Cambridge Pre-U

UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS
Cambridge International Level 3 Pre-U Certificate
Principal Subject

FURTHER MATHEMATICS **9795/01**

Paper 1 Further Pure Mathematics **May/June 2011**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF20)

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READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an electronic calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 120.

This document consists of **4** printed pages.

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International Examinations

[Turn over

1 Given that the matrix $\mathbf{A} = \begin{pmatrix} 2 & k \\ 1 & -3 \end{pmatrix}$, where k is real, is such that $\mathbf{A}^3 = \mathbf{I}$, find the value of k and the numerical value of $\det \mathbf{A}$. [4]

2 The cubic equation $x^3 + x^2 + 7x - 1 = 0$ has roots α, β and γ.

(i) Show that $\alpha^2 + \beta^2 + \gamma^2 = -13$. [3]

(ii) State what can be deduced about the nature of these roots. [2]

3 **(i)** Express $\mathrm{f}(r-1) - \mathrm{f}(r)$ as a single algebraic fraction, where $\mathrm{f}(r) = \dfrac{1}{(2r+1)^2}$. [1]

(ii) Hence, using the method of differences, show that

$$\sum_{r=1}^{n} \frac{r}{(4r^2-1)^2} = \frac{n(n+1)}{2(2n+1)^2}$$

for all positive integers n. [4]

4 **(i)** On a single diagram, sketch the graphs of $y = \tanh x$ and $y = \cosh x - 1$, and use your diagram to explain why the equation $\mathrm{f}(x) = 0$ has exactly two roots, where

$$\mathrm{f}(x) = 1 + \tanh x - \cosh x.$$ [3]

(ii) The non-zero root of $\mathrm{f}(x) = 0$ is α.

(a) Verify that $1 < \alpha < 1.5$. [1]

(b) Taking $x_1 = 1.25$ as an initial approximation to α, use the Newton-Raphson iterative method to find x_3, giving your answer to 5 decimal places. [4]

5 Find the general solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y = 8x^2$. [7]

6 Consider the set S of all matrices of the form $\begin{pmatrix} p & p \\ p & p \end{pmatrix}$, where p is a non-zero rational number.

(i) Show that S, under the operation of matrix multiplication, forms a group, G. [5]

(ii) Find a subgroup of G of order 2 and show that G contains no subgroups of order 3. [4]

7 Sketch the curve with equation $y = \dfrac{x^2 + 4x}{2x - 1}$, justifying all significant features. [11]

8 **(i)** Determine the two values of k for which the system of equations

$$\begin{aligned} x + 2y + 3z &= 4 \\ 2x + 3y + kz &= 9 \\ x + ky + 6z &= 1 \end{aligned}$$

has no unique solution. [3]

(ii) Show that the system is consistent for one of these values of k and inconsistent for the other. [4]

9 **(a)** The points A, B and C have position vectors

$$\mathbf{a} = \begin{pmatrix} 19 \\ 3 \\ 10 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 12 \\ 7 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{c} = \begin{pmatrix} 5 \\ 15 \\ 3 \end{pmatrix}$$

respectively, and O is the origin. Calculate the volume of the tetrahedron $OABC$. [3]

(b) **(i)** The plane Π_1 has equation $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix}$. Determine an equation for Π_1 in the form $\mathbf{r.n} = d$. [4]

(ii) A second plane, Π_2, has equation $\mathbf{r.} \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = 13$. Find a vector equation for the line of intersection of Π_1 and Π_2. [4]

10 **(i)** Use de Moivre's theorem to show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [2]

(ii) The sequence $\{u_n\}$ is such that $u_0 = 1$, $u_1 = \cos\theta$ and, for $n \geqslant 1$,

$$u_{n+1} = (2\cos\theta)u_n - u_{n-1}.$$

(a) Determine u_2 and u_3 in terms of powers of $\cos\theta$ only. [2]

(b) Suggest a simple expression for u_n, the nth term of the sequence, and prove it for all positive integers n using induction. [6]

11 **(i)** Let $I_n = \int_0^{\frac{1}{6}\pi} \sec^n t \, dt$ for positive integers n. Prove that, for $n \geqslant 2$,

$$(n-1)I_n = \frac{2^{n-2}}{(\sqrt{3})^{n-1}} + (n-2)I_{n-2}. \quad [5]$$

(ii) The curve with parametric equations $x = \tan t$, $y = \frac{1}{2}\sec^2 t$, for $0 \leqslant t \leqslant \frac{1}{6}\pi$, is rotated through 2π radians about the x-axis to form a surface of revolution of area S. Show that $S = \pi I_5$ and evaluate S exactly. [10]

12 The complex number z_1 is such that $z_1 = a + \mathrm{i}b$, where a and b are positive real numbers.

(i) Given that $z_1^2 = 2 + 2\mathrm{i}$, show that $a = \sqrt{\sqrt{2} + 1}$ and find the exact value of b in a similar form. [5]

The complex number z_2 is such that $z_2 = -a + \mathrm{i}b$.

(ii) **(a)** Determine $\arg z_2$ as a rational multiple of π.
[You may use the result $\tan\left(\frac{1}{8}\pi\right) = \sqrt{2} - 1$.] [2]

(b) The point P_n in an Argand diagram represents the complex number z_2^n, for positive integers n. Find the least value of n for which P_n lies on the half-line with equation

$$\arg(z) = \tfrac{1}{4}\pi. \qquad [3]$$

13 **(i)** **(a)** Given that $t = \tan x$, prove that $\dfrac{2}{2 - \sin 2x} = \dfrac{1 + t^2}{1 - t + t^2}$. [2]

(b) Hence determine the value of the constant k for which

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\tan^{-1}\left(\frac{1 - 2\tan x}{\sqrt{3}}\right)\right\} = \frac{k}{2 - \sin 2x}. \qquad [4]$$

(ii) The curve C has cartesian equation $x^2 - xy + y^2 = 72$.

(a) Determine a polar equation for C in the form $r^2 = \mathrm{f}(\theta)$, and deduce the polar coordinates $(r,\ \theta)$, where $0 \leqslant \theta < 2\pi$, of the points on C which are furthest from the pole O. [7]

(b) Find the exact area of the region of the plane in the first quadrant bounded by C, the x-axis and the line $y = x$. Deduce the total area of the region of the plane which lies inside C and within the first quadrant. [5]