UNIVERSITY OF CAMBRIDGE INTERNATIONAL EXAMINATIONS

Pre-U Certificate

MARK SCHEME for the May/June 2011 question paper

for the guidance of teachers

9795 FURTHER MATHEMATICS

9795/01 Paper 1 (Further Pure Mathematics), maximum raw mark 120

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

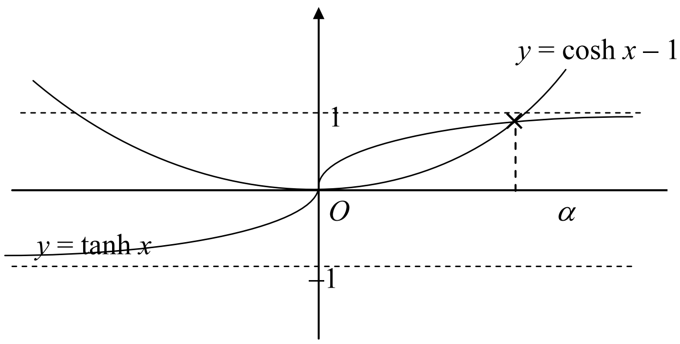
Mark schemes must be read in conjunction with the question papers and the report on the examination.

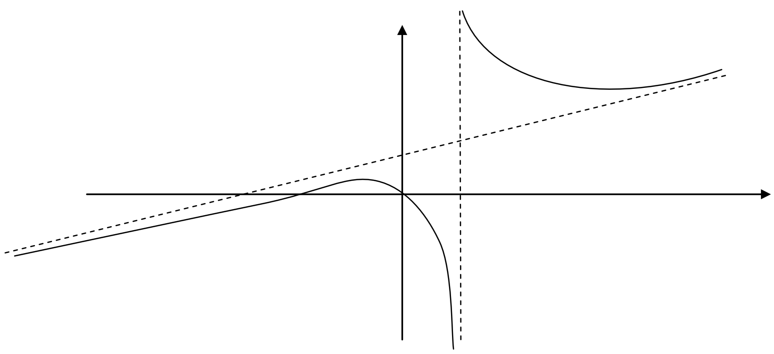
- Cambridge will not enter into discussions or correspondence in connection with these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2011 question papers for most IGCSE, Pre-U, GCE Advanced Level and Advanced Subsidiary Level syllabuses and some Ordinary Level syllabuses.

