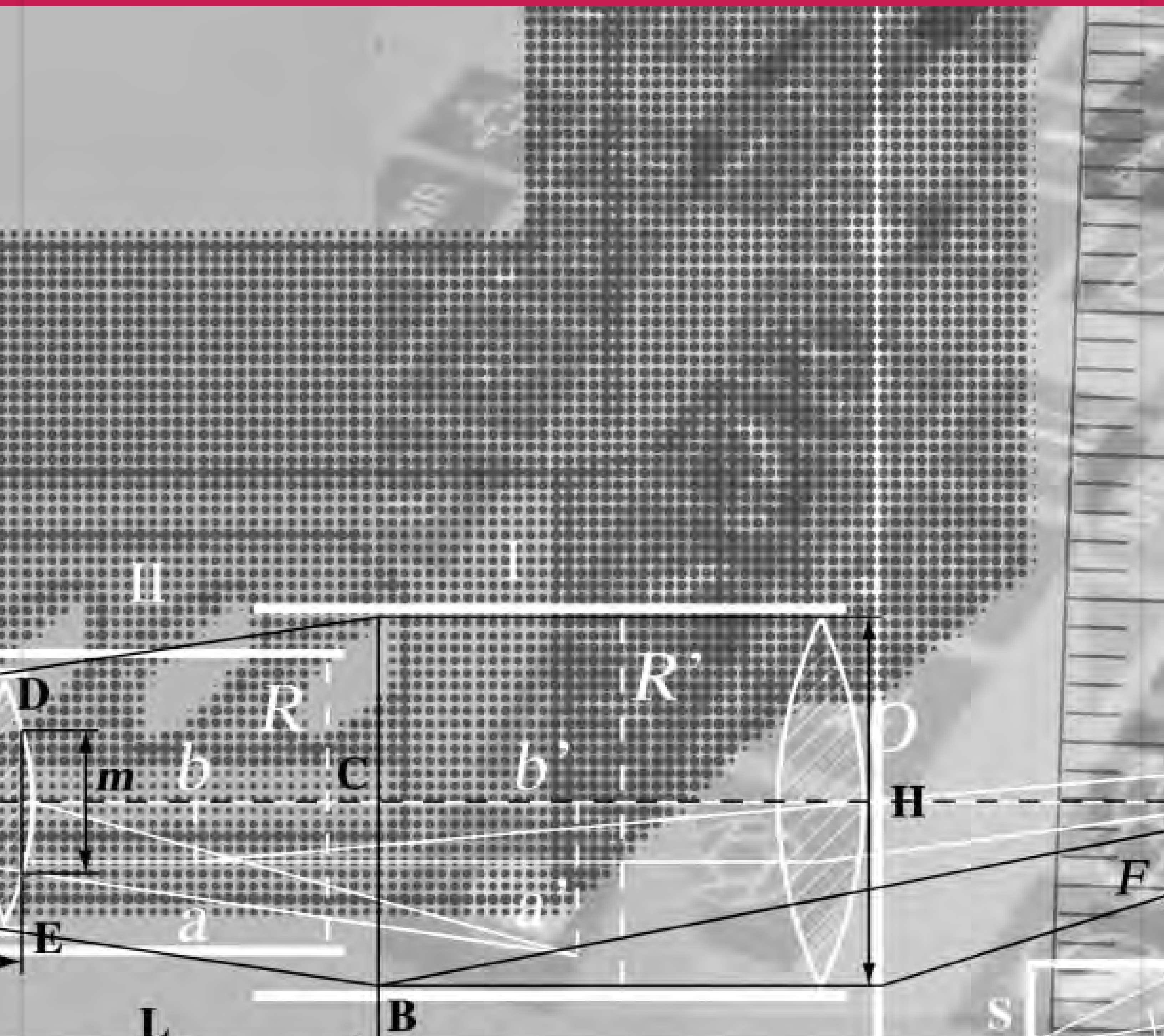
Cambridge Pre-U Specimen Papers
and Mark Schemes

Cambridge International Level 3
Pre-U Certificate in
FURTHER MATHEMATICS

For use from 2008 onwards



UNIVERSITY *of* CAMBRIDGE
International Examinations

Cambridge
Pre-U

Specimen Materials

Further Mathematics (9795)

Cambridge International Level 3
Pre-U Certificate in Further Mathematics (Principal)

For use from 2008 onwards

QAN 500/3829/1

Support

CIE provides comprehensive support for all its qualifications, including the Cambridge Pre-U. There are resources for teachers and candidates written by experts. CIE also endorses a range of materials from other publishers to give a choice of approach. More information on what is available for this particular syllabus can be found at www.cie.org.uk

Syllabus Updates

This booklet of specimen materials is for use from 2008. It is intended for use with the version of the syllabus that will be examined in 2010, 2011 and 2012. The purpose of these materials is to provide Centres with a reasonable idea of the general shape and character of the planned question papers in advance of the first operational examination.

If there are any changes to the syllabus CIE will write to centres to inform them. The syllabus and these specimen materials will also be published annually on the CIE website (www.cie.org.uk/cambridgepreu). The version of the syllabus on the website should always be considered as the definitive version.

Further copies of this, or any other Cambridge Pre-U specimen booklet, can be obtained by either downloading from our website www.cie.org.uk/cambridgepreu

or contacting:
Customer Services, University of Cambridge International Examinations,
1 Hills Road, Cambridge CB1 2EU
Telephone: +44 (0)1223 553554
Fax: +44 (0)1223 553558
E-mail: international@cie.org.uk

UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **For Examination from 2010**

SPECIMEN PAPER

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF16)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 120.

