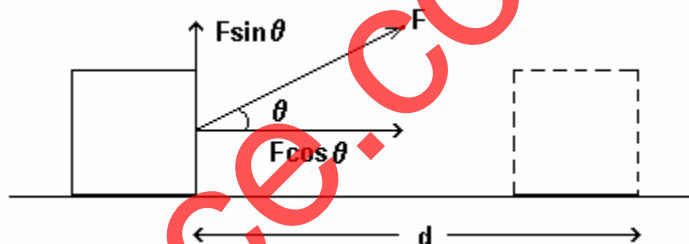


5.1 Work

“The product of force and displacement (in the direction of force), during which the force is acting, is defined as work.”

When 1 N force is applied on a particle and the resulting displacement of the particle, in the direction of the force, is 1 m, the work done is defined as 1 J (joule). The dimensional formula of work is $M^1 L^2 T^{-2}$.

The displacement may not be in the direction of the applied force in all cases. In the figure shown, the displacement, d , makes an angle θ with the applied force, F . According to the definition of force,



Work, W = force \times displacement in the direction of the force

$$= F (d \cos \theta) = (F \cos \theta) (d)$$

$$= (\text{the component of force in the direction of displacement}) \times (\text{displacement})$$

(i) For $\theta = \pi/2$, work $W = 0$, even if F and d are both non-zero. In uniform circular motion, the centripetal force acting on a particle is perpendicular to its displacement. Hence, the work done due to centripetal force during such a motion is zero.

(ii) If $\theta < \pi/2$, work done is positive and is said to be done on the object by the force.

(iii) If $\pi/2 < \theta < \pi$, work done is negative and is said to be done by the object against the force.

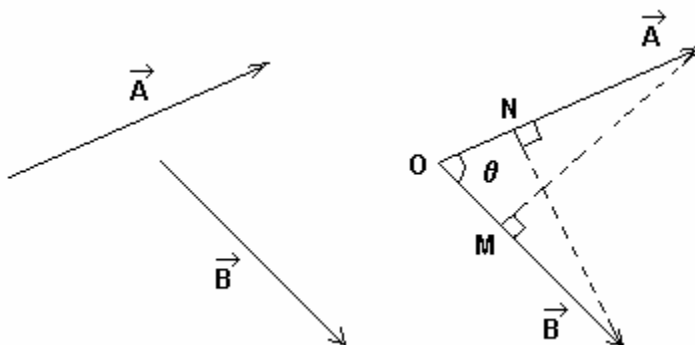
5.2 Scalar product of two vectors

The scalar product of two vectors, \vec{A} and \vec{B} , also known as the dot product, is written by putting a dot (\cdot) between the two vectors and is defined as:

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta = AB \cos \theta, \text{ where } \theta \text{ is the angle between the two vectors.}$$

To obtain the scalar product of \vec{A} and \vec{B} , they are to be drawn from a common point, O , with the same magnitudes and directions as shown in the figure.

M is the foot of perpendicular from the head of \vec{A} to \vec{B} . OM ($= A \cos \theta$) is the magnitude of projection of \vec{A} on \vec{B} .



Similarly, N is the foot of perpendicular from the head of \vec{B} to \vec{A} and $ON (= B \cos \theta)$ is the magnitude of projection of \vec{B} on \vec{A} .

$$\begin{aligned} \therefore \vec{A} \cdot \vec{B} &= AB \cos \theta = B (A \cos \theta) = (B)(OM) \\ &= (\text{magnitude of } \vec{B})(\text{magnitude of projection of } \vec{A} \text{ on } \vec{B}) \end{aligned}$$

or

$$\begin{aligned} \vec{A} \cdot \vec{B} &= AB \cos \theta = A (B \cos \theta) = (A)(ON) \\ &= (\text{magnitude of } \vec{A})(\text{magnitude of projection of } \vec{B} \text{ on } \vec{A}) \end{aligned}$$

Thus, scalar product of two vectors is equal to the product of magnitude of one vector with the magnitude of projection of second vector on the direction of the first vector.