UNIVERSITY of CAMBRIDGE International Examinations

1		Attempt at $\mathbf{A}^2$ and $\mathbf{A}^3$	**M1**
		$\mathbf{A}^2 = \begin{pmatrix} k+4 & -k \\ -1 & k+9 \end{pmatrix}$, $\mathbf{A}^3 = \begin{pmatrix} k+8 & k^2+7k \\ k+7 & -4k-27 \end{pmatrix}$ ≥ 3 entries of $\mathbf{A}^3$ ✓	**A1**
		$\mathbf{A}^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow k = -7$ All entries of $\mathbf{A}^3$ must be ✓ if done this way Otherwise, allow just one key element checked (since "given")	**A1**
		$\det\begin{pmatrix} 2 & -7 \\ 1 & -3 \end{pmatrix} = -6 - k = 1$ **ft** numerical value consistent with their k	**B1**
			[4]
		ALTERNATIVE	
		$\det(\mathbf{A}^3) = (\det \mathbf{A})^3$	**M1**
		$\mathbf{A}^3 = \mathbf{I} \Rightarrow \det \mathbf{A} = 1$	**A1**
		Det $\mathbf{A} = -6 - k$	**B1**
		$k = -7$	**A1**
			[4]
2	**(i)**	Noting $\alpha + \beta + \gamma = -1$ and $\alpha\beta + \beta\gamma + \gamma\alpha = 7$ $(\alpha\beta\gamma = 1)$	**B1**
		$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = -13$ **GIVEN ANSWER** legit.	**M1**
			A1
			[3]
		ALTERNATIVE Substitute $x = \sqrt{y}$ to find eqn. with roots $\alpha^2, \beta^2, \gamma^2$	**M1**
		$y^3 + 13y^2 + 51y - 1 = 0$	**A1**
		$\alpha^2 + \beta^2 + \gamma^2 = (-13) / 1 = -13$	**A1**
			[3]
	(ii)	Eqn. has at least one non-real (complex) root	**B1**
		one real and two complex (conjugate) roots	**B1**
			[2]
3	**(i)**	$f(r-1) - f(r) = \dfrac{1}{(2r-1)^2} - \dfrac{1}{(2r+1)^2} = \dfrac{8r}{(4r^2-1)^2}$ (denom^{r}. may be factorised)	**B1**
			[1]
	(ii)	$\displaystyle\sum_{r=1}^{n} \frac{r}{(4r^2-1)^2} = \frac{1}{8}\sum_{r=1}^{n}\{f(r-1) - f(r)\}$ Use of this result …	**M1**
		$= \dfrac{1}{8}\displaystyle\sum_{r=1}^{n}\{(f(0)-f(1)) + (f(1)-f(2)) + \ldots + (f(n-1)-f(n))\}$ & cancelling	**M1**
		$= \dfrac{1}{8}\{f(0) - f(n)\} = \dfrac{1}{8}\left(1 - \dfrac{1}{(2n+1)^2}\right)$	**A1**
		$= \dfrac{1}{8}\left(\dfrac{(4n^2+4n+1)-1}{(2n+1)^2}\right)$ Common denom^{r}. with squaring attempted in num^{r}.	
		$= \dfrac{n(n+1)}{2(2n+1)^2}$ **GIVEN ANSWER** from correct working	**A1**
			[4]

4 **(i)**  Grad. of tanh$x$ should be 1	**B1** **B1**	
Curves cross twice (at $x = 0$ and $x = \alpha$) so there are 2 roots to $\tanh x = \cosh x - 1$ i.e. $1 + \tanh x - \cosh x = 0$	**B1**	[3]
(ii) **(a)** $f(1)\,f(1.5) = 0.22\ldots \times (-0.45\ldots) < 0$ $\Rightarrow 1 < \alpha < 1.5$ by the "Change-of-Sign" Rule	**B1**	[1]
(b) $f'(x) = \text{sech}^2 x - \sinh x$	**B1 B1**	
Use of $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ at least once	**M1**	
$x_1 = 1.25$, $x_2 = 1.219\,625\,3$, $x_3 = 1.218\,76$ to 5 d.p.	**A1**	[4]
5 Aux.Eqn. $m^2 + 1 = 0 \Rightarrow m = \pm i \Rightarrow$ Comp.Fn. is $y_c = A\cos x + B\sin x$	**M1** **A1**	
For Part.Intgrl. trying $y = ax^2 + bx + c$ (with at least a non-zero)	**M1**	
$\dfrac{dy}{dx} = 2ax + b$, $\dfrac{d^2y}{dx^2} = 2a$		
Diff^g. their y_p to find y' and y'' and subst^g. into the given d.e.	**M1**	
Equating terms to find a, b, c (with at least a and c non-zero)	**M1**	
$a = 8$, $b = 0$, $c = -16$; i.e. $y_p = 8x^2 - 16$	**A1**	
Gen.Soln. is $y = A\cos x + B\sin x + 8x^2 - 16$ **ft** their $y_c + y_p$ provided y_c has 2 arb. consts. and y_p has none	**B1**	[7]

