Cambridge International Examinations
Cambridge Pre-U Certificate

FURTHER MATHEMATICS (PRINCIPAL) **9795/01**

Paper 1 Further Pure Mathematics **For Examination from 2016**

SPECIMEN MARK SCHEME

3 hours

MAXIMUM MARK: 120

The syllabus is approved for use in England, Wales and Northern Ireland as a Cambridge International Level 3 Pre-U Certificate.

This document consists of **10** printed pages.

Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

The following abbreviations may be used in a mark scheme:

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)
aef Any equivalent form
art Answers rounding to
cwo Correct working only (emphasising that there must be no incorrect working in the solution)
ft Follow through from previous error is allowed
o.e. Or equivalent

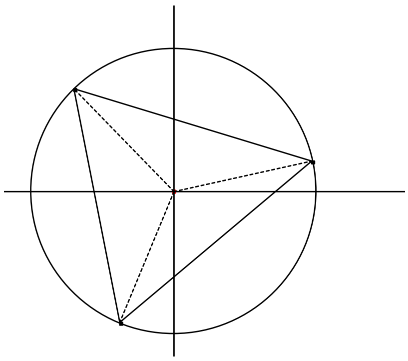
1		$\sum_{r=1}^{n}\left(r^2-r+1\right)=\sum_{r=1}^{n}r^2-\sum_{r=1}^{n}r+\sum_{r=1}^{n}1$ Splitting summation and use of given results	M1
		$=\frac{1}{6}n(n+1)(2n+1)-\frac{1}{2}n(n+1)+n$ 1st for Σr^2; 2nd for Σr & $\Sigma 1 = n$	B1 B1
		$=\frac{1}{3}n(n^2+2)$ legitimately	A1
2		$A = k\int(\sin\theta+\cos\theta)^2\,\mathrm{d}\theta$ including squaring attempt; ignore limits and $k \neq \frac{1}{2}$	M1
		$=\frac{1}{2}\int(1+\sin 2\theta)\,\mathrm{d}\theta$ for use of the double-angle formula **OR** integration of $\sin\theta\cos\theta$ as $k\sin^2\theta$ or $k\cos^2\theta$	B1
		$=\frac{1}{2}\left[\theta-\frac{1}{2}\cos 2\theta\right]_0^{\pi/2}$ **ft** (constants only) in the integration; MUST be 2 separate terms	A1
		$=\frac{1}{4}\pi+\frac{1}{2}$	A1
3	(i)	$y=(\sinh x)^{\frac{1}{2}} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}(\sinh x)^{-\frac{1}{2}}.\cosh x$ **OR** $y^2=\sinh x \Rightarrow 2y\frac{\mathrm{d}y}{\mathrm{d}x}=\cosh x$	M1 A1
		$=\frac{\sqrt{1+y^4}}{2y}$	A1
	(ii)	$\int\frac{2y}{\sqrt{1+y^4}}\,\mathrm{d}y=\int 1.\mathrm{d}x$ By separating variables in **(i)**'s answer	M1
		$\Rightarrow x=\int\frac{2y}{\sqrt{1+y^4}}\,\mathrm{d}y$	A1
		But $x=\sinh^{-1}y^2$ so $\int\frac{2t}{\sqrt{1+t^4}}\,\mathrm{d}x=\sinh^{-1}(t^2)+C$ condone missing " $+C$ "	A1
		ALT.1 Set $t^2=\sinh\theta$, $2t\,\mathrm{d}t=\cosh\theta\,\mathrm{d}\theta$ **M1** Full substitution $\int\frac{2t}{\sqrt{1+t^4}}\,\mathrm{d}t=\int\frac{\cosh\theta}{\sqrt{1+\sinh^2\theta}}\,\mathrm{d}\theta$ **A1** $=\int 1.\mathrm{d}\theta=\theta=\sinh^{-1}(t^2)$ **A1**	
		ALT.2 Set $t^2=\tan\theta$, $2t\,\mathrm{d}t=\sec^2\theta\,\mathrm{d}\theta$ **M1** Full substitution $\int\frac{2t}{\sqrt{1+t^4}}\,\mathrm{d}t=\int\frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta$ **A1** $=\int\sec\theta.\mathrm{d}\theta$ $=\ln\lvert\sec\theta+\tan\theta\rvert=\ln\lvert t^2+\sqrt{1+t^4}\rvert$ **A1**	