The scalar product of vectors is zero if the angle between the vectors $\theta = \pi/2$, positive if $0 \leq \theta < \pi/2$ and negative if $\pi/2 < \theta \leq \pi$.

Properties of scalar product

(1) Commutative law:

$$\vec{A} \cdot \vec{B} = AB \cos \theta = BA \cos \theta = \vec{B} \cdot \vec{A}$$

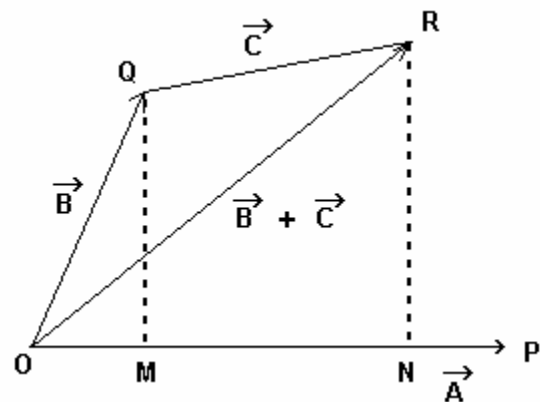
Thus, scalar product of two vectors is commutative.

(2) Distributive law:

$$\vec{OP} = \vec{A}, \quad \vec{OQ} = \vec{B} \text{ and } \vec{OR} = \vec{C}$$

as shown in the figure. Now,

$$\begin{aligned} \vec{A} \cdot (\vec{B} + \vec{C}) &= |\vec{A}| |\text{projection of } \vec{B} + \vec{C} \text{ on } \vec{A}| \\ &= |\vec{A}| (ON) = |\vec{A}| (OM + MN) \\ &= |\vec{A}| (OM) + |\vec{A}| (MN) \\ &= |\vec{A}| |\text{proj. of } \vec{B} \text{ on } \vec{A}| + |\vec{A}| |\text{proj. of } \vec{C} \text{ on } \vec{A}| \\ &= \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \end{aligned}$$



Thus, scalar product of two vectors is distributive with respect to summation.

(3) If $\vec{A} \parallel \vec{B}$, $\theta = 0^\circ$, $\vec{A} \cdot \vec{B} = AB \cos 0^\circ = AB$ and

$$\vec{A} \cdot \vec{A} = |\vec{A}| |\vec{A}| = A^2 \quad \therefore |\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$$

Thus, magnitude of a vector is equal to the square root of scalar product of the vector with itself.

(4) If $\vec{A} \perp \vec{B}$, $\theta = 90^\circ \therefore \vec{A} \cdot \vec{B} = AB \cos 90^\circ = 0$

Thus, the scalar products of two mutually perpendicular vectors is zero.

(5) Scalar products of unit vectors in Cartesian co-ordinate system:

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \quad \text{and} \quad \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

(6) Scalar product in terms of Cartesian components of vectors:

If $\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$, then

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$$

$$= A_x B_x + A_y B_y + A_z B_z$$

(7) $\vec{A} \cdot \vec{B} = AB \cos \theta$

$$\therefore \cos \theta = \frac{\vec{A} \cdot \vec{B}}{AB} = \frac{A_x B_x + A_y B_y + A_z B_z}{\sqrt{A_x^2 + A_y^2 + A_z^2} \sqrt{B_x^2 + B_y^2 + B_z^2}}$$

This formula is used to find the angle between two vectors.

5.3 Work done by a variable force

‘A force varying with position or time is known as the variable force.’