6 **(i)** Good attempt to multiply 2 matrices of the appropriate form: $\begin{pmatrix} p & p \\ p & p \end{pmatrix}\begin{pmatrix} q & q \\ q & q \end{pmatrix}$	**M1**	
"Closure" noted or implied by correct product matrix $= \begin{pmatrix} 2pq & 2pq \\ 2pq & 2pq \end{pmatrix} \in S$	**A1**	
Statement that $\times_M$ known to be associative	**B1**	
Alt. $[(p)(q)]\,(r) = (p)\,[(q)(r)] = (4pqr)$ shown		
Identity is $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $(\in S)$	**B1**	
$\begin{pmatrix} p & p \\ p & p \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4p} & \frac{1}{4p} \\ \frac{1}{4p} & \frac{1}{4p} \end{pmatrix}$ $(\in S$ as $p \neq 0)$	**B1**	
… and $(S, \times_M)$ is a group since all four group axioms are satisfied		[5]
(ii) Attempt to look for a self-inverse element; i.e. solving $p = \dfrac{1}{4p}$ **ft** their $(p)^{-1}$ and **E**	**M1**	
$p = -\frac{1}{2}$ and noting that $H = \{\mathbf{E}, \mathbf{A}\}$ where $\mathbf{E} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$, $\mathbf{A} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$	**A1**	
Looking for $\{\mathbf{E}, \mathbf{B}, \mathbf{B}^2\}$ where $\mathbf{B}^3 = \mathbf{E}$; i.e. solving $(4p^3) = \frac{1}{2}$	**M1**	
Explaining carefully that $p^3 = \frac{1}{8} \Leftrightarrow p = \frac{1}{2}$ and no such $\mathbf{B}$ $(\neq \mathbf{E})$ exists	**A1**	
		[4]
7 $y = \dfrac{x^2+4x}{2x-1} = \dfrac{\frac{1}{2}x(2x-1)+\frac{9}{4}(2x-1)+\frac{9}{4}}{2x-1} = \frac{1}{2}x+\frac{9}{4}+\dfrac{\frac{9}{4}}{2x-1}$	**M1**	
For $y = \frac{1}{2}x + c$ For $c = \frac{9}{4}$ (ignore remd^r. term)	**A1 A1**	
Vertical asymptote $x = \frac{1}{2}$ noted or clear from graph	**B1**	
$\dfrac{dy}{dx} = \dfrac{(2x-1).(2x+4)-(x^2+4x).2}{(2x-1)^2}$ Diff^g. and setting num^r. = 0	**M1**	
Solving a quadratic in x $(x^2 - x - 2 = 0)$	**M1**	
TPs at (– 1, 1) and (2, 4) One each; or one for both x's ✓ but y's missing	**A1 A1**	
Crossing-points on the axes at (0, 0) and (– 4, 0)	**B1**	
 General shape	**B1**	
All correct	**B1**	
		[11]

8	**(i)**	No unique soln. $\Leftrightarrow \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & k \\ 1 & k & 6 \end{vmatrix} = 0$	**M1**
		Gaining and solving a quadratic eqn. in k	**M1**
		$0 = -(k^2 - 8k + 15) = -(k-3)(k-5) \Rightarrow k = 3, 5$	**A1**
			[3]
	(ii)	$k = 3 \Rightarrow \begin{matrix} x+2y+3z=4 \\ 2x+3y+3z=9 \\ x+3y+6z=1 \end{matrix} : \begin{matrix} 2\times(\mathbf{1})-(\mathbf{2}) \Rightarrow & y+3z=-1 \\ (\mathbf{3})-(\mathbf{1}) \Rightarrow & y+3z=-3 \end{matrix}$	
		Subst^g. back and eliminating one variable; inconsistency correctly shown	**M1** **A1**
		$k = 5 \Rightarrow \begin{matrix} x+2y+3z=4 \\ 2x+3y+5z=9 \\ x+5y+6z=1 \end{matrix} : \begin{matrix} 2\times(\mathbf{1})-(\mathbf{2}) \Rightarrow & y+z=-1 \\ (\mathbf{3})-(\mathbf{1}) \Rightarrow & 3y+3z=-3 \end{matrix}$	
		Subst^g. back and eliminating one variable; consistency correctly shown	**M1** **A1**
			[4]
		ALTERNATIVE (whole qn.)	
		$\left(\begin{matrix} 1 & 2 & 3 \\ 2 & 3 & k \\ 1 & k & 6 \end{matrix}\middle\vert\begin{matrix} 4 \\ 9 \\ 1 \end{matrix}\right) \to \left(\begin{matrix} 1 & 2 & 3 \\ 0 & -1 & k-6 \\ 0 & k-2 & 3 \end{matrix}\middle\vert\begin{matrix} 4 \\ 1 \\ -3 \end{matrix}\right) \begin{matrix} \\ R_2' = R_2 - 2R_1 \\ R_3' = R_3 - R_1 \end{matrix}$ convincing attempt at Gaussian elimination	**M1**
		$\to \left(\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 6-k \\ 0 & k-2 & 3 \end{matrix}\middle\vert\begin{matrix} 4 \\ -1 \\ -3 \end{matrix}\right)$ or by *Cramer's Rule*	
		$\to \left(\begin{matrix} 1 & 2 & 3 \\ 0 & 1 & 6-k \\ 0 & 0 & k^2-8k+15 \end{matrix}\middle\vert\begin{matrix} 4 \\ -1 \\ k-5 \end{matrix}\right) \quad R_3' = R_3 - (k-2)R_2$ Final R_2 Final R_3	**A1** **A1**
		$k^2 - 8k + 15 = 0$ for no unique solution $\quad k = 3, 5$	**M1** **A1**
		Noting $k = 3 \Rightarrow R_3 = 0\ 0\ 0 \mid -2$ giving inconsistency	**B1**
		Noting $k = 5 \Rightarrow R_3 = 0\ 0\ 0 \mid 0$ giving consistency	**B1**
			[7]