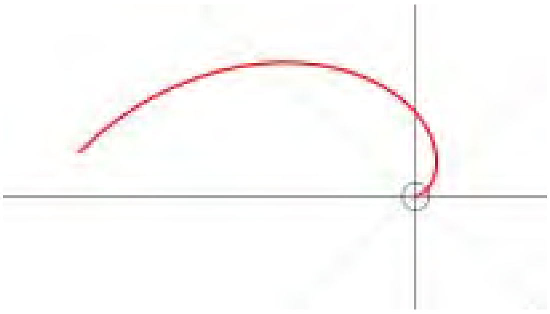
4	(i)	$y = \dfrac{x+1}{x^2+3} \Rightarrow y.x^2 - x + (3y-1) = 0$ Creating a quadratic in x	M1
		For real x, $1 - 4y(3y-1) \geqslant 0$ Considering the discriminant	M1
		$12y^2 - 4y - 1 \leqslant 0$ Creating a quadratic **inequality**	M1
		For real x, $(6y+1)(2y-1) \leqslant 0$ Factorising/solving a 3-term quadratic	M1
		$-\frac{1}{6} \leqslant y \leqslant \frac{1}{2}$ CAO	A1
	(ii)	$y = \frac{1}{2}$ substituted back $\Rightarrow \frac{1}{2}(x^2 - 2x + 1) = 0 \Rightarrow x = 1$ $[y = \frac{1}{2}]$	M1 A1
		$y -\frac{1}{6} =$ substituted back $\Rightarrow -\frac{1}{6}(x^2 + 6x + 9) = 0 \Rightarrow x = -3$ $[y = -\frac{1}{6}]$	M1 A1
5	(i) (a)	$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$	B1
	(b)	$\begin{pmatrix} \cos\beta & \sin\beta \\ \sin\beta & -\cos\beta \end{pmatrix}$	B1
	(ii)	$\begin{pmatrix} \cos\phi & \sin\phi \\ \sin\phi & -\cos\phi \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ multiplication of 2 reflection matrices. Correct order.	M1 M1
		$= \begin{pmatrix} \cos\phi\cos\theta + \sin\phi\sin\theta & \cos\phi\sin\theta - \sin\phi\cos\theta \\ \sin\phi\cos\theta - \cos\phi\sin\theta & \cos\phi\cos\theta + \sin\phi\sin\theta \end{pmatrix}$	
		$= \begin{pmatrix} \cos(\phi-\theta) & -\sin(\phi-\theta) \\ \sin(\phi-\theta) & \cos(\phi-\theta) \end{pmatrix}$ Use of the addition formulae; correctly done	M1 A1
		… giving a Rotation (about O) through $(\phi - \theta)$	M1 A1
6	(i)	Possible orders are 1, 2, 3, 4, 6 & 12	B1
		By *Lagrange's Theorem*, the order of an element divides the order of the group (since the order of an element ≡ the order of the subgroup generated by that element)	B1
	(ii)	E.g. $y = xyx \Rightarrow y \,.\, x^2y = xyx \,.\, x^2y$ by ③	M1
		$= xy \,.\, x^3 \,.\, y = xy \,.\, y^2 \,.\, y$ by ②	M1
		$= x \,.\, y^4 = x \,.\, (y^2)^2$ [by ②]	
		$= x \,.\, (x^3)^2 = x \,.\, e$ by ①	
		2 **M**'s for first, correct uses of 2 different conditions; the **A** for the 3rd condition used to clinch the result.	A1
	(iii)	Proving G not abelian: [e.g. by $xyx = y$ but $x^2 \neq e$] $\Rightarrow$ G not cyclic **OR** establishing a contradiction	B1 B1

