14.1 Periodic and Oscillatory motion

Motion of a system at regular interval of time on a definite path about a definite point is known as a periodic motion, e.g., uniform circular motion of a particle.

To and fro motion of a system on a linear path is called an oscillatory motion, e.g., motion of the bob of a simple pendulum.

14.2 Simple harmonic motion

This is the simplest type of periodic motion which can be understood by considering the following example.

Suppose a body of mass m is suspended at the lower end of a massless elastic spring obeying Hooke's law which is fixed to a rigid support in the vertical position. The spring elongates by length Δl and attains equilibrium as shown in Fig. (b)

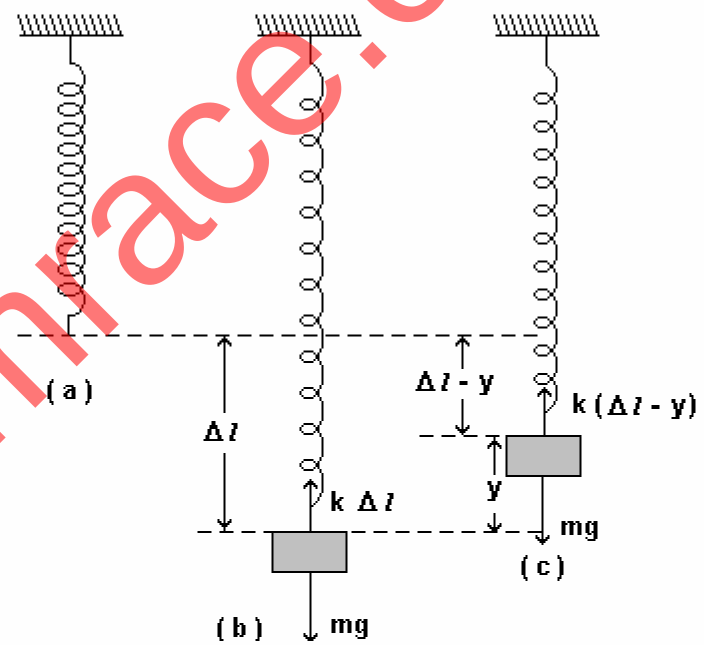
Here two forces act on the body.

(1) Its weight, mg, downwards and

(2) the restoring force developed in the spring, k Δl , upwards, where k = force constant of the spring.

For equilibrium, mg = k Δl ... (1)

The spring is constrained to move in the vertical direction only.



Spring-block system

Now, suppose the body is given some energy in its equilibrium condition and it undergoes displacement y in the upward direction as shown in Fig. (c).

Two forces act on the body in this displaced condition also.

(1) Its weight, mg, downwards and

(2) the restoring force developed in the spring, k (Δl - y), upwards.

The resultant force acting on the body in this condition is given by

F = - mg + k (Δl - y) ... ... (2)

From equations (1) and (2),

F = - ky

Displacement:

The distance of the body at any instant from the equilibrium point is known as its displacement at that instant. The displacements along the positive Y-axis are taken as positive and those on the negative Y-axis are taken as negative.

In the equation, $F = -ky$, F is negative when y is positive and vice versa. Thus, the resultant force acting on the body is proportional to the displacement and is directed opposite to the displacement, i.e., towards the equilibrium point.

Differential equation of simple harmonic motion (SHM)

According to Newton's second law of motion,

$$F = ma = m\frac{dv}{dt} = m\frac{d^2y}{dt^2} = -ky \quad \text{(for spring-type oscillator as above)}$$

$$\therefore \quad \frac{d^2y}{dt^2} = -\frac{k}{m}y = -\omega_0^2 y \quad \left(\text{taking } \frac{k}{m} = \omega_0^2\right)$$

$$\therefore \quad \frac{d^2y}{dt^2} + \omega_0^2 y = 0$$

This is the differential equation of SHM.