9	(a)	Attempt at scalar triple product: $\mathbf{a} \bullet \mathbf{b} \times \mathbf{c}$ or $\mathbf{a} \times \mathbf{b} \bullet \mathbf{c}$	M1
		NB $\mathbf{b} \times \mathbf{c} = \begin{pmatrix} 36 \\ -41 \\ 145 \end{pmatrix}$ and $\mathbf{a} \times \mathbf{b} = \begin{pmatrix} -73 \\ 139 \\ 97 \end{pmatrix}$	
		Use of $V = \frac{1}{6} \lvert \mathbf{a} \bullet \mathbf{b} \times \mathbf{c} \rvert$ formula	M1
		Answer $335\frac{1}{6}$ or $\frac{2011}{6}$	A1
			[3]
	(b) (i)	$\mathbf{n} = \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -21 \\ 0 \end{pmatrix}$ accept $\pm \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$ etc. Alt approach possible that eliminates and from the original eqn.	M1 A1
		$\mathbf{r} \bullet \mathbf{n} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} = 1 = d$ **ft** incorrect **n**	M1 A1
			[4]
	(ii)	$-x + 3y = 1$, $x + 4y + 7z = 13$ Set $y = \lambda \Rightarrow$ Parametrisation attempt	M1
		ft 1st eqn. from (i) $x = 3\lambda - 1,\ z = 2 - \lambda$	A1 A1
		$\mathbf{r} = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ or any other correct vector line eqn. form **ft**	B1
		MUST have **r** = at the start (allow r =)	
			[4]
		ALTERNATIVE	
		d.v. = $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix} = \begin{pmatrix} 21 \\ 7 \\ -7 \end{pmatrix}$ accept $\pm \begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$ etc. **ft** $\begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}$	M1 A1
		Finding any point on the line; e.g. (– 1, 0, 2), (2, 1, 1), (5, 2, 0), etc.	B1
		Answer as above **ft** point and d.v.	A1
			[4]