7	**(i)**	$\cos 4\theta + \mathrm{i}\sin 4\theta = (c + \mathrm{i}s)^4$ Use of *de Moivre's Theorem*	M1
		$= c^4 + 4c^3.\mathrm{i}s + 6c^2.\mathrm{i}^2s^2 + 4c.\mathrm{i}^3s^3 + \mathrm{i}^4s^4$ Binomial expansion attempted	M1
		$\cos 4\theta = c^4 - 6c^2s^2 + s^4$ and $\sin 4\theta = 4c^3s - 4cs^3$ Equating Re & Im parts	M1
		$\tan 4\theta = \dfrac{\sin 4\theta}{\cos 4\theta} = \dfrac{4c^3s - 4cs^3}{c^4 - 6c^2s^2 + s^4}$	M1
		Dividing throughout by c^4 to get $\dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$ legitimately	A1
	(ii)	$t = \frac{1}{5} \Rightarrow \tan 4\theta = \frac{120}{119}$	B1
		$\tan\left(\frac{1}{4}\pi + \tan^{-1}\frac{1}{239}\right) = \dfrac{1 + \frac{1}{239}}{1 - \frac{1}{239}} = \frac{120}{119}$	M1 A1
		Noting that this is $\tan(4\tan^{-1}\frac{1}{5})$ so that $4\tan^{-1}\frac{1}{5} = \frac{1}{4}\pi + \tan^{-1}\frac{1}{239}$	A1
8	**(i)**	Substituting $x = 1$, $\mathrm{f}(1) = 2$ and $\mathrm{f}'(1) = 3$ into **(*)** $\Rightarrow \mathrm{f}''(1) = 5$	M1 A1
	(ii)	$\{x^2\mathrm{f}'''(x) + 2x\mathrm{f}''(x)\} + \{(2x-1)\mathrm{f}''(x) + 2\mathrm{f}'(x)\} - 2\mathrm{f}'(x) = 3\mathrm{e}^{x-1}$ Product Rule used twice; at least one bracket correct	M1 A1
		Substituting $x = 1$, $\mathrm{f}'(1) = 3$ and $\mathrm{f}''(1) = 5$ into this $\Rightarrow \mathrm{f}'''(1) = -12$ **ft** their $\mathrm{f}''(1)$	M1 A1
	(iii)	$\mathrm{f}(x) = \mathrm{f}(1) + \mathrm{f}'(1)(x-1) + \frac{1}{2}\mathrm{f}''(1)(x-1)^2 + \frac{1}{6}\mathrm{f}'''(1)(x-1)^3 + \ldots$ Use of the Taylor series	M1
		$= 2 + 3(x-1) + \frac{5}{2}(x-1)^2 - 2(x-1)^3 + \ldots$ 1st two terms CAO; 2nd two terms **ft (i) & (ii)'s answers**	A1 A1
	(iv)	Substituting $x = 1.1 \Rightarrow \mathrm{f}(1.1) \approx 2.323$ to 3d.p. CAO	M1 A1
9	**(i)**	$\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 3xy^4$ is a *Bernouilli (differential) equation*	
		$u = \dfrac{1}{y^3} \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} = -\dfrac{3}{y^4} \times \dfrac{\mathrm{d}y}{\mathrm{d}x}$	B1
		Then $\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 3xy^4$ becomes $-\dfrac{3}{y^4} \times \dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{3}{y^3} = -9x \Rightarrow \dfrac{\mathrm{d}u}{\mathrm{d}x} - 3u = -9x$ **AG**	M1 A1

	(ii)	**Method 1**	
		IF is e^{-3x}	M1 A1
		$\Rightarrow ue^{-3x} = \int -9xe^{-3x}dx$	M1
		$= 3xe^{-3x} - \int 3e^{-3x}dx$ Use of "parts"	M1
		$= (3x+1)e^{-3x} + C$	A1
		General solution is $u = 3x + 1 + Ce^{3x}$ **ft**	B1
		$\Rightarrow y^3 = \dfrac{1}{3x+1+Ce^{3x}}$ **ft**	B1
		Using $x = 0, y = \frac{1}{2}$ to find C $C = 7$ or $y^3 = \dfrac{1}{3x+1+7e^{3x}}$	M1 A1
		Method 2	
		Auxiliary equation $m - 3 = 0 \Rightarrow u_C = Ae^{3x}$ is the complementary function	M1 A1
		For particular integral try $u_P = ax + b$, $u_P' = a$	M1
		Substituting $u_P = ax + b$ and $u_P' = a$ into the d.e. and comparing terms	M1
		$a - 3ax - 3b = -9x \Rightarrow a = 3, b = 1$ i.e. $u_P = 3x + 1$	A1
		General solution is $u = 3x + 1\ Ae^{3x}$ **ft** particular integral + complementary function provided particular integral has no arbitrary constants and complementary function has one	B1
		$\Rightarrow y^3 = \dfrac{1}{3x+1+Ae^{3x}}$ **ft**	B1
		Using $x = 0, y = \dfrac{1}{2}$ to find A $A = 7$ or $y^3 = \dfrac{1}{3x+1+7e^{3x}}$	M1 A1
10	(i)	Substituting $\begin{pmatrix} 1+3\lambda \\ -3+4\lambda \\ 2+6\lambda \end{pmatrix}$ into plane equation; i.e. $\begin{pmatrix} 1+3\lambda \\ -3+4\lambda \\ 2+6\lambda \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} = k$	M1
		OR any point on line (since "given")	
		$k = 2 + 6\lambda + 18 - 24\lambda + 6 + 18\lambda = 26$	A1

