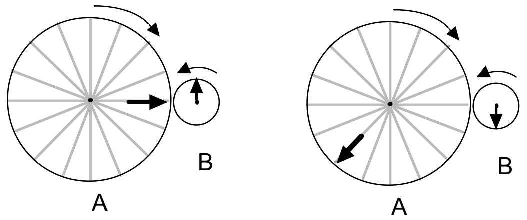
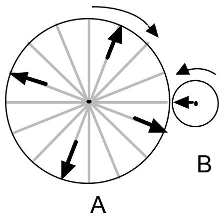
Item	Answer
Inside circle	400 cm^2
Isosceles grid	(a) There are many possibilities, all 5 units (the length of AB) from either A or B. For example, (1,6), (2,3), (6,1), (9,2), (10,3), (11,6) and so on (Note that the perpendicular bisector of AB does not pass through any point that has whole number coordinates) (b) Any two not used in part (a)
Javelin A	$y = 2x + 8$
Javelin B	$y = 2x - 20$
Last poster	16
Lineup	P = 90, Q = 105
Loopy-do	74 cm
Madbag	210
Mazy	250 cm
MeanN	–499
Meanset	5, 10, 11, 12, 12; other possibilities based upon (in ascending order) a, b, 11, 12, 12 where $a + b = 15$ and $a \neq b$
Meanstreet	9, 15, 45 (in any order)
Meet	(16, 15)
Midseq	$7m$
Moussey	7 (6 if it is assumed that the recipe should not be split)

Item	**Answer**
Multitude	(a) 24 and 35 or 25 and 36 or a multiple of 30 added to both, eg, 54 and 65 , 84 and 95 etc, or 55 and 66 , 85 and 96 etc, (b) 28 and 72 or 56 and 44 or 84 and 16
Pair de deux	$(\frac{1}{2}, 1)$ $(1\frac{1}{2}, -\frac{1}{2})$......... $(6, -2)$
Peculiar	$p = 9$, $q = 1$
Pecuniary	218, 82
Pointillism	
Pqr	$p = 8$, $q = 4$, $r = 2$
Put the numbers in	(a) Any answer which works eg, $(50 - 5) - (30 - 20) = 35$ (b) Either of: $(60 - 20) - (80 - 50) = 10$ $(50 - 20) - (80 - 60) = 10$
Repeater	38
Roller	60 cm
Rollover	BA = 3cm, AC = 4.5cm
Rooting range	19,980

Item	**Answer**
Scalefactor	(a)  (b)  (c) 2
Seesaw	+ 5%
Sevendiff	31, 31, 38 in any order
Shaperone	$252\,\text{cm}^2$
Shares	Petra = £60, Stephan = £180
Side by side	42.5 cm
Skywalker	£3.85
Smallfry	89
Sold out	$\frac{1}{2}(m+n)^2 = \frac{1}{2}m^2 + mn + \frac{1}{2}n^2$

Item	**Answer**
Spinalot	 **or**  The layout of numbers on any individual spinner can, of course, be re-arranged
Stamper	3×15 p stamps, 2×20 p stamps
Stretcher	(a) $\frac{1}{2}x(x+y)$ (b) 75% (c) 50% When $y = 0$, the white triangle has a height of x and is half the area of the square
Sum and difference	(a) 101, 103 (b) Two numbers less than 200 that differ by 150. eg, 199 & 49, 198 & 48, 197& 47, 196 & 46 etc, down to 151 & 1
Summertime	2, 3, 5 in any order
Sweet rapper	$m - n + 1$
Tape length	45 cm

Item	Answer
Tendency	$10 \times 10 \times 10 = 1000$ So, to make a product less than 1000, the factors cannot all be 10 or all greater than 10, so at least one factor must be less than 10
Terms	(a) 45 seconds (b) 550
Tgrid	90 cm
Three, four, five	(a) Any multiple of 12 greater than 100 eg, 108, 120, 132,… (b) Any product of 5 with a prime number over 3 or a product of them, eg, 25 (5×5), 35 (5×7), 55 (5×11) … 175 (5×55) This can also be thought of 5 or –5 plus any multiple of 30: 35, 65, 95, 125…; 25, 55, 85, 115…
Threesquare	5 cm
Toto	12, 12, 6 in any order
Towerism	54 cm
Tribubble	196 cm^2
Two-tri	16 cm
V-boats	4
Weighup	7.5 kg