10	**(i)**	$\cos 3\theta = \text{Re}(\cos 3\theta + i\sin 3\theta) = \text{Re}(c + is)^3$ *Use* of de Moivre's Thm. $= c^3 - 3cs^2 = c^3 - 3c(1 - c^2) = 4c^3 - 3c$ **ANSWER GIVEN**	**M1** **A1** [2]
	(ii)	**(a)** $u_2 = 2c^2 - 1$ and $u_3 = 4c^3 - 3c$	**B1 B1** [2]
		(b) Conjecture $u_n = \cos(n\theta)$ 1 Base case, "true for $n = 0$ or 1 (or 2 or 3)", may be taken as read Assuming $u_n = \cos(n\theta)$ for at least $n = k$ and $n = k - 1$ Using given r.r. to generate $u_{k+1} = 2\cos\theta\cos(k\theta) - \cos([k-1]\theta)$ $= 2\cos\theta\cos(k\theta) - \{\cos(k\theta)\cos\theta + \sin(k\theta)\sin\theta\}$ $= \cos\theta\cos(k\theta) - \sin\theta\sin(k\theta)$ OR $\{\cos(k+1)\theta - \cos(k-1)\theta\} - \cos(k-1)\theta$ $= \cos([k+1]\theta)$ If statement is true for $n = 0, 1, 2, \ldots, k$ (but allow assumption for one term only here) then it is also true for $n = k + 1$. Proof follows by (strong) induction	**B1** **M1** **M1** **M1** **A1** **B1** [6]
11	**(i)**	$I_n = \int_0^{\frac{1}{6}\pi} \sec^n t \, dt = \int_0^{\frac{1}{6}\pi} \sec^{n-2} t . \sec^2 t \, dt$ Correct splitting *and* use of parts	**M1**
		$= \left[\sec^{n-2} t . \tan t\right]_0^{\frac{1}{6}\pi} - \int_0^{\frac{1}{6}\pi} \tan t.(n-2)\sec^{n-3} t . \sec t . \tan t \, dt$	**A1** **A1**
		$I_n = \left(\frac{2}{\sqrt{3}}\right)^{n-2} . \frac{1}{\sqrt{3}} - (n-2)\int_0^{\frac{1}{6}\pi} \sec^{n-2} t.\left(\sec^2 t - 1\right) dt$ Subst^g. for tan …	
		$= \frac{2^{n-2}}{\left(\sqrt{3}\right)^{n-1}} - (n-2)\{I_n - I_{n-2}\}$ … and reverting to I's	**M1**
		$(n-1)I_n = \frac{2^{n-2}}{\left(\sqrt{3}\right)^{n-1}} + (n-2)I_{n-2}$ **ANSWER GIVEN**	**A1** [5]
	(ii)	$\dot{x}^2 + \dot{y}^2 = \left(\sec^2 t\right)^2 + \left(\sec^2 t . \tan t\right)^2$ Attempted $= \sec^4 t\left(1 + \tan^2 t\right) = \sec^6 t$ Use of $S = \int 2\pi y\sqrt{\dot{x}^2 + \dot{y}^2} \, dt$ with $y = \frac{1}{2}\sec^2 t$ and their $\dot{x}$ and $\dot{y}$ $= \int 2\pi . \frac{1}{2}\sec^2 t . \sec^3 t \, dt = \pi I_5$	**M1** **A1** **M1** **A1** [4]
		$I_1 = \ln\left[\sec t + \tan t\right]$ $= \ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \ln 1 = \frac{1}{2}\ln 3$ Then, using the R.F., $2I_3 = \frac{2}{3} + \frac{1}{2}\ln 3 \Rightarrow I_3 = \frac{1}{3} + \frac{1}{4}\ln 3$ Using the R.F. again, $4I_5 = \frac{8}{9} + 3\left(\frac{1}{3} + \frac{1}{4}\ln 3\right)$ … leading to $S = \frac{1}{144}\left(68 + 27\ln 3\right)\pi$ or exact equivalent	**M1** **A1** **M1** **A1** **M1** **A1** [6]