To obtain the solution of the above differential equation is to obtain y as a function of t such that on twice differentiating y w.r.t. t, we get back the same function y with a negative sign. Both the sine and the cosine functions possess such a property.

Hence, taking $y = A_1 \sin \omega_0 t + A_2 \cos \omega_0 t$ as a possible solution and differentiating twice w.r.t. t,

$$\frac{dy}{dt} = A_1 \omega_0 \cos \omega_0 t - A_2 \omega_0 \sin \omega_0 t \quad \text{and}$$

$$\frac{d^2y}{dt^2} = -A_1 \omega_0^2 \sin \omega_0 t - A_2 \omega_0^2 \cos \omega_0 t$$

$$= -\omega_0^2 (A_1 \sin \omega_0 t + A_2 \cos \omega_0 t) = -\omega_0^2 y$$

Thus, $y_t = A_1 \sin \omega_0 t + A_2 \cos \omega_0 t$ is the solution of the differential equation and is known as its general solution, where y_t is the displacement of simple harmonic oscillator (SHO) at time t.

Taking $A_1 = A \cos \phi$ and $A_2 = A \sin \phi$,

$$y_t = A \cos \phi \sin \omega_0 t + A \sin \phi \cos \omega_0 t$$

$$\therefore \quad y_t = A \sin (\omega_0 t + \phi) \quad \text{is the solution of the differential equation.}$$

A and ϕ are the constants of the equation whose values depend upon the initial position and initial velocity of the system. The equation gives displacement as a sinusoidal function which is periodic. Hence, the motion of the object represented by this equation is periodic on a linear path about $y = 0$ between $y = -A$ and $y = A$. Such a motion is known as simple harmonic motion (SHM).

Definition of SHM:

"The periodic motion of a body about a fixed point, on a linear path, under the influence of the force acting towards the fixed point and proportional to the displacement of the body from the fixed point, is called a simple harmonic motion."

The body performing SHM is known as a simple harmonic oscillator (SHO).

14.3 Amplitude, Period, Frequency, Angular frequency, Phase

Amplitude: The maximum displacement of the body executing SHM on either side of the mean position is called the amplitude of the SHO.

Phase: $\theta = \omega_0 t + \phi$ is the phase at time t of SHO performing SHM according to the equation $y_t = A \sin(\omega_0 t + \phi)$. At $t = 0$, $\theta = \phi$ which is known as initial phase, epoch or phase constant of the given SHM. The position and direction of motion of SHO at any time can be known from its phase.

Period: Displacement of an SHO at instant t is $y_t = A \sin(\omega_0 t + \phi)$. As the period of sine function is 2π radian, we have

$$y_t = A \sin(\omega_0 t + \phi + 2\pi) = A \sin\left[\omega_0\left(t + \frac{2\pi}{\omega_0}\right) + \phi\right]$$

$\therefore \quad T = \dfrac{2\pi}{\omega_0}$ is the period or the time taken to complete one oscillation by the oscillator.

Putting $\omega_0 = \sqrt{\dfrac{k}{m}}$, $\quad T = 2\pi\sqrt{\dfrac{k}{m}}$.

This is the period for any SHM. In the case of spring-block system, heavier the mass more the period and slower the oscillations. Also, if the spring is hard, its force constant k is large, the period is less and oscillations are faster.

Frequency and Angular frequency: The number of oscillations performed by the oscillator in 1 second is known as the frequency f_0 of the oscillator. Its unit is s^{-1} or hertz (Hz) in honour of the scientist Hertz.