Item	**Answer**
Wheelie bin	First and second parts  Third part 
Yogourtician	First offer is better since if P = price of 500 g of yoghourt, 1st offer gives 1500 g for 2P 2nd offer gives 1000 g for 1.5P ie, 1500 g for 2.25P Alternatively, 1st offer of 3 pots for 2P means 1 pot costs 2/3 P 2nd offer 2pots for 3/2 P means 1 pot for 3/4 P 2/3 < 3/4, so first offer is better.

8 Classification of Problems

1.1 Introduction

The final two sections of the resource contain tables of information about each of the problem solving items.

Within those tables, there are:

Problem solving classification

One of the five strategies is identified for each problem, and is the one that the problem is most susceptible to – but is rarely the only way the problem can be approached. In section 9, the 90 questions are ordered by this classification.

Content area classification

This lists elements of content that correspond to the mathematics teaching it would be reasonable to extend by using the problem. Sometimes there are two options, since the problem would fit in to teaching on either topic. In section 10, the 90 problems are ordered by content area.

Tier

Although problem solving strategies apply across different levels, few if any students will benefit from using all the problems in this resource as they stand. Students who are taking the Foundation tier paper in GCSE may not be able to do some of the problems because they have mathematics in them that is too demanding; students who are taking the Higher tier paper might find that some of the questions are just not problems for them. So, while the purpose of problem solving work in both tiers can focus on the five strategies, different problems will be needed to do so. For that reason, the problems in this resource have all been allocated a suggested tier, as an indication of what might be used with whom.

The classification indicates which questions are suitable only for Foundation and Higher tiers (with the easiest and hardest questions highlighted as part of this), but also includes the classification "Either" for a number of the problems. These are the problems that might be suitable for either tier – representing the harder questions for the Foundation tier, and the easier questions for the Higher tier.

The tier attribution is not a prescription, of course, but a guide, and many of the problems can be adjusted to some extent, ie, made more difficult or more straightforward by amending the mathematical content or their complexity. By that means, a question suitable for the tier other than that specified can often be derived from the one given.

A related point is that, since many Higher tier students will not see some of the Foundation tier problems as problems, there is at times reference in the resource to "problem solving at this level" as an indication of different presuppositions about the knowledge that is being brought to bear, and therefore where the need for a strategy may exist for some students but not for others.

9 Classification Table by Strategy

Strategy Identification	**Item**	**Tier**
Set out classes	Cubical	Foundation
	Crate-ivity	Either
	Highroller	Higher
	Repeater	Higher
Work back – familiar	Apple crumble	Foundation (easiest)
	Bunch of pens	Foundation
	V-boats	Foundation
	Half take	Either
	Hotel	Either
	Flight cost	Higher
Work back – unfamiliar	Put the numbers in	Foundation
	Last poster	Either
	Madbag	Either
	Pointillism	Either
	Scalefactor	Either
	Javelin A	Higher
	Pair de deux	Higher
	Javelin B	Higher (hardest)
	Form	Higher (hardest)
	Sold out	Higher (hardest)

Strategy Identification	**Item**	**Tier**
Find an example to fit	Skywalker	Foundation (easiest)
	Sum and difference	Foundation (easiest)
	Coin double	Foundation
	Double trouble	Foundation
	Factory square	Foundation
	Fivegrand	Foundation
	Lineup	Foundation
	Multitude	Foundation
	Sevendiff	Foundation
	Stamper	Foundation
	Summertime	Foundation
	Three, four, five	Foundation
	Toto	Foundation
	Digitification	Either
	Meanset	Either
	Pecuniary	Either
	Spinalot	Either
	Weighup	Either
	Pqr	Higher
	Rooting range	Higher