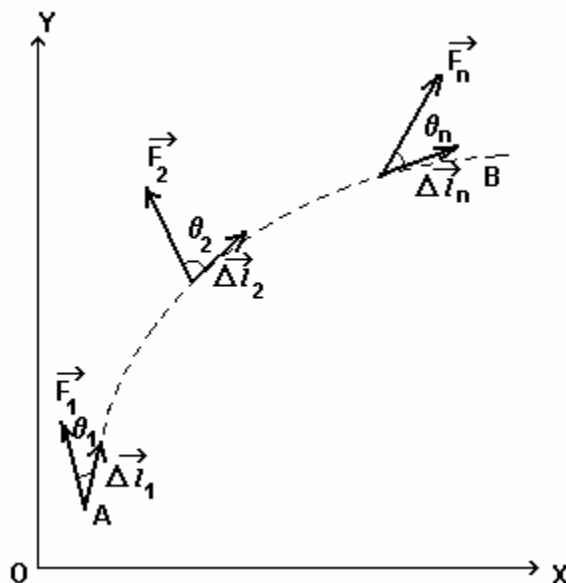
To find work done under a variable force, consider a two-dimensional motion of a particle moving along a curved path AB as shown in the figure under the effect of a variable force. The curved path is divided into

infinitesimally small line segments $\vec{\Delta l}_1, \vec{\Delta l}_2, \dots$

, $\vec{\Delta l}_n$.

Let $\vec{F}_1, \vec{F}_2, \dots, \vec{F}_n$ be the forces acting on the particle at line segments, $\vec{\Delta l}_1, \vec{\Delta l}_2, \dots,$

$\vec{\Delta l}_n$ respectively. As the line segments are very small, the force over each segment can be considered constant.



The total work as the particle moves from A to B can be obtained as the sum of the work done for different line segments as under.

Total work, $W = \vec{F}_1 \cdot \vec{\Delta l}_1 + \vec{F}_2 \cdot \vec{\Delta l}_2 + \vec{F}_n \cdot \vec{\Delta l}_n$

$$= \sum_A^B \vec{F}_i \cdot \Delta \vec{l}_i, \quad i = 1, 2, 3, \dots$$

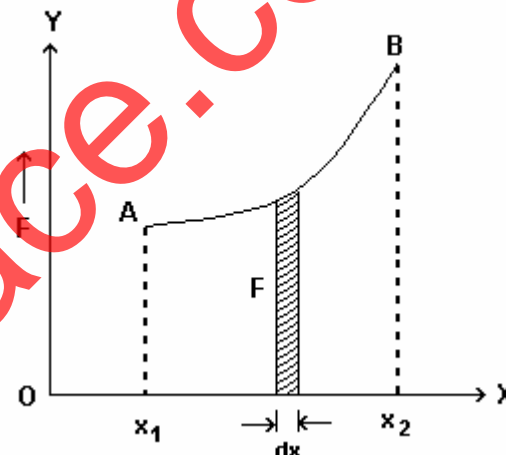
If $\lim_{|\Delta \vec{l}| \rightarrow 0}$, then $W = \int_A^B \vec{F} \cdot d\vec{l} = \int_A^B F \cos \theta dl$

For one-dimensional motion of the particle along X-axis and force acting in the direction of motion as shown in the figure,

$$\text{Work, } W = \int_A^B F dx \cos 0^\circ = \int_{x_1}^{x_2} F dx$$

where x_1 and x_2 are the x-coordinates of A and B respectively.

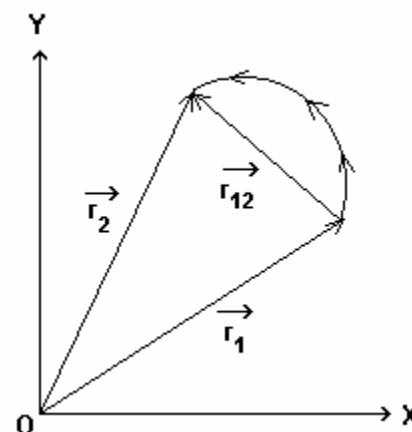
It can be seen from the figure that for a small displacement dx , the force F is almost constant and hence work done is Fdx which is the area of the strip. Total work for motion of the particle from A to B is the sum of areas of such strips which is the area under the curve between $x = x_1$ and $x = x_2$.



