6 - DYNAMICS OF A SYSTEM OF PARTICLES

6.1 Centre of mass

The centre of mass of a system of n particles having masses m_1, m_2, ... , m_n and position vectors $\vec{r}_1$, $\vec{r}_2$, ... , $\vec{r}_n$ respectively is defined as

$$\vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{m_1 + m_2 + \ldots + m_n} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n}{M}, \quad \text{where}$$

$M = m_1 + m_2 + \ldots + m_n$ = the total mass of the system.

$$\therefore \quad M\vec{r}_{cm} = m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots + m_n\vec{r}_n$$

Assuming that the mass remains constant, the time derivative of the above equation gives

$$M\frac{d\vec{r}_{cm}}{dt} = m_1\frac{d\vec{r}_1}{dt} + m_2\frac{d\vec{r}_2}{dt} + \ldots + m_n\frac{d\vec{r}_n}{dt}$$

$$\therefore \quad M\vec{v}_{cm} = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n = \vec{P}$$

Here $\vec{P}$, which is the vector sum of the linear momenta of the particles of the system, is called the total linear momentum of the system and is the product of the total mass and velocity of the centre of mass of the system.

Differentiating the above equation w.r.t. time, we get

$$M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} + \ldots + \frac{d\vec{p}_n}{dt} = \frac{d\vec{P}}{dt}$$

$$= \vec{F}_1 + \vec{F}_2 + \ldots + \vec{F}_n = \vec{F}$$

$$= m_1\vec{a}_1 + m_2\vec{a}_2 + \ldots + m_n\vec{a}_n = M\vec{a}_{cm}$$

Here, $\vec{F}_1$, $\vec{F}_2$, ..., $\vec{F}_n$ are the forces acting on the particles of the system producing the accelerations $\vec{a}_1$, $\vec{a}_2$, ... , $\vec{a}_n$ respectively and $\vec{F}$ is the resultant force.

$$\therefore \quad M\frac{d\vec{v}_{cm}}{dt} = \frac{d\vec{P}}{dt} = \vec{F} = M\vec{a}_{cm}$$

Two kinds of forces act on the particle of a system:

(i) Internal forces between particles of the system and
(ii) external forces.

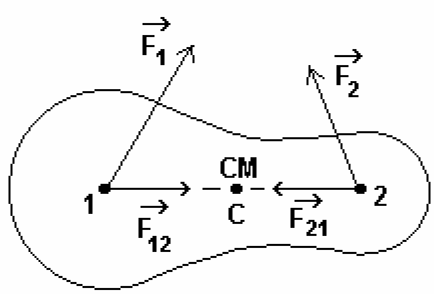

Fig. (a)

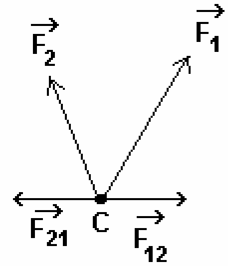

Fig. (b)

The external forces $\vec{F}_1$ $\vec{F}_2$ are acting on particles 1 and 2 of a system of particles as shown in the figure (a). $\vec{F}_{12}$ and $\vec{F}_{21}$ are the mutual forces of interaction between the particles 1 and 2. All these forces can be considered to be acting on the centre of mass (C.M.) as shown in figure (b).

According to Newton's third law of motion, the internal forces being equal and opposite cancel each other. Thus the system moves under the effect of the resultant external force $\vec{F}$ only as if the whole mass of the system were concentrated at its centre of mass.

This resultant external force $\vec{F}$ is equal to the rate of change of total linear momentum of the system. This is Newton's second law of motion for a system of particles which was derived with the help of Newton's third law of motion. This is known as inter-dependence of Newton's laws of motion.

6.2 Law of conservation of linear momentum

If the resultant external force on the system of particles is zero,

$$M\vec{a}_{cm} = \frac{d\vec{P}}{dt} = \vec{F} = 0. \qquad \therefore \quad \vec{P} = \vec{p}_1 + \vec{p}_2 + \dots + \vec{p}_n = \text{constant.}$$

Also, $\vec{a}_{cm} = 0 \Rightarrow$ velocity of the centre of mass remains constant.