1 The region R of an Argand diagram is defined by the inequalities

$$0 \leqslant \arg(z + 4\mathrm{i}) \leqslant \tfrac{1}{4}\pi \quad \text{and} \quad |z| \leqslant 4.$$

Draw a clearly labelled diagram to illustrate R. [4]

2 It is given that

$$\mathrm{f}(n) = 7^n(6n + 1) - 1.$$

By considering $\mathrm{f}(n+1) - \mathrm{f}(n)$, prove by induction that $\mathrm{f}(n)$ is divisible by 12 for all positive integers n. [6]

3 Solve exactly the equation

$$5\cosh x - \sinh x = 7,$$

giving your answers in logarithmic form. [6]

4 Write down the sum

$$\sum_{n=1}^{2N} n^3$$

in terms of N, and hence find

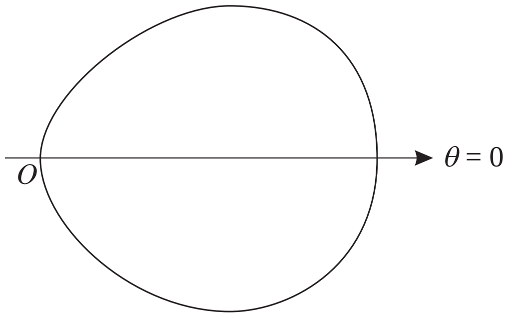
$$1^3 - 2^3 + 3^3 - 4^3 + \ldots - (2N)^3$$

in terms of N, simplifying your answer. [6]

5 Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 6\frac{\mathrm{d}y}{\mathrm{d}x} + 9y = 72\mathrm{e}^{3x}.$$ [7]

6



The diagram shows a sketch of the curve C with polar equation $r = a\cos^2\theta$, where a is a positive constant and $-\frac{1}{2}\pi \leqslant \theta \leqslant \frac{1}{2}\pi$.

(i) Explain briefly how you can tell from this form of the equation that C is symmetrical about the line $\theta = 0$ and that the tangent to C at the pole O is perpendicular to the line $\theta = 0$. [2]

(ii) The equation of C may be expressed in the form $r = \frac{1}{2}a(1 + \cos 2\theta)$. Using this form, show that the area of the region enclosed by C is given by

$$\frac{1}{16}a^2 \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} (3 + 4\cos 2\theta + \cos 4\theta)\,\mathrm{d}\theta,$$

and find this area. [6]

7 The equation

$$8x^3 + 12x^2 + 4x - 1 = 0$$

has roots α, β, γ. Show that the equation with roots $2\alpha + 1$, $2\beta + 1$, $2\gamma + 1$ is

$$y^3 - y - 1 = 0.$$ [3]

The sum $(2\alpha + 1)^n + (2\beta + 1)^n + (2\gamma + 1)^n$ is denoted by S_n. Find the values of S_3 and S_{-2}. [5]

8 The curve C has equation

$$y = \frac{x^2 - 2x - 3}{x + 2}.$$

(i) Find the equations of the asymptotes of C. [4]

(ii) Draw a sketch of C, which should include the asymptotes, and state the coordinates of the points of intersection of C with the x-axis. [5]

9 Given that $w_n = 3^{-n}\cos 2n\theta$ for $n = 1, 2, 3, \ldots$, use de Moivre's theorem to show that

$$1 + w_1 + w_2 + w_3 + \ldots + w_{N-1} = \frac{9 - 3\cos 2\theta + 3^{-N+1}\cos 2(N-1)\theta - 3^{-N+2}\cos 2N\theta}{10 - 6\cos 2\theta}. \qquad [7]$$

Hence show that the infinite series

$$1 + w_1 + w_2 + w_3 + \ldots$$

is convergent for all values of θ, and find the sum to infinity. [2]

10 **(a)** Find the inverse of the matrix $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & -1 \\ 3 & 8 & 2 \end{pmatrix}$, and hence solve the set of equations

$$\begin{aligned} x + 3y + 4z &= -5, \\ 2x + 5y - z &= 10, \\ 3x + 8y + 2z &= 8. \end{aligned}$$ [5]

(b) Find the value of k for which the set of equations

$$\begin{aligned} x + 3y + 4z &= -5, \\ 2x + 5y - z &= 15, \\ 3x + 8y + 3z &= k, \end{aligned}$$

is consistent. Find the solution in this case and interpret it geometrically. [5]

11 A group G has distinct elements $e, a, b, c, \ldots$, where e is the identity element and $\circ$ is the binary operation. Prove that if

$$a \circ a = b, \qquad b \circ b = a$$

then the set of elements $\{e, a, b\}$ forms a subgroup of G. [5]

Prove that if

$$a \circ a = b, \qquad b \circ b = c, \qquad c \circ c = a$$

then the set of elements $\{e, a, b, c\}$ does not form a subgroup of G. [5]

12 With respect to an origin O, the points A, B, C, D have position vectors

$$2\mathbf{i} - \mathbf{j} + \mathbf{k}, \quad \mathbf{i} - 2\mathbf{k}, \quad -\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}, \quad -\mathbf{i} + \mathbf{j} + 4\mathbf{k},$$

respectively. Find

(i) a vector perpendicular to the plane OAB, [2]

(ii) the acute angle between the planes OAB and OCD, correct to the nearest 0.1°, [3]

(iii) the shortest distance between the line which passes through A and B and the line which passes through C and D, [4]

(iv) the perpendicular distance from the point A to the line which passes through C and D. [3]

13 Given that $y = \cos\{\ln(1+x)\}$, prove that

(i) $(1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\sin\{\ln(1+x)\}$, [1]

(ii) $(1+x)^2\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + (1+x)\dfrac{\mathrm{d}y}{\mathrm{d}x} + y = 0$. [2]

Obtain an equation relating $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. [2]

Hence find Maclaurin's series for y, up to and including the term in x^3. [4]

Verify that the same result is obtained if the standard series expansions for $\ln(1+x)$ and $\cos x$ are used. [3]

14 Let $I_n = \displaystyle\int_1^{\mathrm{e}} (\ln x)^n \,\mathrm{d}x$, where n is a positive integer. By considering $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(x(\ln x)^n\right)$, or otherwise, show that

$$I_n = \mathrm{e} - nI_{n-1}.$$ [4]