For a particle moving under the effect of a constant force on a curved path from the point \vec{r}_1 to \vec{r}_2 as shown in

the figure, work done, $W_{12} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = \vec{F} \cdot (\vec{r}_2 - \vec{r}_1)$
 (since, \vec{F} is constant)

Thus, the work done by the constant force is the scalar product of the force and the displacement vector of the particle.



5.4 kinetic energy

“The capacity of a body to do work, by virtue of its motion, is known as the kinetic energy of the body.”

More the speed of the object, more is its kinetic energy. The net force acting on a body produces acceleration or deceleration in its motion thereby changing its velocity and kinetic energy. Also, work is done by the net force on the body while causing its displacement.

The work, W , done by the force, \vec{F} , on a particle of mass, m , causing acceleration, \vec{a} and

displacement, \vec{d} is given by

$$W = \vec{F} \cdot \vec{d} = m \vec{a} \cdot \vec{d} \dots \dots \dots (1)$$

Using the result, $v^2 - v_0^2 = 2 \vec{a} \cdot \vec{d}$ in the above equation,

$$W = m \left(\frac{v^2 - v_0^2}{2} \right) = \frac{1}{2} mv^2 - \frac{1}{2} mv_0^2, \text{ where } v_0 \text{ and } v \text{ are the speeds before and after the application of force respectively.}$$

$$= K - K_0 = \text{change in kinetic energy} = \Delta K,$$

where K_0 and K are the initial and final kinetic energies. The unit of kinetic energy is the same as that of work, i.e., joule.

Thus, “the work done by the resultant force on a body, in the absence of dissipative forces, is equal to the change in the kinetic energy of the body.”

This statement is known as the work-energy theorem.

When a particle performs uniform circular motion, the centripetal force acting on it is perpendicular to its tangential instantaneous displacement and hence no work is done by the centripetal force and the speed and kinetic energy of the particle do not change.

If the body is displaced in the direction of the force acting on it, work is done by the force on the body and is positive. The kinetic energy of the body increases in this case. If the displacement of the body is against the force acting on it, work is done by the body and is negative. The kinetic energy of the body decreases in this case.

Also, kinetic energy, $K = \frac{1}{2} mv^2 = \frac{m^2 v^2}{2m} = \frac{p^2}{2m}$, where p = linear momentum of the body.

5.5 Potential energy

“The capacity of a body to do work, by virtue of its position in a force field or due to its configuration is known as the potential energy of the body.”

Gravitational potential energy:

The gravitational acceleration, g , due to the Earth’s gravitational force can be taken as constant for heights much smaller as compared to the radius of the Earth.

If a body of mass m moves from height y_1 to height y_2 in the Earth’s gravitational field,

the force on the body = $-mg \hat{j}$ and its displacement = $(y_2 - y_1) \hat{j}$

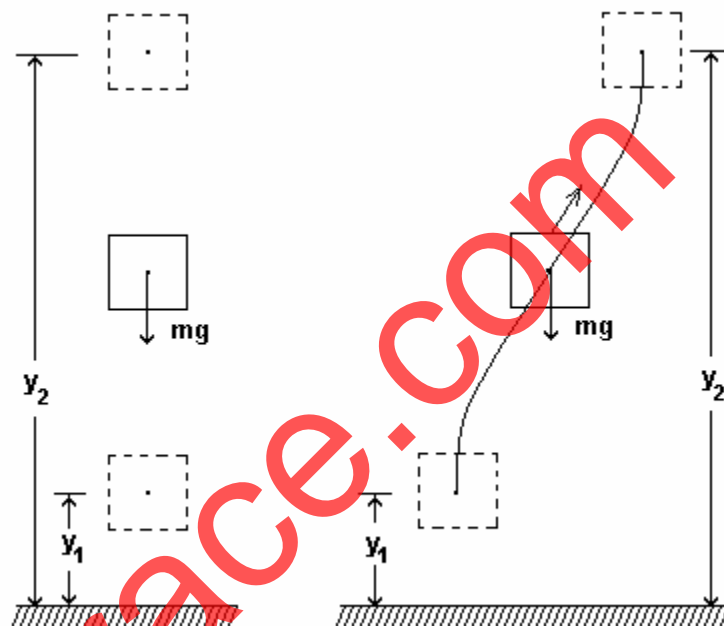
$$\begin{aligned} \therefore \text{the work done, } W &= \vec{F} \cdot \vec{d} \\ &= -mg \hat{j} \cdot (y_2 - y_1) \hat{j} \\ &= mgy_1 - mgy_2 \dots \dots (1) \end{aligned}$$