Strategy Identification	Item	Tier
Find key relationships	Bugeye	Foundation (easiest)
	Mazy	Foundation (easiest)
	Roller	Foundation (easiest)
	Two-tri	Foundation (easiest)
	Five times	Foundation
	Shares	Foundation
	Tape length	Foundation
	Threesquare	Foundation
	Abacus	Either
	Arwick 40	Either
	Bouncy-bouncy	Either
	Charterly	Either
	Club sandwich	Either
	Gang of four	Either
	Happylappy	Either
	Isosceles grid	Either
	Loopy-do	Either
	Moussey	Either
	Peculiar	Either
	Side by side	Either
	Tendency	Either
	Tgrid	Either
	Towerism	Either
	3Rex	Higher
	Boxclever	Higher
	Eye test	Higher
	Inside circle	Higher
	MeanN	Higher
	Seesaw	Higher
	Shaperone	Higher
	Terms	Higher
	Wheelie bin	Higher
	Stretcher	Higher (hardest)

Strategy Identification	Item	Tier
Find mathematics features	April 1st	Either
	Explain 7	Either
	Rollover	Either
	Smallfry	Either
	Tribubble	Either
	Yogourtician	Either
	Cuboid ratio	Higher
	Cupid	Higher
	Ex-cube-me	Higher
	Expand	Higher
	Fire rescue	Higher
	Graphy	Higher
	Meanstreet	Higher
	Meet	Higher
	Midseq	Higher
	Sweet rapper	Higher
	Perp perp	Higher (hardest)

10 Classification Table by Content Area

Content area		Item	Tier	Strategy classification
N	Basic algebra	Explain 7	Either	Find mathematical features
		Boxclever	Higher	Find key relationships
		Eye test	Higher	Find key relationships
		Midseq	Higher	Find mathematical features
		Shaperone	Higher	Find key relationships
		Sweet rapper	Higher	Find mathematical features
		Stretcher	Higher (hardest)	Find key relationships
N	Basic number	Sum and difference	Foundation (easiest)	Find an example to fit
		Double trouble	Foundation	Find an example to fit
		Fivegrand	Foundation	Find an example to fit
		Multitude	Foundation	Find an example to fit
		Three, four, five	Foundation	Find an example to fit
		Factory square	Foundation	Find an example to fit
		Five times	Foundation	Find key relationships
		Put the numbers in	Foundation	Work back – unfamiliar
		Sevendiff	Foundation	Find an example to fit
		Summertime	Foundation	Find an example to fit
		Toto	Foundation	Find an example to fit
		Smallfry	Either	Find mathematical features
N	Equations	Weighup	Either	Find an example to fit
		Pqr	Higher	Find an example to fit

Content area		Item	Tier	Strategy classification
N	Fractions and decimals	Shares	Foundation	Find key relationships
		Half take	Either	Work back – familiar
		Side by side	Either	Find key relationships
N	Linear graphs	Javelin A	Higher	Work back – unfamiliar
		Meet	Higher	Find mathematical features
		Javelin B	Higher (hardest)	Work back – unfamiliar
N	Number in context	Apple crumble	Foundation (easiest)	Work back – familiar
		Mazy	Foundation (easiest)	Find key relationships
		Roller	Foundation (easiest)	Find key relationships
		Skywalker	Foundation (easiest)	Find an example to fit
		Bunch of pens	Foundation	Work back – familiar
		Coin double	Foundation	Find an example to fit
		Lineup	Foundation	Find an example to fit
		Stamper	Foundation	Find an example to fit
		V-boats	Foundation	Work back – familiar
		Arwick 40	Either	Find key relationships
		Club sandwich	Either	Find key relationships
		Happylappy	Either	Find key relationships
		Hotel	Either	Work back – familiar
		Last poster	Either	Work back – unfamiliar
		Loopy-do	Either	Find key relationships
		Side by side	Either	Find key relationships
N	Percentages	Bouncy-bouncy	Either	Find key relationships
		Seesaw	Higher	Find key relationships