Let $J_n = \dfrac{I_n}{n!}$. Show that

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + \frac{1}{10!} = \frac{1}{\mathrm{e}}(1 + J_{10}).$$ [6]

It can be shown that

$$\sum_{r=2}^{n} \frac{(-1)^r}{r!} = \frac{1}{\mathrm{e}}(1 + (-1)^n J_n)$$

for all positive integers n. Deduce the sum to infinity of the series

$$\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots ,$$

justifying your conclusion carefully. [3]

BLANK PAGE

UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **For Examination from 2010**

SPECIMEN MARK SCHEME

3 hours

MAXIMUM MARK: 120

This document consists of **5** printed pages and **1** blank page.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1	Show circle with centre O and radius 4 Show half-line from $-4\mathrm{i}$ upwards Show line at angle $\frac{1}{4}\pi$, i.e. passing through point $z = 4$ Indicate correct segment as region R	B1 B1 B1 B1	**4**
2	State or imply correct form for $\mathrm{f}(n+1)$ Obtain correct factorised simplification, e.g. $\mathrm{f}(n+1) - \mathrm{f}(n) = 7^n(36n+48)$ State $\mathrm{f}(1) = 48$ Express $\mathrm{f}(n+1)$ as $\mathrm{f}(n) + 12 \times 7^n(3n+4)$ Conclude that $\mathrm{f}(n)$ divisible by 12 $\Rightarrow$ $\mathrm{f}(n+1)$ divisible by 12 Complete the induction proof correctly	M1 A1 B1 M1 A1 A1	**6**
3	Rewrite equation in terms of e^x and e^{-x} Obtain simplified quadratic, e.g. $2(\mathrm{e}^x)^2 - 7\mathrm{e}^x + 3 = 0$ Solve quadratic for e^x Obtain $\mathrm{e}^x = 3$ and $\frac{1}{2}$ State answers $x = \ln 3$ and $x = \ln\frac{1}{2}$, or exact equivalents	M1 A1 M1 A1 A1, A1	**6**
4	Substitute $2N$ for n in standard formula $\frac{1}{4}n^2(n+1)^2$ Obtain $N^2(2N+1)^2$ Express given series as $1^3 + 2^3 + \dots + (2N)^3 - 2\left[2^3 + 4^3 + \dots + (2N)^3\right]$ State (at any stage) that $2^3 + 4^3 + \dots + (2N)^3 = 2^3(1^3 + 2^3 + \dots + N^3)$ Obtain expression $N^2(2N+1)^2 - 16 \times \frac{1}{4}N^2(N+1)^2$, or equivalent State answer as $-4N^3 - 3N^2$, or factorised equivalent	M1 A1 M1 B1 A1$\surd$ A1	**6**
5	Solve auxiliary equation $m^2 + 6m + 9 = 0$ Obtain solution $m = -3$ (repeated root) State complementary function as $(A + Bx)\mathrm{e}^{-3x}$ State correct form $\lambda\mathrm{e}^{3x}$ for particular integral Substitute PI completely in differential equation and equate coefficients Obtain $\lambda = 2$ State general solution $y = (A + Bx)\mathrm{e}^{-3x} + 2\mathrm{e}^{3x}$	M1 A1 A1 M1 M1 A1 A1	**7**
6	**(i)** Justify symmetry via $\cos^2\theta = \cos^2(-\theta)$ or equivalent Justify direction of tangent at O via $r = 0$ when $\theta = (\pm)\frac{1}{2}\pi$	B1 B1	**2**
	(ii) State or imply formula $\frac{1}{2}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} r^2\,\mathrm{d}\theta$, with correct limits Obtain $r^2 = \frac{1}{4}a^2(1 + 2\cos 2\theta + \cos^2 2\theta)$ Use double-angle formula $\cos^2 2\theta = \frac{1}{2}(1 + \cos 4\theta)$ Obtain given answer $\frac{1}{16}a^2\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}(3 + 4\cos 2\theta + \cos 4\theta)\,\mathrm{d}\theta$ correctly State indefinite integral $3\theta + 2\sin 2\theta + \frac{1}{4}\sin 4\theta$ Obtain answer $\frac{3}{16}\pi a^2$	B1 B1 M1 A1 B1* B1(dep*)	**6**