Here, the work depends on the initial and final positions of the body and not on the path followed. Also, there is no loss of mechanical energy if friction and air resistance are ignored. In this case, the force is known as conservative force and the field as conservative field.

If v_1 and v_2 are velocities of the body at heights y_1 and y_2 respectively, then according to work-energy theorem,

The work done, $W = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 = mgy_1 - mgy_2$ [from equation (1) above]

Here, mgy_1 and mgy_2 are the potential energies of the body at heights y_1 and y_2 respectively from the surface of the Earth due to the Earth's gravitational field. Surface of the Earth is the reference level from where potential energy is calculated. Besides, the Earth's surface, any other level can be chosen as reference level.



Thus, gravitational potential energy of a body of mass m at height h from the surface of the Earth is $U = mgh$.

From the above equation, $\frac{1}{2}mv_1^2 + mgy_1 = \frac{1}{2}mv_2^2 + mgy_2$

Thus, in a conservative field, the mechanical energy, E , which is the sum of kinetic energy ($K = \frac{1}{2}mv^2$) and potential energy ($U = mgh$) remains constant.

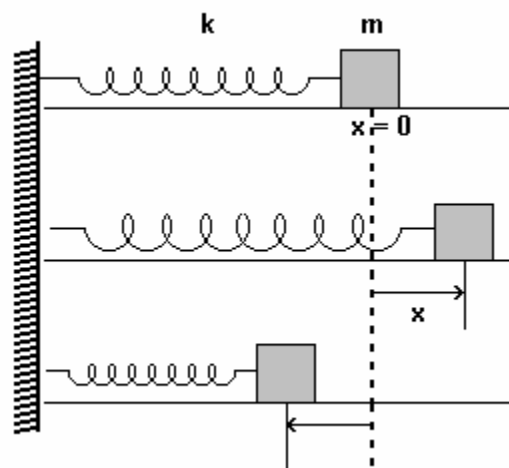
$\therefore E = K + U$.

Thus, the mechanical energy of an isolated system of bodies is conserved in a conservative field. This statement is known as the law of conservation of mechanical energy.

5.6 Elastic potential energy: (Potential energy due to the configuration of a system)

One end of an elastic spring of negligible mass and obeying Hooke's law is tied with a rigid wall and to the other end a block of mass m is tied. The block is at $x = 0$ when the spring is not extended or compressed. On displacing the block, the restoring force is produced in the spring which tries to restore the block to its original position.

The restoring force is directly proportional to the displacement of the block and is in a direction opposite to the displacement.



$$\therefore F \propto -x \quad \text{or} \quad F = -kx,$$

where k is the force constant of the spring which is defined as the force required to pull or compress the spring by unit displacement. Its unit is N/m and its dimensional formula is $\text{M}^1 \text{L}^0 \text{T}^{-2}$.

Work done by the applied force on the spring is,

$$W = \int_0^x kx dx = k \int_0^x x dx = k \left[\frac{x^2}{2} \right]_0^x = \frac{1}{2} kx^2$$

This work done on the spring is stored in the form of elastic potential energy of the spring. Taking the potential energy to be zero for $x = 0$, for change in length equal to x , the potential stored in the spring will be

$$U = \frac{1}{2} kx^2$$

Relation between force and potential energy

The change in kinetic energy, ΔK , of the particle is equal to work, W , done on it by force, F , in displacing it through a small distance, Δx .

$$\therefore \Delta K = W = F \Delta x$$

But, by the law of conservation of mechanical energy,

$$\Delta K + \Delta U = 0 \Rightarrow F \Delta x + \Delta U = 0 \quad \therefore F = -\frac{\Delta U}{\Delta x}$$

\therefore instantaneous value of $F = -\lim_{\Delta x \rightarrow 0} \frac{\Delta U}{\Delta x} = -\frac{dU}{dx}$ This equation holds only in the case of conservative forces.