Obviously, $f_0 = \dfrac{1}{T}$

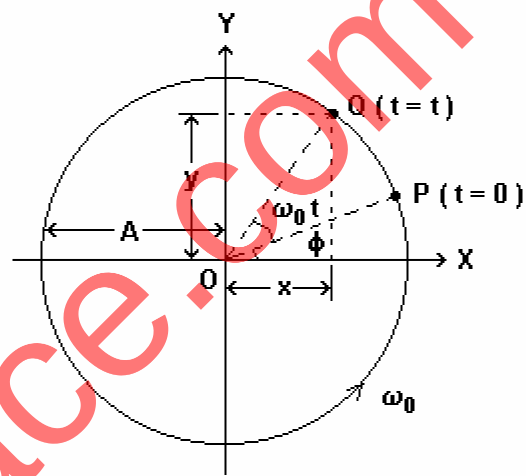
$\omega_0 = 2\pi f_0 = \dfrac{2\pi}{T}$ is the angular frequency of the oscillator. Its unit is rad / s.

14.4 Uniform circular motion and SHM

Consider a particle moving with a constant angular speed ω_0 in an anticlockwise direction on a circular path having centre O and radius A as shown in the figure.

At time t = 0, its angular position w.r.t. the reference line OX is $\angle POX = \phi$.

At time t = t, having undergone angular displacement $\omega_0 t$ reaching Q from P, its angular position is $\angle QOX = \omega_0 t + \phi$.



Uniform circular motion and SHM

The co-ordinates of point Q are

$x = A \cos(\omega_0 t + \phi)$ and ... ... (1)

$y = A \sin(\omega_0 t + \phi)$. ... ... (2)

As the particle moves on the circular path, its feet of perpendiculars on X- and Y- axes move as per the equations (1) and (2) and their motion is simple harmonic.

Thus, a given SHM can be described as the projected motion of a particle, known as the reference particle, performing an appropriate uniform circular motion on the diameter of the circle known as the reference circle. The radius of the reference circle is equal to the amplitude of the corresponding SHO and the angular speed of the reference particle is equal to the angular frequency of the SHO. Also, the angular position of the reference particle w.r.t. the reference line at any time is equal to the phase of the SHO at that time.

Combining two SHMs with phase difference of $\pi / 2$ and same amplitude results in uniform circular motion and if the amplitudes are different, the motion is on an elliptical path. Combining SHMs in different ways, different types of motion can be obtained.

14.5 Displacement, velocity and acceleration of SHO

Displacement: The equation for the displacement of SHO is

$y = A \sin(\omega_0 t + \phi)$.

Velocity:

Differentiating with respect to time, we get velocity,

$$v = \frac{dy}{dt} = A\,\omega_0 \cos(\omega_0 t + \phi) \quad \dots \quad (1)$$

$$= \pm A\,\omega_0 \sqrt{1 - \sin^2(\omega_0 t + \phi)} = \pm\,\omega_0 \sqrt{A^2 - A^2 \sin^2(\omega_0 t + \phi)}$$

$$= \pm\,\omega_0 \sqrt{A^2 - y^2}$$

Velocity of SHO, v, is positive when it is moving along positive y-direction and negative when it is moving along negative y-direction.

At $y = 0$ (equilibrium point), $v = \pm A\,\omega_0$ (which is maximum velocity).

At $y = \pm A$ (end points), $v = 0$.

The velocity of SHO and its corresponding reference particle are the same every time the SHO is at the equilibrium point.

Acceleration:

Differentiating equation (1) with respect to time, we get acceleration,

$$a = \frac{dv}{dt} = \frac{d^2y}{dt^2} = -A\,\omega_0^2 \sin(\omega_0 t + \phi) = -\,\omega_0^2\, y$$

At $y = 0$ (equilibrium point), $a = 0$.