7	State relationship between new and old roots as $y = 2x + 1$	B1	
	Substitute $x = \dfrac{y-1}{2}$ in given cubic for x and simplify	M1	
	Obtain given cubic in y correctly	A1	**3**
	State $S_1 = 0$	B1	
	Deduce $S_3 = S_1 + 3 = 3$	B1	
	EITHER: Use $S_1 - S_{-1} - S_{-2} = 0$	M1	
	Evaluate $S_{-1} = \dfrac{-1}{1} = -1$	M1	
	Obtain $S_{-2} = 1$ correctly	A1	
	OR: Substitute $z = \dfrac{1}{y}$ to obtain cubic in z	M1	
	Evaluate sums of squares of roots for $z^3 + z^2 - 1 = 0$	M1	
	Obtain $S_{-2} = (-1)^2 - 2 \times 0 = 1$ correctly	A1	**5**
8	**(i)** State that $x = -2$ is an asymptote	B1	
	Attempt to express $\dfrac{x^2 - 2x - 3}{x+2}$ in quotient-remainder form	M1	
	Obtain correct expression $x - 4 + \dfrac{5}{x+2}$	A1	
	State that $y = x - 4$ is an asymptote	A1	**4**
	(ii) State (or label on sketch) intersections with x-axis at $(-1, 0)$ and $(3, 0)$	B1	
	Show asymptotes on sketch located correctly	B1, B1	
	Show right-hand branch correctly located, with correct approaches to asymptotes	B1	
	Show left-hand branch correctly located, with correct approaches to asymptotes	B1	**5**
9	Consider complex series $1 + \frac{1}{3}z + \left(\frac{1}{3}z\right)^2 + \ldots \left(\frac{1}{3}z\right)^{N-1}$, where $z = e^{2i\theta}$ or equivalent	M1	
	Use GP sum formula to obtain $\dfrac{1 - \left(\frac{1}{3}z\right)^N}{1 - \frac{1}{3}z}$, or equivalent	A1	
	Use appropriate $1 - \frac{1}{3}z^*$ to produce real denominator	M1	
	Obtain $\dfrac{9\left\{1 - \left(\frac{1}{3}z\right)^N\right\}\left\{1 - \frac{1}{3}z^*\right\}}{10 - 6\cos 2\theta}$, or equivalent	A1	
	Expand the numerator, and simplify the term involving $z^N z^*$ appropriately	M1	
	Calculate the real part of the numerator	M1	
	Obtain the given answer $\dfrac{9 - 3\cos 2\theta + 3^{-N+1}\cos 2(N-1)\theta - 3^{-N+2}\cos 2N\theta}{10 - 6\cos 2\theta}$ correctly	A1	**7**
	State that terms in 3^{-N+1} and 3^{-N+2} tend to zero as $N \to \infty$	M1	
	State sum to infinity is $\dfrac{9 - 3\cos 2\theta}{10 - 6\cos 2\theta}$	A1	**2**
10	**(a)** Obtain correct value 1 for the determinant of the matrix	B1	
	Show or imply correct process for obtaining inverse matrix	M1	
	Obtain correct inverse matrix $\begin{pmatrix} 18 & 26 & -23 \\ -7 & -10 & 9 \\ 1 & 1 & -1 \end{pmatrix}$	A1	
	Form product of inverse matrix and RHS column vector	M1	
	Obtain correct solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -14 \\ 7 \\ -3 \end{pmatrix}$	A1	**5**
	(b) Add the first two equations, and obtain $k = 10$	B1	
	Solve a pair of the equations simultaneously	M1	
	Obtain $z = t$, $y = -25 - 9t$, $x = 70 + 23t$, or any equivalent form	A1, A1	
	State that the solution represents the common line of intersection of three planes	A1	**5**

11	State that valid group table requires $a \circ b = b \circ a = e$	M1	
	Show (e.g.) that $a \circ (a \circ b) = (a \circ a) \circ b = b \circ b = a$	M1	
	Deduce that $a \circ b = e$	A1	
	Show similarly (e.g.) that $b \circ (b \circ a) = b$	M1	
	Deduce that $b \circ a = e$	A1	**5**
	EITHER: State that valid group table requires either $a \circ b = e$ or $a \circ c = e$	M1	
	Assume $a \circ b = e$ and deduce that $a \circ c = b$	M1	
	State that $a \circ c$ and $a \circ a$ are both equal to b and obtain contradiction	A1	
	Assume instead that $a \circ c = e$ and deduce that $b \circ c = a$	M1	
	Obtain corresponding contradiction	A1	
	[Alternative ways of obtaining contradictions are possible]		
	OR: State that there are precisely 2 distinct groups of order 4	M2	
	State that one of these has 1 self-inverse element	A1	
	State that the other of these has 3 self-inverse elements	A1	
	Conclude that the set in question cannot be a group as it has no such elements	A1	**5**
12	**(i)** Attempt vector product $\overrightarrow{OA} \times \overrightarrow{OB}$, or equivalent	M1	
	Obtain answer $2\mathbf{i} + 5\mathbf{j} + \mathbf{k}$	A1	**2**
	(ii) State that normal vector for plane OCD is $10\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ (or e.g. half of this)	B1	
	Use the scalar product of the two normal vectors	M1	
	Obtain answer 55.8°	A1	**3**
	(iii) Calculate the common perpendicular $\mathbf{n} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{c})$	M1	
	Obtain $\mathbf{n} = -4\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$, or any multiple of this	A1	
	Calculate $\dfrac{\lvert(\mathbf{c} - \mathbf{a}).\mathbf{n}\rvert}{\lvert\mathbf{n}\rvert}$ or $\dfrac{\lvert(\mathbf{d} - \mathbf{b}).\mathbf{n}\rvert}{\lvert\mathbf{n}\rvert}$, or equivalent	M1	
	Obtain answer $\dfrac{11}{\sqrt{6}}$, or equivalent	A1	**4**
	(iv) Calculate $(\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{c})$ or $(\mathbf{d} - \mathbf{a}) \times (\mathbf{d} - \mathbf{c})$, or equivalent	M1	
	Divide the magnitude of this by the magnitude of $(\mathbf{d} - \mathbf{c})$	M1	
	Obtain answer $\frac{1}{2}\sqrt{86}$, or equivalent	A1	**3**
	[In all parts of the question, longer methods can score full credit if carried out correctly.]		
13	**(i)** Derive or verify given answer $(1+x)\dfrac{dy}{dx} = -\sin\{\ln(1+x)\}$ correctly	B1	**1**
	(ii) State $(1+x)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = \dfrac{-\cos\{\ln(1+x)\}}{1+x}$	B1	
	Obtain given answer $(1+x)^2\dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 0$	B1	**2**
	Differentiate the equation in part **(ii)**, including use of product rule	M1	
	Obtain $(1+x)^2\dfrac{d^3y}{dx^3} + 3(1+x)\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = 0$	A1	**2**
	Substitute $x = 0$ to evaluate y and its first three derivatives	M1	
	Obtain correct values 1, 0, −1, 3	A1	
	Use the numerical derivatives to produce terms of the Maclaurin series	M1	
	Obtain $1 - \frac{1}{2}x^2 + \frac{1}{2}x^3$	A1	**4**
	State either $\cos\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \ldots\right)$ or $1 - \frac{1}{2}\{\ln(1+x)\}^2 + \ldots$	B1	
	Obtain $1 - \frac{1}{2}\left(x - \frac{1}{2}x^2 + \ldots\right)^2 + \ldots$	B1	
	Obtain $1 - \frac{1}{2}x^2 + \frac{1}{2}x^3$ correctly	B1	**3**