In the case of a spring, the potential energy, $U = \frac{1}{2} kx^2$

$$\therefore F = -\frac{dU}{dx} = -\frac{1}{2} k(2x) = -kx$$

5.7 Power

“The time rate of doing work is known as power.” or “Power is defined as the work done per unit time.”

If ΔW is the work done in time interval Δt , the average power in time interval Δt is

$$\langle P \rangle = \frac{\Delta W}{\Delta t} \quad \text{and} \quad \text{instantaneous power at time } t \text{ is}$$

$$P = \lim_{\Delta t \rightarrow 0} \frac{\Delta W}{\Delta t} = \frac{dW}{dt}$$

If dW is the work done by a force \vec{F} during the displacement \vec{dr} , then

$$P = \frac{dW}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}, \text{ where } \vec{v} \text{ is the instantaneous velocity at time } t.$$

Power is a scalar quantity like work and energy. Its unit is watt (W). 1 W = 1 J/s. Its dimensional formula is $M^1 L^2 T^{-3}$.

1 kilowatt (kW) = 10^3 watt (W) and 1 megawatt (MW) = 10^6 W.

In British system, unit of power is horsepower (hp). 1 hp = 746 W.

1 kilowatt-hour (kWh) is the electrical unit of energy (work) as the product of power and time.

1 unit of electrical energy = 1 kWh = 3.6×10^6 J.

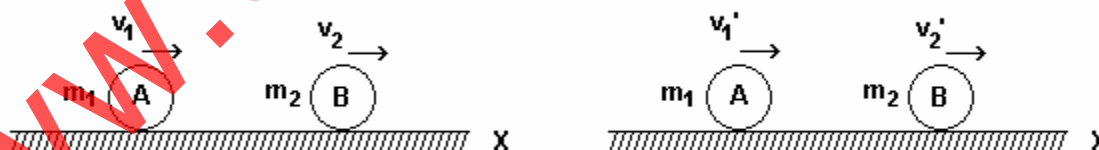
5.8 Elastic and inelastic collisions

- In elastic collision, total linear momentum and total kinetic energy of the colliding bodies are conserved.
- In inelastic collision, total linear momentum of the colliding bodies is conserved, but part or whole of the kinetic energy is lost in other forms of energy.
- In both, elastic as well as inelastic collisions, total energy (energy of all forms, mechanical, internal, sound, etc.) and total linear momentum are conserved.

Inelastic collision in one dimension:

Partly inelastic collision:

Suppose a sphere A of mass m_1 moving with velocity v_1 in X-direction collides with another sphere of mass m_2 moving in the same direction with velocity v_2 . ($v_1 > v_2$)



Let v_1' and v_2' be their velocities in the same (X) direction after the collision which we want to find.

According to the law of conservation of linear momentum,

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$\text{or, } m_1 (v_1 - v_1') = m_2 (v_2 - v_2') \dots \dots (1)$$

Now, a parameter known as coefficient of restitution, e, is defined as under:

$$\text{Co-efficient of restitution, } e = \frac{v_2' - v_1'}{v_1 - v_2} = \frac{\text{velocity of separation after the collision}}{\text{velocity of approach before the collision}}$$

For partly inelastic collision, $0 < e < 1$.

$$\therefore v_2' - v_1' = e(v_1 - v_2) \dots \dots \dots (2)$$

Solving equations (1) and (2) for v_1' and v_2' , we get

$$v_1' = \frac{m_1 - m_2 e}{m_1 + m_2} v_1 + \frac{(1 + e)m_2}{m_1 + m_2} v_2 \quad \text{and} \quad v_2' = \frac{(1 + e)m_1}{m_1 + m_2} v_1 - \frac{m_1 e - m_2}{m_1 + m_2} v_2$$

Thus, v_1' and v_2' can be calculated from the above equations.