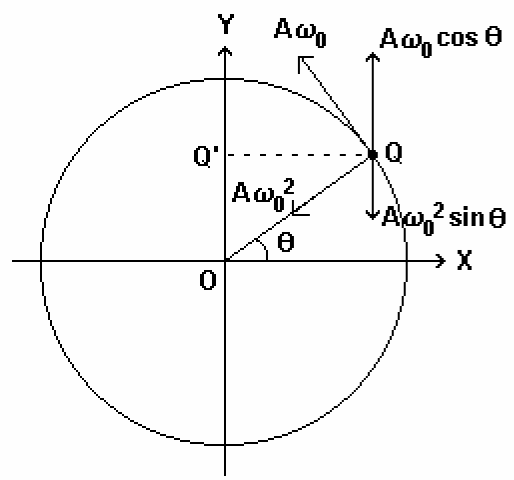
At $y = \pm A$ (end points), $a = \mp\, \omega_0^2 A$.

The acceleration of SHO and its corresponding reference particle are the same every time the SHO is at either of the end points.

Note:

The velocity of the SHO can also be found by taking the component of linear velocity $A\omega_0$ of the reference particle in the corresponding direction (here Y-axis), i.e., $A\,\omega_0 \cos\theta$ as shown in the figure.

Similarly the component of acceleration $A\,\omega_0^2$ of the reference particle in the corresponding direction (here Y-axis) is $A\,\omega_0^2 \sin\theta$ which is the magnitude of acceleration of the SHO.



Velocity and acceleration of SHO

14.7 Simple pendulum

" A system of a small massive body suspended by a light, inextensible string from a rigid (fixed) support and capable of oscillating in one vertical plane only is known as a simple pendulum."

Mass of the pendulum, m, is supposed to be concentrated at the centre of the suspended body called bob of the pendulum (figure on the next page).

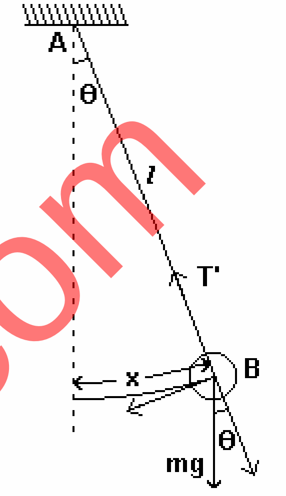
The distance of the centre of the bob from the point of suspension A is called the length (l) of the simple pendulum. At some instant, the bob of the pendulum is at B and the string makes an angle θ with the vertical.

The pendulum oscillates on the circular arc of radius l in a vertical plane as shown in the figure.

Two forces act on the bob of the pendulum.

(1) Weight of the bob = mg, in the downward direction and

(2) tension in the string T', in the direction BA.



The torque about A due to T' is zero as its line of action passes through A. The torque due to weight, mg, about A is

$$\vec{\tau} = \vec{l} \times m\vec{g} = -\, l\, mg \sin\theta$$

But, $\tau = I\alpha = m l^2 \alpha$ and $\alpha = \frac{d\omega_0}{dt} = \frac{d^2\theta}{dt^2}$

$$\therefore \quad m l^2 \frac{d^2\theta}{dt^2} = -\, l\, mg \sin\theta \quad \ldots \ldots \ldots \ldots \ldots \ldots (1)$$

For small θ (in radian), linear displacement of the bob on the curved path is x and

$$\sin\theta \approx \theta = \frac{x}{l}$$

Putting this value of sin θ in equation (1), we get

$$m l^2 \frac{d^2 (x / l)}{dt^2} = -\, l\, mg \frac{x}{l}$$

$$\therefore \quad \frac{d^2 x}{dt^2} = -\frac{g}{l} x .$$

This is the differential equation of SHM.

$$\therefore \quad \frac{g}{l} = \omega_0^2 = \frac{4\pi^2}{T^2}$$

$$\therefore \quad T = 2\pi \sqrt{\frac{l}{g}}$$

This is the expression of the period of the simple pendulum. Its value does not depend on the mass of the bob of the pendulum.

- The period of the spring-block type of SHO does not change when taken to a different planet as the values of m and k appearing in the expression of its period do not change.

- The period of simple pendulum increases on a planet where the value of 'g' is less and the pendulum clock taken there loses time, whereas its period decreases on the planet where the value of 'g' is more and the pendulum clock gains time when taken to that planet.