14	Differentiate $x(\ln x)^n$ as a product	M1	
	Obtain $(\ln x)^n + n(\ln x)^{n-1}$	A1	
	Deduce $\int_1^e (\ln x)^n \, dx + n\int_1^e (\ln x)^{n-1} \, dx = \left[x(\ln x)^n\right]_1^e$	M1	
	Obtain given result correctly	A1	**4**
	State or imply (at any stage) that $J_n = \frac{e}{n!} - J_{n-1}$	B1	
	Relate J_{10} or I_{10} to J_9 or I_9 respectively	M1	
	Continue the process downwards	M1	
	State $J_{10} = e\left(\frac{1}{10!} - \frac{1}{9!} + \ldots + \frac{1}{2!}\right) - J_1$	A1	
	Evaluate J_1 (or I_1) as 1	B1	
	Rearrange and obtain given result	A1	**6**
	State that the sum to infinity is $\frac{1}{e}$	B1	
	State that J_n tends to zero because $n!$ becomes large while I_n remains bounded	B1, B1	**3**

BLANK PAGE

UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/02**

Paper 2 Further Applications of Mathematics **For Examination from 2010**

SPECIMEN PAPER

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF16)

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
Where a numerical value for the acceleration due to gravity is needed, use $10\,\mathrm{m\,s^{-2}}$.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 120.

UNIVERSITY of CAMBRIDGE International Examinations

Section A: Mechanics (59 marks)

1 A car of mass 1500 kg has a maximum speed of $24\,\mathrm{m\,s^{-1}}$ when moving along a straight horizontal road with its engine working at its full power of 18 kW. The resistance to motion is proportional to the speed. The car is moving up a straight road inclined at $\sin^{-1}\left(\frac{1}{25}\right)$ to the horizontal with the engine working at full power. Find the acceleration of the car when its speed is $10\,\mathrm{m\,s^{-1}}$. [6]

2 At noon a radar operator on board a ship, which is travelling due north at $30\,\mathrm{km\,h^{-1}}$, detects another vessel 40 km away on a bearing of 060°. The other vessel is travelling at $20\,\mathrm{km\,h^{-1}}$ on a bearing of 300°. Find, to the nearest minute, the time at which the ship and the other vessel are closest together. [7]

3 A goods lift starts from rest at A and rises vertically. It comes to rest at B having moved a distance of 40 m. The motion of the lift is modelled as simple harmonic with period 10 s.

(i) Find the time taken for the lift to move the first 16 m and the speed at the end of that time. [6]

(ii) Show that, during the motion, a crate inside the lift will not leave the floor of the lift. [3]

4 Starting from rest, a cyclist sets off along a horizontal road, pedalling so that there is a forward force acting of constant magnitude T N. When her speed is $v\,\mathrm{m\,s^{-1}}$, the resistance to motion has magnitude kv^2 N, where k is a positive constant. Show that, for the motion,

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = T - kv^2,$$

where m kg is the mass of the cyclist and her bicycle, and x m is the distance she has travelled. [1]

By solving this differential equation, show that

$$v^2 = \frac{T}{k}\left(1 - \mathrm{e}^{-\frac{2kx}{m}}\right).$$ [7]

Along this road the cyclist's speed approaches a limiting value. Find this value in terms of the given quantities. [2]

5 A small bead B of mass m rests at the lowest point A of the inside of a fixed smooth hollow sphere with centre O and radius a. The bead is given a horizontal velocity u, and in the subsequent motion air resistance is neglected.

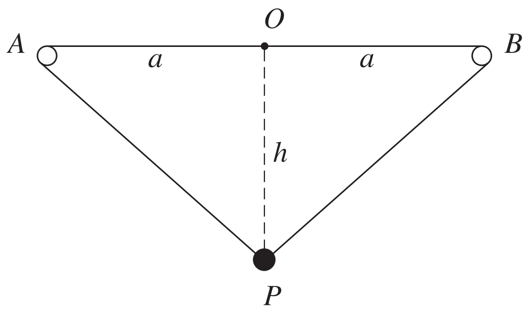
(i) Show that, when the bead is on the inner surface of the sphere with angle $AOB = \theta$, the normal contact force acting on the bead has magnitude

$$m\left(\frac{u^2}{a} + 3g\cos\theta - 2g\right). \qquad [4]$$

(ii) Show that, if $u < \sqrt{(2ag)}$, the bead does not reach the level of O. [2]

(iii) For the case $u = 2\sqrt{(ag)}$, find the speed of the bead when it loses contact with the sphere. Find also the greatest height above the level of A reached in the subsequent motion. [You may assume that the bead reaches its greatest height before hitting the sphere.] [6]

6



The point O is mid-way between two small smooth pegs A and B which are fixed at the same horizontal level a distance $2a$ apart. Two light elastic strings, each of natural length a and modulus of elasticity λ, have one end fixed at O and are attached at the other end to a particle P of mass m. One of the strings passes over peg A and the other passes over peg B. The particle hangs in equilibrium at a distance h vertically below O, as shown in the diagram. Express the tension in each string in terms of λ, a and h, and show that

$$h = \frac{mga}{2\lambda}. \qquad [5]$$

The particle is held at O, and released from rest. In the subsequent motion any resistances may be neglected.

(i) Express in terms of λ and a the total elastic potential energy in the strings at the instant when the particle is released. [2]

(ii) Show that, when the particle is at its lowest point, $OP = 2h$. [4]

(iii) Express in terms of m, g, a and λ the speed of P as it passes the equilibrium position. [4]

Section B: Probability (61 marks)

7 The continuous random variable Y has cumulative distribution function given by

$$\mathrm{F}(y) = \begin{cases} 0 & y < 0, \\ 1 - (1-y)^3 & 0 \leqslant y \leqslant 1, \\ 1 & y > 1. \end{cases}$$

Find

(i) $\mathrm{P}(Y < 0.5)$, [1]

(ii) the lower quartile of Y, [2]

(iii) the probability density function of Y, [2]

(iv) $\mathrm{E}(Y)$. [2]

8 **(i)** In an experiment, a fair coin is tossed 80 times. Use an appropriate normal distribution to estimate the probability that the number of heads obtained is at least 50. [5]