	(ii)	Working with vector $\begin{pmatrix} 10+2m \\ 2-6m \\ 3m-43 \end{pmatrix}$.	B1
		Substituting into the plane equation: $\begin{pmatrix} 10+2m \\ 2+6m \\ 3m-43 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} = k$	M1
		Solving a linear equation in m: $20 + 4m - 12 + 36m + 9m - 129 = 26$	M1
		$m = 3 \Rightarrow Q = (16, -16, -34)$	A1
		Shortest distance is $\|m\| \left\| \begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix} \right\| = 21$ or $PQ = \sqrt{6^2 + 18^2 + 9^2} = 21$	A1
	(iii)	Finding 3 points in the plane: e.g. $A(1, -3, 2)$, $B(4, 1, 8)$, $C(10, 2, -43)$	M1
		Then 2 vectors in (// to) plane: e.g. $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$, $\overrightarrow{AC} = \begin{pmatrix} 9 \\ 5 \\ -45 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 6 \\ 1 \\ -51 \end{pmatrix}$ **OR B1 B1** for any two vectors in the plane	M1
		Vector product of any two of these to get normal to plane: $\begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix}$ (any non-zero multiple)	M1 A1
		$d = \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix} \bullet$ (any position vector) $= \begin{pmatrix} 10 \\ -9 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}$ e.g. $= 39$ $\Rightarrow 10x - 9y + z = 39$ CAO (aef)	M1 A1
		ALTERNATE SOLUTION	
		$ax + by + cz = d$ contains $\begin{pmatrix} 1+3\lambda \\ -3+4\lambda \\ 2+6\lambda \end{pmatrix}$ and $\begin{pmatrix} 10 \\ 2 \\ -43 \end{pmatrix}$	
		... so $a + 3a\lambda + 4b\lambda - 3b + 2c + 6c\lambda = d$ and $10a + 2b - 43c = d$	M1 B1
		Then $a - 3b + 2c = d$ and $3a + 4b + 6c = 0$ (λ terms) i.e. equating terms	M1
		Eliminating (e.g.) c from 1st two equations $\Rightarrow 9a + 10b = 0$	M1
		Choosing $a = 10$, $b = -9 \Rightarrow c = 1$ and $d = 39$ i.e. $10x - 9y + z = 39$ CAO	M1 A1

11	**(i)**	$\lvert w \rvert = \sqrt{\left(\sqrt{3}-1\right)^2 + \left(\sqrt{3}+1\right)} = \sqrt{4-2\sqrt{3}+4+2\sqrt{3}} = \sqrt{8}$ or $2\sqrt{2}$	M1 A1
		$\arg(w) = \tan^{-1}\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}+1}{\sqrt{3}+1}\right) = \tan^{-1}\left(2+\sqrt{3}\right) = \frac{5}{12}\pi$	M1 A1
	(ii) (a)	$z^3 = \left(2\sqrt{2},\ \frac{5}{12}\pi\right),\ \left(2\sqrt{2},\ \frac{29}{12}\pi\right),\ \left(2\sqrt{2},\ -\frac{19}{12}\pi \text{ or } \frac{53}{12}\pi\right)$	
		$\sqrt[3]{\lvert w \rvert}$; $\frac{\arg(w)}{3}$ These method marks can be earned for just the first root	M1 M1
		$\Rightarrow z = \left(\sqrt{2},\ \frac{5}{36}\pi\right),\ \left(\sqrt{2},\ \frac{29}{36}\pi\right),\ \left(\sqrt{2},\ -\frac{19}{36}\pi\right)$ **A** marks for the 2nd & 3rd roots: r e^(iθ) forms equally acceptable	A1 A1
	(b)	z_1, z_2, z_3 the roots of $z^3 - 0.z^2 + 0.z - w = 0$ $\Rightarrow z_1 z_2 z_3 = w = \left(\sqrt{3}-1\right) + i\left(\sqrt{3}+1\right)$ **ALT.** Multiplying the 3 roots together in any form	M1 A1
	(c)	 Three points in approx. correct places	M1
		All equally spaced around a circle, centre O, radius $\sqrt{2}$	M1
		(Explained that Δ_1 equilateral)	A1
		$l = 2\times\sqrt{2}\sin\left(\frac{1}{2}\times\frac{2}{3}\pi\right) = \sqrt{6}$ or by the *Cosine Rule*	M1 A1
	(d)	$k = \exp\left\{-i.\frac{5}{36}\pi\right\}$ or $\exp\left\{-i.\frac{29}{36}\pi\right\}$ or $\exp\left\{i.\frac{19}{36}\pi\right\}$	B1