14.8 Damped oscillations

SHM is an ideal situation. In fact, there is always a resistive force offered by the medium. e.g., air resistance in case of oscillating pendulum and internal frictional forces as in the case of a vibrating tuning fork.

Energy lost in doing work against the resistive and frictional forces is mostly dissipated in the form of heat. The mechanical energy of SHO is E = $\frac{1}{2} k A^2$, where A is the amplitude of its oscillations. This shows that the amplitude of the oscillator decreases gradually due to dissipation of its energy. Such oscillations are called damped oscillations.

It is experimentally found that the resistive force acting on the oscillator opposing its motion is directly proportional to the velocity for small velocities.

$\therefore$ $F_v = - bv$, where b is a constant called the damping coefficient. Its unit is N-s / m.

Two forces act on the damped oscillator.

(1) Restoring force = - ky and (2) resistive force = - bv = $- b \frac{dy}{dt}$

According to Newton's second law, $m \frac{d^2y}{dt^2} = - ky - b \frac{dy}{dt}$

$$\therefore \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m} y = 0 \quad \dots \dots \dots \dots \dots \dots \dots \quad (1)$$

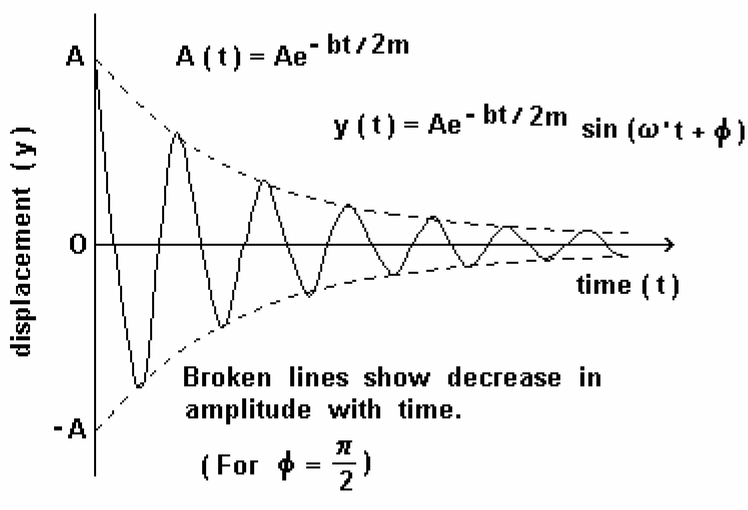
This is the differential equation for damped oscillations. Its solution for $\left[\frac{k}{m} > \left(\frac{b}{2m} \right)^2 \right]$ is $y_t = A e^{-\frac{bt}{2m}} \sin(\omega' t + \phi)$, where angular frequency of damped oscillations,

$$\omega' = \sqrt{\frac{k}{m} - \left(\frac{b}{2m} \right)^2}$$

Here, A and ϕ are the constants of the solution and their values depend on the initial conditions.

$A_t = A e^{-\frac{bt}{2m}}$ is the amplitude of the damped oscillator at time t which decreases exponentially with time.

The graph of displacement, $y_t \rightarrow$ time t is shown in the figure where the broken lines show the decrease in the amplitude with time.



Graph of displacement $\rightarrow$ time in damped oscillations

Putting $A_t = A e^{-\frac{bt}{2m}}$, the expression for mechanical energy of damped oscillator is

$E_t = \frac{1}{2} k A^2 e^{-\frac{bt}{m}}$ for small damping $\left(\frac{b}{\sqrt{2km}} << 1 \right)$ which shows that the mechanical energy also decreases exponentially with time.

14.9 Natural oscillations, Forced oscillations and Resonance

The oscillations of an oscillator in the absence of resistive forces are known as natural oscillations and their frequency as natural frequency (f_0), e.g., the natural angular frequency of the simple pendulum is $\omega_0 = \sqrt{\frac{g}{l}}$. An oscillator can have more than one natural frequency.