(ii) The experiment described in part **(i)** is repeated on 60 occasions. Use an appropriate approximation to estimate the probability that at least 50 heads are obtained on at most two occasions. [4]

9 A population has variance σ^2; $X_1, X_2, \ldots, X_n$ is a random sample drawn from the population, and $\overline{X} = \frac{1}{n}(X_1 + X_2 + \ldots + X_n)$ denotes the sample mean. Show that $\mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}$. [3]

A librarian travels to work by train each weekday morning, a journey which takes X minutes, where X is a random variable which is normally distributed with standard deviation 2 minutes. A random sample of 25 journeys gave $\Sigma x = 1450$. Calculate a symmetrical 98% confidence interval for the librarian's mean journey time, giving your limits to 2 decimal places. [3]

Comment on a claim made by the librarian that, on average, the train journey time is at least one hour. [1]

Find the smallest sample size so that a symmetrical 95% confidence interval for the mean journey time has a width of at most one minute. [3]

10 A keyboard operator estimates that, when inputting a page of data, she makes exactly one error with probability $\frac{2}{9}$ and exactly two errors with probability $\frac{1}{30}$. Assuming that the number of errors that she makes when inputting a page of data has a Poisson distribution, estimate the mean, and verify that the Poisson distribution with this mean produces probabilities close to the keyboard operator's estimates. [5]

Using this Poisson distribution, and assuming independence between pages, find the conditional probability that she makes more than 3 errors when inputting two pages of data given that she makes at least 1 error. [5]

11 An unbiased octahedral die has one face numbered 1, one face numbered 4, three faces numbered 2 and three faces numbered 3. When the die is thrown once on to a horizontal table the score X is the number on the face in contact with the table.

(i) Show that the probability generating function of X can be written in the form $\frac{1}{8}t(1+t)^3$. [3]

(ii) Use the probability generating function to find the mean and variance of X. [5]

(iii) The die is thrown four times and the sum of the scores obtained is Y. Write down the probability generating function of Y, and find $\mathrm{P}(Y = 10)$. [4]

12 Two friends, Ali and Bernard, leave their homes at points A and B respectively each day and travel towards each other's homes along the same road AB. The point C is halfway along AB. Four random variables are defined as follows.

L_A : the time at which Ali leaves A;
T_A : the time that Ali takes to travel from A to C;
L_B : the time at which Bernard leaves B;
T_B : the time that Bernard takes to travel from B to C.

These variables are independent and have normal distributions with means and standard deviations given in the following table.

	Mean	Standard deviation
L_A	8 o'clock	5 minutes
T_A	10 minutes	1 minute
L_B	2 minutes past 8	4 minutes
T_B	9 minutes	2 minutes

(i) Find the probability that Ali reaches C later than 12 minutes past 8. [4]

(ii) Show that the probability that Ali reaches C before Bernard is 0.56, correct to 2 significant figures. [5]

(iii) State one place in your working where you have used the fact that the variables are independent. [1]

Assuming that the time that Ali takes to travel from A to B is equal to $2T_A$, calculate the probability that Ali arrives at B at least 10 minutes after Bernard leaves B. [3]

BLANK PAGE

UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/02**

Paper 2 Further Applications of Mathematics **For Examination from 2010**

SPECIMEN MARK SCHEME

3 hours

MAXIMUM MARK: 120

This document consists of **5** printed pages and **1** blank page.

UNIVERSITY *of* CAMBRIDGE
International Examinations

1	State that resistance at speed 24 is $\frac{18\,000}{24}$ (= 750)	M1	
	Calculate resistance at speed 10 as $\frac{10}{24} \times 750$	M1	
	State or imply correct value 312.5	A1	
	Use Newton II for motion up the hill	M1	
	State correct equation, e.g. $\frac{18\,000}{10} - 312.5 - \frac{1}{25} \times 1500g = 1500a$	A1	
	Obtain answer $0.592\,\text{m s}^{-2}$	A1	**6**
2	*EITHER*: Express position vectors in terms of t and subtract	M1	
	Obtain relative displacement $\mathbf{r} = (20\sqrt{3} - 10\sqrt{3}t)\mathbf{i} + (20 - 20t)\mathbf{j}$, or equivalent	A1	
	Subtract velocity vectors	M1	
	Obtain relative velocity $\mathbf{v} = -10\sqrt{3}\mathbf{i} - 20\mathbf{j}$, or equivalent	A1	
	Use condition $\mathbf{r.v} = 0$ for closest approach	M1	
	Obtain $-600 + 300t - 400 + 400t = 0$ (giving $t = 1.429$)	A1	
	Deduce that closest approach occurs at 1326 hrs	A1	
	OR: Use correct velocity triangle (sides 30, 20 and included angle 60°)	M1	
	Obtain relative speed $\sqrt{700}$	A1	
	Use appropriate trigonometry to find direction of relative velocity	M1	
	Obtain correct angle 79.1066 . . . °, or equivalent	A1	
	Identify relevant angle 19.1066 . . . ° for closest approach calculation	M1	
	Calculate time as $\frac{40\cos 19.1066\ldots^\circ}{\sqrt{700}}$	A1	
	Deduce that closest approach occurs at 1326 hrs	A1	**7**
3	**(i)** State or imply $\omega = \frac{2\pi}{10}$	B1	
	Use equation $x = a\cos\omega t$, or equivalent	M1	
	Solve with $x = 4$, $a = 20$ to obtain $\omega t = \cos^{-1}(0.2)$	A1	
	Obtain answer 2.18 s	A1	
	Calculate speed from $a\omega\sin\omega t$, or equivalent	M1	
	Obtain answer $12.3\,\text{m s}^{-1}$	A1	**6**
	(ii) State or imply 3-term Newton II equation $R - mg = mf$	M1	
	Identify greatest downwards acceleration as $\omega^2 \times a$	M1	
	Justify given result, via $g = 10 > 7.89... = \omega^2 a$	A1	**3**
4	Use Newton II and $v\frac{\mathrm{d}v}{\mathrm{d}x}$ for acceleration to obtain given DE correctly	B1	**1**
	Separate the variables correctly, e.g. $\int \mathrm{d}x = \int \frac{mv}{T - kv^2}\,\mathrm{d}v$	B1	
	Attempt integration of both sides	M1*	
	Obtain both x and $-\frac{m}{2k}\ln(T - kv^2)$, or equivalent	A1	
	Use $x = 0$, $v = 0$ to evaluate a constant of integration, or as limits	M1(dep*)	
	Obtain $x = \frac{m}{2k}\ln T - \frac{m}{2k}\ln(T - kv^2)$, or equivalent	A1	
	Transform equation to exponential form, using correct methods	M1(dep*)	
	Obtain given answer $v^2 = \frac{T}{k}\left(1 - \mathrm{e}^{-2kx/m}\right)$ correctly	A1	**7**
	State or imply that $\mathrm{e}^{-2kx/m} \to 0$ as $x \to \infty$ or that $v\frac{\mathrm{d}v}{\mathrm{d}x} \to 0$ as $x \to \infty$	M1	
	Obtain limiting value $\sqrt{\left(\frac{T}{k}\right)}$	A1	**2**