Content area		Item	Tier	Strategy classification
N	Properties of number, including place value	Digitification	Either	Find an example to fit
		Tendency	Either	Find key relationships
		Rooting range	Higher	Find an example to fit
N	Ratio and proportion	Shares	Foundation	Find key relationships
		Tape length	Foundation	Find key relationships
		Club sandwich	Either	Find key relationships
		Moussey	Either	Find key relationships
		Yogourtician	Either	Find mathematical features
		Cuboid ratio	Higher	Find mathematical features
		Inside circle	Higher	Find key relationships
		Fire rescue	Higher	Find mathematical features
		Meet	Higher	Find mathematical features
		Shaperone	Higher	Find key relationships
		Wheelie bin	Higher	Find key relationships
N	Sequences	Lineup	Foundation	Find an example to fit
		Charterly	Either	Find key relationships
		Midseq	Higher	Find mathematical features
		Pair de deux	Higher	Work back - unfamiliar
		Terms	Higher	Find key relationships
N	Simultaneous equations	Peculiar	Either	Find key relationships
		Towerism	Either	Find key relationships
		Cupid	Higher	Find mathematical features
		Expand	Higher	Find mathematical features
		Meanstreet	Higher	Find mathematical features

Content area		Item	Tier	Strategy classification
N	Symbols and formulae	Sevendiff	Foundation	Find an example to fit
		Toto	Foundation	Find an example to fit
		Explain 7	Either	Find mathematical features
		Pecuniary	Either	Find an example to fit
		Towerism	Either	Find key relationships
		Flight cost	Higher	Work back – familiar
N	Quadratic equations	Expand	Higher	Find mathematical features
		Graphy	Higher	Find mathematical features
		Form	Higher (hardest)	Work back – unfamiliar
G	Area	Tgrid	Either	Find key relationships
		Tribubble	Either	Find mathematical features
		Inside circle	Higher	Find key relationships
		Stretcher	Higher (hardest)	Find key relationships
G	Co-ordinates	Abacus	Either	Find key relationships
		Isosceles grid	Either	Find key relationships
		3Rex	Higher	Find key relationships
G	Perimeter	Bugeye	Foundation (easiest)	Find key relationships
		Two-tri	Foundation (easiest)	Find key relationships
		Threesquare	Foundation	Find key relationships
		Gang of four	Either	Find key relationships
		Tgrid	Either	Find key relationships
		Eye test	Higher	Find key relationships

Content area		Item	Tier	Strategy classification
G	Pythagoras' theorem	Perp perp	Higher (hardest)	Find mathematical features
		Sold out	Higher (hardest)	Work back – unfamiliar
G	Three-dimensional shapes	Cubical	Foundation	Set out cases
G	Transformations and similarity	Pointillism	Either	Work back – unfamiliar
		Scalefactor	Either	Work back – unfamiliar
		Perp perp	Higher (hardest)	Find mathematical features
G	Two-dimensional shapes	Roller	Foundation (easiest)	Find key relationships
		Crate-ivity	Either	Set out cases
		Isosceles grid	Either	Find key relationships
		Rollover	Either	Find mathematical features
G	Units of measurement	April 1st	Either	Find mathematical features
G	Volume and surface area	Boxclever	Higher	Find key relationships
		Cuboid ratio	Higher	Find mathematical features
		Ex-cube-me	Higher	Find mathematical features
S	Averages and range	Meanset	Either	Find an example to fit
		Meanstreet	Higher	Find mathematical features
		MeanN	Higher	Find key relationships
		Sweet rapper	Higher	Find mathematical features
S	Organising data	Repeater	Higher	Set out cases

Content area		Item	Tier	Strategy classification
S	Probability	Madbag	Either	Work back – unfamiliar
		Spinalot	Either	Find an example to fit
		Highroller	Higher	Set out cases