In reality, the amplitudes of oscillations decrease exponentially with time due to damping forces. To sustain natural oscillations, some external periodic force must be applied to the oscillator. The oscillations under the influence of some external periodic force are known as forced oscillations.

The differential equation of forced oscillations under the external periodic force, $F_0 \sin \omega t$, where ω is the frequency of the external force is given by

$$m \frac{d^2y}{dt^2} = - ky - b \frac{dy}{dt} + F_0 \sin \omega t$$

$$\therefore \quad \frac{d^2y}{dt^2} + \frac{b}{m}\frac{dy}{dt} + \frac{k}{m} y = \frac{F_0}{m} \sin \omega t$$

$$\therefore \quad \frac{d^2y}{dt^2} + r \frac{dy}{dt} + \omega_0^2 y = a_0 \sin \omega t \qquad \left(\text{putting } \frac{b}{m} = r, \quad \frac{k}{m} = \omega_0^2 \text{ and } \frac{F_0}{m} = a_0 \right).$$

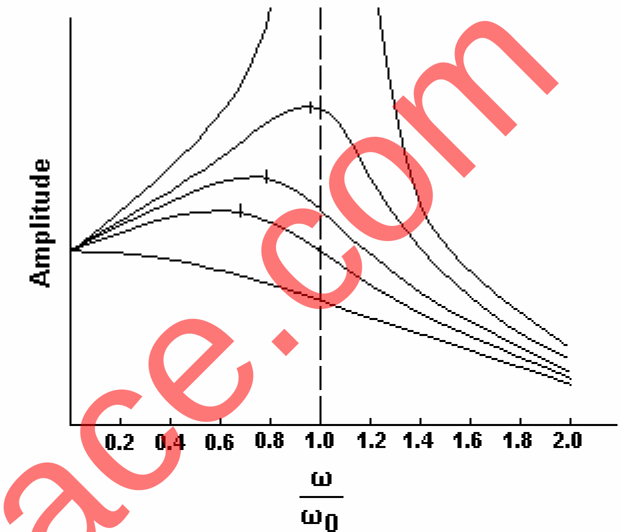
This is the differential equation of forced oscillation in the presence of damping and its solution is given as

$$y = A \sin (\omega t + \alpha), \quad \text{where } A = \frac{a_0}{\left[(\omega_0^2 - \omega^2)^2 + r^2 \omega^2 \right]^{\frac{1}{2}}} \quad \text{and} \quad \alpha = \tan^{-1} \frac{\omega y_0}{v_0}.$$

The amplitude of the oscillator is maximum when the value of $(\omega_0^2 - \omega^2)^2 + r^2\omega^2$ is minimum. It can be proved mathematically that this minimum value is reached when $\omega = \sqrt{\omega_0^2 - \frac{r^2}{2}}$. This phenomenon is known as resonance. The value of ω for which resonance occurs and the amplitude becomes maximum is known as the resonant angular frequency.

The curves for amplitude $\rightarrow \frac{\omega}{\omega_0}$ for different values of b are shown in the figure on the next page. The amplitude becomes infinite for b = 0 which is an ideal condition. For different curves, the amplitude is not maximum when $\frac{\omega}{\omega_0} = 1$, but it is maximum when it is close to 1 for small damping.

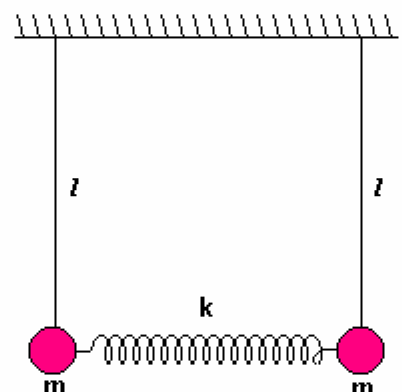
Mechanical systems may have more than one resonant frequencies. When the frequency of the external periodic force is close to the natural frequency, the system oscillates with a very large amplitude and it may break or collapse. This is the reason why soldiers are instructed to march out of pace on the bridge. While designing a bridge, care is taken so that its natural frequency is not close to the frequency of the external force due to gusts of wind.