Completely inelastic collision:

Putting $e = 0$ for completely inelastic collision in the above equations,

$$v_1' = v_2' = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} = v \quad (\text{the common velocity of the spheres after collision})$$

Thus, in completely inelastic collision, the colliding bodies move jointly with a common velocity.

Here, the kinetic energy before collision, $K_i = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$ and

the kinetic energy after collision, $K_f = \frac{1}{2} (m_1 + m_2) v^2 = \frac{1}{2} (m_1 + m_2) \left\{ \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \right\}^2$

$$\therefore K_f - K_i = - \frac{m_1 m_2 (v_1 - v_2)^2}{2(m_1 + m_2)}$$

The negative sign indicates that the kinetic energy decreases during inelastic collision.

Elastic collision in one dimension:

Putting $e = 1$ for elastic collision in equation (2), we get $v_2' - v_1' = (v_1 - v_2)$

$$\therefore v_1 + v_1' = v_2 + v_2' \dots \dots \dots (3)$$

Multiplying equations (1) and (3), we get

$$m_1 (v_1^2 - v_1'^2) = m_2 (v_2^2 - v_2'^2). \quad \text{Multiplying by } \frac{1}{2} \text{ and rearranging,}$$

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

This shows that the kinetic energy is conserved for $e = 1$. Hence, elastic collision can also be defined as the one in which $e = 1$.

Special cases of elastic collision:

Taking $e = 1$ for elastic collision, the velocities of the spheres after the collision are given by

$$v_1' = \frac{m_1 - m_2}{m_1 + m_2} v_1 + \frac{2 m_2}{m_1 + m_2} v_2 \quad \text{and} \quad v_2' = \frac{2 m_1}{m_1 + m_2} v_1 - \frac{m_1 - m_2}{m_1 + m_2} v_2$$

(1) For $m_1 \gg m_2$, neglecting m_2 as compared to m_1 in the above equations, we get

$$v_1' \approx v_1 \quad \text{and} \quad v_2' \approx 2v_1 - v_2$$

This shows that the larger sphere continues to move with the same velocity, whereas the velocity of the smaller sphere increases. If the smaller sphere was at rest, it moves with twice the velocity of the larger sphere after the collision.

(2) For $m_2 \gg m_1$, neglecting m_1 as compared to m_2 in the above equations, we get

$$v_2' \approx v_2 \quad \text{and} \quad v_1' \approx 2v_2 - v_1$$

In this case also, the larger sphere continues to move with the same velocity, but the velocity of the smaller sphere changes.

- (i) If the smaller sphere were moving with twice the speed of the larger sphere, it becomes stationary after the collision.
- (ii) If the smaller sphere were moving with velocity less than twice the velocity of the larger sphere, it continues to move in the same direction, but with decreased speed.
- (iii) If the smaller sphere were moving with velocity more than twice the velocity of the larger sphere, it rebounds and starts moving in the opposite direction with the velocity given as above.

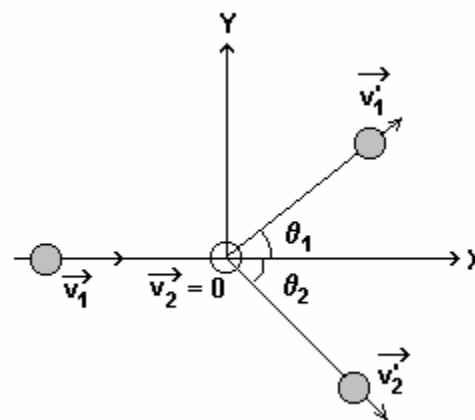
If the larger sphere was stationary, it remains stationary. In this case, the smaller sphere rebounds and moves with the same speed in the reverse direction.