12	(i)	$I_n = \int_0^3 x^{n-1}\left(x\sqrt{16+x^2}\right)\,\mathrm{d}x$ Correct splitting *and* use of parts	M1
		$= \left[x^{n-1}.\dfrac{(16+x^2)^{3/2}}{3}\right]_0^3 - \int_0^3 (n-1)x^{n-2}\,\dfrac{(16+x^2)^{3/2}}{3}\,\mathrm{d}x$	A1
		$= 3^{n-2}\cdot 125 - \left(\dfrac{n-1}{3}\right)\int_0^3 x^{n-2}(16+x^2)\sqrt{16+x^2}\,\mathrm{d}x$ Method to get 2nd integral of correct form	M1
		$= 3^{n-2}.125 - \left(\dfrac{n-1}{3}\right)\{16I_{n-2} + I_n\}$ [i.e. reverting to I's in 2nd integral **ft**]	M1
		$\Rightarrow 3\,I_n = 3^{n-1}.125 - 16(n-1)\,I_{n-2} - (n-1)\,I_n$ Collecting up I_ns	M1
		$(n+2)\,I_n = 125 \times 3^{n-1} - 16(n-1)\,I_{n-2}$ **AG**	A1
	(ii) (a)	 Spiral (with r increasing)	B1
		From O to just short of $\theta = \pi$	B1
	(b)	$r = \frac{1}{4}\theta^4 \Rightarrow \dfrac{\mathrm{d}r}{\mathrm{d}\theta} = \theta^3$ and $r^2 + \left(\dfrac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = \frac{1}{16}\theta^8 + \theta^6$	M1 A1
		$L = \int_0^3 \frac{1}{4}\theta^3\sqrt{16+\theta^2}\ \left(= \frac{1}{4}I_3\right)$	M1 A1
		Now $I_1 = \left[\dfrac{1}{3}(16+x^2)^{3/2}\right]_0^3 = \frac{61}{3}$	B1
		and $5I_3 = 125\times 9 - 16\times 2\left(\frac{61}{3}\right) = \frac{1423}{3}$ or $474\frac{1}{3}$ Use of given reduction formula	M1
		so that $L = \frac{1}{20}\times\frac{1423}{3} = \frac{1423}{60}$ or $23\frac{43}{60}$ or awrt 23.7 **ft** only from suitable $k\,I_3$	A1

13		Base-line case: for $n = 5$, 13579 R_5 = 1508 7 6269 contains a string of (5 – 4 = 1) 7s	B1
		13579 R_6 = 1508776269, 13579 R_7 = 15087776269, etc. or form of 1st & last 4 digits	B1
		Assume that, for some $k \geqslant 5$, 13579 R_k = 1508 $\underbrace{77\ldots7}_{(k-4)\text{ 7's}}$ 6269. Induction hypothesis	M1
		Then, for $n = k + 1$, 13579 R_{k+1} = 13579(10R_k + 1)	M1
		Give the **M** mark for the key observation that $R_{k+1} = 10R_k + 1$ **or** $10^k + R_k$, even if not subsequently used.	
		= 1508 $\underbrace{77\ldots7}_{(k-4)\text{ 7's}}$ 62690 + 13579 = 1508 $\underbrace{77\ldots7}_{(k-4+1)\text{ 7's}}$ 76269	A1
		which contains a string of (k – 4 + 1) 7s, as required. Proof follows by induction (usual round-up).	A1