Resonance curve

14.10 Coupled oscillations

The figure shows two pendulums connected by an elastic spring. Obviously, they cannot oscillate independently of each other. They are called coupled oscillators (more appropriately coupled pendula) and their oscillations are known as coupled oscillations. The constituent particles of solids also undergo coupled oscillations.



Coupled oscillators (Coupled pendula)

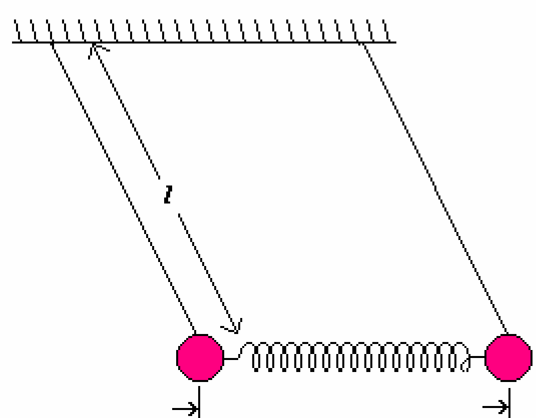
Oscillations of coupled oscillators are complex and not always simple harmonic, i.e., their displacements x_1 and x_2 cannot be expressed in the form of sine or cosine functions. But by suitable transformation of the co-ordinate system, they can be expressed in the form of equations of SHM as under.

$X_1 = A \sin(\omega_1 t + \phi_1)$ ... ... (1) and

$X_2 = B \sin(\omega_2 t + \phi_2)$ ... ... (2),

where $X_1 = x_1 + x_2$ and $X_2 = x_1 - x_2$.

ω_1 and ω_2 are normal frequencies and oscillations given by X_1 and X_2 with these frequencies are the normal modes of vibrations of the coupled oscillators. This oscillator has two normal modes as only two co-ordinates are present. With proper selection of initial conditions, the coupled oscillator can be oscillated in any one of these two modes.



Normal mode

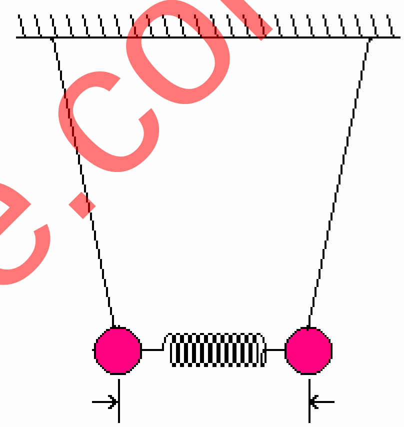
If at t = 0, $x_1 = x_2$, i.e., both the oscillators are given equal displacements in the same direction, then B = 0 from equation (2).

The coupled oscillator will oscillate with angular frequency $\omega_1 = \sqrt{\frac{g}{l}}$ according to equation (1). As shown in the figure (previous page), both the oscillators undergo equal displacements in the same direction in the same time. Hence the length of the spring does not change. So in this mode, the oscillators oscillate independently of each other as if the spring is not present.

Next, if at t = 0, $x_1 = -x_2$, i.e., both the oscillators are given equal displacements in mutually opposite directions and released, then A = 0 from equation (1).

The coupled oscillator will oscillate with angular frequency $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$ according to equation (2).

Both these types of oscillations are the normal modes of oscillations of the given coupled oscillator. If the initial conditions were different from the above two conditions, then the oscillations of each oscillator would be complex. However, in such a situation, the displacements of both the oscillators can be represented as a linear combination of the above two equations as the function of time.



Normal mode