"The total linear momentum of a system of particles remains constant in the absence of the resultant external force." This statement is known as the law of conservation of linear momentum.

In the absence of the resultant external force, the individual momenta, $\vec{p}_1$, $\vec{p}_2$, etc. of different particles of the system can change, but their vector sum, i. e., the total linear momentum remains constant. Also, as $\vec{a}_{cm} = 0$, the centre of mass of the system continues to move with uniform velocity. This law of conservation of linear momentum is universal. It s equally true for the systems made up of planets and the systems made up of protons, electrons, etc.

Suppose a chemical bomb lying at rest explodes without any external force and its fragments are thrown in the air. Although the initial kinetic energy of the bomb was zero, its fragments possess some kinetic energy. This mechanical energy is produced by conversion of chemical energy of the bomb. However, the law of conservation of momentum still holds good as no resultant external force acted on the bomb when it exploded. Thus, although the individual fragments possess momentum in different directions, the vector sum of their momenta would be zero as the initial momentum of the bomb was zero.

6.3 Centre of mass of a rigid (solid) body

- A system of particles in which the relative positions of particles remain invariant, irrespective of how large a force is applied to it, is called a rigid body. All rigid bodies are solid bodies, but all solid bodies may not be rigid bodies. Rigid body is an ideal concept. In fact, no real body is rigid.

- The location of the centre of mass of a rigid body depends on
  (i) the distribution of mass in the body and
  (ii) the shape of the body.

- The centre of mass of a rigid body can be anywhere inside or outside the body. For example, the centre of mass of a disc of uniform mass distribution is at its geometric centre, within its matter, while for a ring it lies at its geometric centre which is outside its matter.

- The centre of mass of symmetric bodies with uniform mass distribution can be easily obtained mathematically, whereas such mathematical derivation is difficult, if the body is not symmetric.

A solid object can be considered as made up of a large number of small mass elements, dm_1, dm_2, ... , dm_n having position vectors, $\vec{r}_1$, $\vec{r}_2$, ... , $\vec{r}_n$ respectively. By definition,

$$\vec{r}_{cm} = \frac{dm_1 \vec{r}_1 + dm_2 \vec{r}_2 + \dots + dm_n \vec{r}_n}{dm_1 + dm_2 + \dots + dm_n} = \frac{\int \vec{r}\, dm}{\int dm} = \frac{\int \vec{r}\, dm}{M}, \quad (M = \text{total mass of the body})$$

In terms of components,

$$x_{cm} = \frac{\int x\, dm}{M}, \quad y_{cm} = \frac{\int y\, dm}{M} \quad \text{and} \quad z_{cm} = \frac{\int z\, dm}{M}$$

If $\rho(x, y, z)$ is the density of the material of the body at point (x, y, z) where a small mass dm having volume dV is located, then

$dm = \rho(x, y, z)\, dV$ and the position vector of centre of mass,

$$\vec{r}_{cm} = \frac{\int \vec{r}\, \rho(x, y, z)\, dV}{\int \rho(x, y, z)\, dV}, \text{ the components of which are}$$

$$x_{cm} = \frac{\int x\, \rho(x, y, z)\, dV}{\int \rho(x, y, z)\, dV}, \quad y_{cm} = \frac{\int y\, \rho(x, y, z)\, dV}{\int \rho(x, y, z)\, dV} \quad \text{and} \quad z_{cm} = \frac{\int z\, \rho(x, y, z)\, dV}{\int \rho(x, y, z)\, dV}$$

If the density is uniform through, $\rho(x, y, z) = \rho = \text{constant}$ and
$\int dV = V = \text{volume of the body}$ which gives

$$x_{cm} = \frac{1}{V}\int x\, dV, \quad y_{cm} = \frac{1}{V}\int y\, dV \quad \text{and} \quad z_{cm} = \frac{1}{V}\int z\, dV \quad \text{where,} \quad dV = dx\, dy\, dz$$

All these integrals are taken over the entire body.