Summarizing the above two cases, when one of the two spheres colliding is much more massive than the other then the velocity of the larger sphere remains almost the same after the collision, whereas the velocity of the smaller sphere after the collision is almost twice the velocity of larger sphere less the velocity of the smaller sphere before the collision.

(3) If $m_1 = m_2$, then $v_1' = v_2$ and $v_2' = v_1$. Thus, in elastic collision of two spheres of equal mass, their velocities get exchanged.

Elastic collision in two dimensions

Suppose a sphere of mass m_1 moving in X-direction with velocity \vec{v}_1 collides elastically with a stationary ($\vec{v}_2 = 0$) sphere of mass m_2 as shown in the figure. After the collision, they move in the directions making angles θ_1 and θ_2 with the X-axis with velocities \vec{v}_1' and \vec{v}_2' respectively.



According to the law of conservation of momentum,

$$m_1 \vec{v}_1 = m_1 \vec{v}_1' + m_2 \vec{v}_2'$$

Equating the X- and Y-components of the momenta

$$m_1 v_1 = m_1 v_1' \cos \theta_1 + m_2 v_2' \cos \theta_2 \dots (1) \text{ and } 0 = m_1 v_1' \sin \theta_1 - m_2 v_2' \sin \theta_2 \dots (2)$$

As the collision is elastic, kinetic energy is conserved,

$$\therefore \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \dots (3)$$

Using the above three equations, any three unknown quantities can be determined.

5.9 Different forms of energy

Besides mechanical energy, some examples of other forms of energy are as under.

Internal Energy:

The sum of vibrational kinetic energy and potential energy due to mutual attraction and repulsion between constituent particles of a substance is known as internal energy of the body. Due to work done against friction, internal energy and hence temperature of the body increases.

Heat or thermal energy:

The kinetic energy of the constituent particles of a body due to their random motion is known as heat or thermal energy of the body.

The difference between internal energy and heat is analogous to mechanical energy and work. When work is done by body A on body B, mechanical energy of body A decreases and that of body B increases. Similarly, when heat is transferred from body A to body B, internal energy of body A decreases and that of body B increases.

Chemical energy:

A stable compound has energy less than its constituent elements in free state. This difference is known as chemical energy or chemical binding energy. In a chemical reaction, if the chemical energy of the products is less than that of the reactants, heat is evolved and the reaction is called exothermic. Here, chemical energy got converted into heat energy. Similarly, if the chemical energy of the products is more than that of the reactants, heat is absorbed and the reaction is called endothermic. Here, heat energy got converted into chemical energy.

Electrical energy:

The energy associated with electric current is known as electrical energy. When electric current is used in a heater, electrical energy is converted into heat energy. Similarly, in a lamp, it is converted into heat and light energy.

Nuclear energy:

The mass of a nucleus is less than the sum of the masses of its constituent protons and neutrons in free state. The energy equivalent to this mass difference is known as nuclear energy or nuclear binding energy. In a nuclear fission reaction, when heavy nuclei like uranium are bombarded by neutrons, they break up into smaller nuclei releasing huge amount of nuclear energy. Such reactions are used in nuclear reactors for producing power and in

atomic weapons. In the Sun and stars, nuclear fusion reaction occurs in which lighter nuclei like protons, deuterons fuse at high temperature to form a helium nucleus releasing huge amount of energy. At the microscopic level, all different forms of energy are in the form of potential and/or kinetic energy.

Equivalence of mass and energy

Albert Einstein gave an equation for inter-conversion of mass and energy as under.

$E = mc^2$, where $c = 3 \times 10^8$ m/s is the velocity of light in vacuum.

Conservation of energy

“The total energy of an isolated system remains constant.” This is the statement of law of conservation of energy. One form of energy may get converted into another form or energy. Energy cannot be created or destroyed. The universe is an isolated system and so the total energy of the universe remains constant.

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