12	**(i)**	$(a + ib)^2 = 2 + 2i \Leftrightarrow a^2 - b^2 = 2$ and $ab = 1$ Squaring & equating Re/Im parts	**M1**
		$a^2 - \frac{1}{a^2} - 2 = 0 \Rightarrow a^4 - 2a^2 - 1 = 0 \Rightarrow (a^2 - 1)^2 = 2$ (or by the quadratic formula) Subst^g. for b (say) and solving a quadratic in a^2	**M1**
		$a = \sqrt{\sqrt{2}+1}$ **(AG)** MUST note that $a^2 > 0$ to explain choice of +ve sq.rt.	**A1**
		Similarly, $b = \sqrt{\sqrt{2}-1}$ from $b^4 + 2b^2 - 1 = 0$ or $b = \frac{1}{a}$	**M1** **A1**
			[5]
	(ii)	**(a)** $z_2 = -\sqrt{\sqrt{2}+1} + i\sqrt{\sqrt{2}-1}$	
		$\arg(z_2) = \pi - \tan^{-1}\sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}}$ Attempt incl^g. rationalising denom^r.	**M1**
		$= \pi - \tan^{-1}\left(\sqrt{2}-1\right) = \pi - \tfrac{1}{8}\pi = \tfrac{7}{8}\pi$	**A1**
			[2]
		(b) $\arg(z_2{}^n) = \tfrac{7}{8}n\pi$	**B1**
		$= \left(2k + \tfrac{1}{4}\right)\pi$	**M1**
		$n = \frac{16k+2}{7}$, giving least $n = 14$	**A1**
		[Condone lack of convincing explanation that this IS the least such n.]	
			[3]

13 **(i)** **(a)** Use of $\sin 2x = \dfrac{2t}{1+t^2}$ in $\dfrac{2}{2-\sin 2x}$, where $t = \tan x$	**M1**
$\dfrac{2}{2-\sin 2x} = \dfrac{2}{2-\left(\frac{2t}{1+t^2}\right)} = \dfrac{1+t^2}{1-t+t^2}$ **ANSWER GIVEN**	**A1**
	[2]
(b) $y = \tan^{-1}\left(1 - \dfrac{2\tan x}{\sqrt{3}}\right) \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{1+\left(\frac{1-2t}{\sqrt{3}}\right)^2} \times \dfrac{-2}{\sqrt{3}}\sec^2 x$	**M1** **A1**
or $\tan y = \ldots$ and use of implicit diffn.	
$= \dfrac{-2}{\sqrt{3}}\left(1+t^2\right) \times \dfrac{3}{4-4t+4t^2}$	**M1**
$= \dfrac{-\sqrt{3}}{2} . \dfrac{1+t^2}{1-t+t^2} = \dfrac{-\sqrt{3}}{2} . \dfrac{2}{2-\sin 2x}$ so that $k = -\sqrt{3}$	**A1**
	[4]
(ii) **(a)** Use of $x = r\cos\theta$, $y = r\sin\theta$ (and $x^2 + y^2 = r^2$)	**M1**
$\Rightarrow r^2 = 72 + r^2 \sin\theta\cos\theta$ i.e. x, y completely (and correctly) eliminated	**M1**
$\Rightarrow r^2 = \dfrac{144}{2-\sin 2\theta}$ or equivalent form	**A1**
$2 - \sin 2\theta \geq 1 \Rightarrow r^2 \leq 144$ so that $r_{\max} = 12$	**M1**
Calculus approach fine as an alternative	**A1**
Then $\sin 2\theta = 1 \Rightarrow \theta = \frac{1}{4}\pi$ or $\frac{5}{4}\pi$	**M1**
No **M ft** from incorrect differentiation	**A1**
	[7]
(b) $A = \frac{1}{2}\int r^2\, d\theta = \int \dfrac{72}{2-\sin 2\theta}\, d\theta$	**M1**
$= 72\left[\dfrac{-1}{\sqrt{3}}\tan^{-1}\left(\dfrac{1-2\tan x}{\sqrt{3}}\right)\right]_0^{\pi/4}$ Use of previous result	**M1**
$= \dfrac{72}{\sqrt{3}}\left(\tan^{-1}\left(\frac{1}{\sqrt{3}}\right) - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right)$ or $\dfrac{72}{\sqrt{3}}\left(\dfrac{\pi}{6} - \left(-\dfrac{\pi}{6}\right)\right)$	**A1**
$= 8\pi\sqrt{3}$ **CAO ft** their k above	**A1**
Area inside C in 1st quad. $= 2A = 16\pi\sqrt{3}$ since C is symmetric in $y = x$ **ft**	**B1**
	[5]