5	**(i)** Use conservation of energy	M1	
	Obtain $v^2 = u^2 - 2ga(1 - \cos\theta)$	A1	
	Use Newton II radially, with acceleration $\dfrac{v^2}{a}$	M1	
	Obtain given answer $R = m\left(\dfrac{u^2}{a} + 3g\cos\theta - 2g\right)$ correctly	A1	**4**
	(ii) Substitute $v = 0$, giving $\cos\theta = 1 - \dfrac{u^2}{2ga}$	M1	
	Relate $u < \sqrt{(2ag)}$ to $\theta < \frac{1}{2}\pi$ and confirm given result	A1	**2**
	(iii) Substitute $R = 0$, $u^2 = 4ag$ and solve for θ	M1	
	Obtain $\cos\theta = -\frac{2}{3}$	A1	
	Deduce that $v = \sqrt{\left(\frac{2}{3}ag\right)}$	A1	
	Calculate vertical component of velocity at loss of contact as $v\sin\theta$	M1	
	Calculate greatest height above loss of contact as $\dfrac{v^2\sin^2\theta}{2g}$, or equivalent	M1	
	Obtain greatest height above A as $\frac{50}{27}a$	A1	**6**
6	Use correct Hooke's law, with attempt at the extension based on the data	M1	
	State $T = \dfrac{\lambda\sqrt{(a^2 + h^2)}}{a}$	A1	
	Resolve vertically at P, i.e. $2T\cos\theta = mg$	M1	
	State $\cos\theta = \dfrac{h}{\sqrt{(a^2 + h^2)}}$	B1	
	Substitute for T and $\cos\theta$ and obtain given result	A1	**5**
	(i) Use correct elastic energy formula, with $x = a$	M1	
	State answer λa	A1	**2**
	(ii) Equate total energy at top and bottom (must include P.E. and E.E.)	M1	
	Obtain equation $\dfrac{\lambda(a^2 + x^2)}{a} - mgx = \lambda a$, or equivalent	A1$\surd$	
	Use relation between m, g, a, h, λ to simplify, and solve for x	M1	
	Obtain $x = 2h$ correctly	A1	**4**
	(iii) Attempt energy equation with P.E., E.E. and K.E. relating two relevant positions	M1*	
	Obtain correct equation, e.g. $\frac{1}{2}mv^2 + \dfrac{\lambda(a^2 + h^2)}{a} - mgh = \lambda a$	A1	
	Eliminate h	M1(dep*)	
	Obtain answer $v = \sqrt{\left(\dfrac{mg^2a}{2\lambda}\right)}$	A1	**4**
7	**(i)** State $\mathrm{P}(Y < 0.5) = \mathrm{F}(0.5) = \frac{7}{8}$	B1	**1**
	(ii) State equation $1 - (1 - y)^3 = \frac{1}{4}$, or equivalent	M1	
	Obtain answer $1 - \sqrt[3]{\left(\frac{3}{4}\right)}$, or equivalent ($\approx 0.091$)	A1	**2**
	(iii) State or imply that $\mathrm{f}(y) = \mathrm{F}'(y)$	M1	
	Obtain answer $3(1 - y)^2$ (for $0 \leqslant y \leqslant 1$, and 0 otherwise)	A1	**2**
	(iv) Evaluate $\displaystyle\int_0^1 3y(1 - y)^2\,\mathrm{d}y$	M1	
	Obtain correct value $\frac{1}{4}$	A1	**2**

8	**(i)** Use $\mu = 40$ and $\sigma = \sqrt{20}$	B1, B1	
	Standardise $\dfrac{49.5 - 40}{\sqrt{20}}$ (with or without continuity correction at this stage)	M1	
	Obtain correct z-value 2.124	A1	
	Obtain correct probability 0.0168 from normal tables	A1	**5**
	(ii) Evaluate $60 \times 0.0168 = 1.008$	B1$\surd$	
	Use Poisson distribution with this mean	M1	
	State or imply correct expression $e^{-1.008}\left(1 + 1.008 + \dfrac{1.008^2}{2!}\right)$	A1$\surd$	
	Obtain correct probability 0.918	A1	**4**
9	State $\text{Var}(\overline{X}) = \dfrac{1}{n^2}\text{Var}(X_1 + X_2 + \ldots + X_n)$	M1	
	Use addition of variances to obtain $\dfrac{1}{n^2}(\sigma^2 + \sigma^2 + \ldots + \sigma^2)$	M1	
	Obtain given answer $\dfrac{\sigma^2}{n}$ correctly	A1	**3**
	State interval of the form $\bar{x} \pm \dfrac{z\sigma}{\sqrt{n}}$	M1	
	Show correct values, i.e. $\dfrac{1450}{25} \pm 2.326 \times \dfrac{2}{\sqrt{25}}$	A1	
	Obtain correct interval (57.07, 58.93)	A1	**3**
	State that the claim seems unjustified, as 60 is outside the calculated interval	B1$\surd$	**1**
	State inequality of the form $2z \times \dfrac{\sigma}{\sqrt{n}} \leqslant 1$	M1	
	Show correct values, i.e. $2 \times 1.96 \times \dfrac{2}{\sqrt{n}} \leqslant 1$	A1	
	Obtain integer answer 62 correctly	A1	**3**
10	State equations $\mu e^{-\mu} = \frac{2}{9}$ and $\frac{1}{2}\mu^2 e^{-\mu} = \frac{1}{30}$	B1, B1	
	Eliminate $e^{-\mu}$ and deduce that $\mu = 0.3$	B1	
	Evaluate both $0.3 \times e^{-0.3}$ and $\frac{1}{2} \times 0.3^2 \times e^{-0.3}$	M1	
	Demonstrate close agreement, e.g. $0.222\,25 \approx 0.222\,22$ and $0.033\,34 \approx 0.033\,33$	A1	**5**
	Use Poisson distribution with mean 0.6	M1	
	Attempt to evaluate $\dfrac{\text{P(more than 3 errors)}}{\text{P(more than 0 errors)}}$	M1	
	State correct numerator, i.e. $1 - e^{-0.6}\left(1 + 0.6 + \dfrac{0.6^2}{2} + \dfrac{0.6^3}{6}\right)$	A1	
	State correct denominator, i.e. $1 - e^{-0.6}$	A1	
	Obtain correct conditional probability 0.007 44	A1	**5**

11 **(i)**	State probabilities $p_1 = \frac{1}{8}$, $p_2 = \frac{3}{8}$, $p_3 = \frac{3}{8}$, $p_4 = \frac{1}{8}$	B1	
	State generating function $\mathrm{G}(t)$ is $\frac{1}{8}t + \frac{3}{8}t^2 + \frac{3}{8}t^3 + \frac{1}{8}t^4$	M1	
	Simplify to given answer $\frac{1}{8}t(1+t^3)$ correctly	A1	**3**
(ii)	Differentiate $\mathrm{G}(t)$	M1	
	Obtain $\frac{1}{8}(1+t)^3 + \frac{3}{8}t(1+t)^2$, or equivalent	A1	
	Substitute $t = 1$ and obtain $\mathrm{E}(X) = \frac{5}{2}$	A1	
	Use correct formula $\mathrm{G}''(1) + \mathrm{G}'(1) - [\mathrm{G}'(1)]^2$	M1	
	Obtain $\mathrm{Var}(X) = \frac{3}{4}$	A1	**5**
(iii)	State that $\mathrm{G}_Y(t) = \left[\frac{1}{8}t(1+t)^3\right]^4$	B1	
	State or imply that the required probability is the coefficient of t^{10} in $\mathrm{G}_Y(t)$	B1	
	Evaluate the coefficient of t^6 in $(1+t)^{12}$, i.e. $\binom{12}{6} = 924$	M1	
	Obtain correct answer $\frac{231}{1024}$, or equivalent	A1	**4**
12 **(i)**	State that the mean arrival time is 0810, or equivalent	B1	
	State that the relevant variance is 26	B1	
	Standardise using 0812, and attempt to find upper tail normal probability	M1	
	Obtain correct answer 0.348	A1	**4**
(ii)	Attempt relevant mean and variance, using all the data in the table	M1	
	Obtain mean of ±1 and variance 46	A1, A1	
	Evaluate $\Phi\left(\frac{1}{\sqrt{46}}\right)$	M1	
	Obtain given answer 0.56 following correct z-value 0.147...	A1	**5**
(iii)	Refer correctly to the variance of a sum or difference used	B1	**1**
	State that the relevant variance is 45	B1	
	Use standardised value $(\pm)\frac{10-18}{\sqrt{45}}$	M1	
	Obtain correct answer 0.884	A1	**3**

BLANK PAGE

MARK SCHEME NOTES

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

1. Marks are of the following three types.

   M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula.

   A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated Method mark is earned (or implied).

   B Mark for a correct result or statement independent of Method marks.

   The marks indicated in the scheme may not be subdivided. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

2. When a part of a question has two or more 'method' steps, the M marks are in principle independent unless the scheme specifically says otherwise; and similarly where there are several B marks allocated. The notation 'dep*' is used to indicate that a particular M or B mark is dependent on an earlier, asterisked, mark in the scheme. Of course, in practice it may happen that when a candidate has once gone wrong in a part of a question, the work from there on is worthless so that no more marks can sensibly be given. On the other hand, when two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

3. The symbol $\sqrt{}$ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A and B marks are given for correct work only — differences in notation are of course permitted. A and B marks are not given for 'correct' answers or results obtained from incorrect working. When A or B marks are awarded for work at an intermediate stage of a solution, there may be various alternatives that are equally acceptable.

4. Where alternative methods of solution, not covered in the mark scheme, are used, full marks will be given for a correct result obtained by any valid method, with equivalent partial credit for equivalent stages. (This does not however apply if candidates are directed in the question to use a particular method.)

5. The following abbreviations may be used in a mark scheme.

   AEF Any Equivalent Form (of answer or result is equally acceptable).

   AG Answer Given on the question paper (so extra care is needed in checking that the detailed working leading to the result is valid).

   BOD Benefit Of Doubt (allowed for work whose validity may not be absolutely plain).

   CAO Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed).

   ISW Ignore Subsequent Working.

   MR Misread.

   PA Premature Approximation (resulting in basically correct work that is numerically insufficiently accurate).

   SOS See Other Solution (the candidate makes a better attempt at the same question).

   SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance).

University of Cambridge International Examinations
1 Hills Road, Cambridge, CB1 2EU, United Kingdom
Tel: +44 1223 553554 Fax: +44 1223 553558
Email: international@cie.org.uk Website: www.cie.org.uk

PART OF THE
CAMBRIDGE ASSESSMENT
GROUP