Solve all problems vectorially:

(1) Obtain the unit vectors perpendicular to each of $\vec{x} = (1, 2, -1)$ and $\vec{y} = (1, 0, 2)$.

$\left[\text{Ans}: \pm \left(\frac{4}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{-2}{\sqrt{29}} \right) \right]$

(2) If α is the angle between two unit vectors $\vec{a}$ and $\vec{b}$, then prove that $|\vec{a} - \vec{b} \cos\alpha| = \sin\alpha$.

(3) If a vector $\vec{r}$ makes with X-axis and Y-axis angles of measures 45° and 60° respectively, then find the measure of the angle which $\vec{r}$ makes with Z-axis.

[Ans: 60° or 120°]

(4) If $\vec{x}$ and $\vec{y}$ are non-collinear vectors of R^3, then prove that $\vec{x}$, $\vec{y}$ and $\vec{x} \times \vec{y}$ are non-coplanar.

(5) If the measure of angle between $\bar{x} = \bar{i} + \bar{j}$ and $\bar{y} = t\bar{i} - \bar{j}$ is $\frac{3\pi}{4}$, then find t.

[Ans: 0]

(6) Show that for any $a \in R$, the directions (2, 3, 5) and (a, a + 1, a + 2) cannot be the same or opposite.

(7) If θ is a measure of angle between unit vectors $\bar{a}$ and $\bar{b}$, prove that $\sin\frac{\theta}{2} = \frac{1}{2}|\bar{a} - \bar{b}|$.

(8) If $\bar{x}$, $\bar{y}$ and $\bar{z}$ are non-coplanar, prove that $\bar{x} + \bar{y}$, $\bar{y} + \bar{z}$ and $\bar{z} + \bar{x}$ are also non-coplanar.

(9) Show that the vectors (1, 2, 1), (1, 1, 4) and (1, 3, - 2) are coplanar. Also express each of these vectors as a linear combination of the other two.

[Ans : $(1, 2, 1) = \frac{1}{2}(1, 1, 4) + \frac{1}{2}(1, 3, -2)$; $(1, 1, 4) = 2(1, 2, 1) - (1, 3, -2)$;
$(1, 3, -2) = 2(1, 2, 1) - (1, 1, 4)$]

[Note: These vectors are collinear besides being coplanar. Hence, any vector of R^3 which is not collinear with them cannot be expressed as a linear combination of these vectors even if it is coplanar with them.]

(10) Show that (1, 1, 0), (1, 0, 1) and (0, 1, 1) are non-coplanar vectors. Also express any vector (x, y, z) of R^3 as a linear combination of these vectors.

[Ans : $(x, y, z) = \frac{x + y - z}{2}(1, 1, 0) + \frac{x - y + z}{2}(1, 0, 1) + \frac{y + z - x}{2}(0, 1, 1)$]

(11) Prove that an angle in a semi-circle is a right angle.

(12) Prove that the three altitudes in a triangle are concurrent.

(13) If A - P - B and $\frac{AP}{PB} = \frac{m}{n}$, then prove that for any point O in space, $n(\overrightarrow{OA}) + m(\overrightarrow{OB}) = (m + n)\overrightarrow{OP}$.

(14) Prove that A (1, 5, 6), B (3, 1, 2) and C (4, - 1, 0) are collinear. Find also the ratio in which A divides $\overline{BC}$ from B.

[Ans: - 2 : 3]

(15) Find in which ratio and at which point does the XY-plane divide $\overline{AB}$ where A is (2, - 2, 1) and B is (1, 4, - 5).

[Ans : 1 : 5 from A at $\left(\frac{11}{6}, -1, 0\right)$]

(16) If A (0, - 1, - 1), B (16, - 3, - 3) and C (- 8, - 1, - 2) are given points, then find the point D (x, y, z) in space so that $\overrightarrow{AB} = \overrightarrow{CD}$.

[Ans: (8, - 3, - 4)]

(17) A (0, - 1, - 4), B (1, 2, 3) and C (5, 4, - 1) are given points. If D is the foot of perpendicular from A on $\overline{BC}$, find its position vector.

[Ans: (3, 3, 1)]

(18) If the position vectors A, B, C of triangle ABC are $\bar{a}$, $\bar{b}$, $\bar{c}$ respectively, then show that the area of triangle ABC = $\frac{1}{2}\left|(\bar{a}\times\bar{b})+(\bar{b}\times\bar{c})+(\bar{c}\times\bar{a})\right|$.

(19) Find the volume of a prism having a vertex at origin O and having coterminous edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, where A is (4, 3, 1), B is (3, 1, 2) and C is (5, 2, 1).

[Ans: 10 cubic units]

(20) Find the volume of tetrahedron having vertices V (1, 1, 3), A (4, 3, 2), B (5, 2, 7) and C (6, 4, 8).

$\left[\text{Ans}: \frac{14}{3} \text{ cubic units}\right]$

(21) If the forces of magnitudes $\sqrt{2}$, 2 and $\sqrt{3}$ units are applied to a particle in the directions of vectors (- 1, 0, 1), (1, 0, 1) and (1, 1, - 1) respectively, then find the magnitude and direction of the resultant force.

$\left[\text{Ans}: \sqrt{5},\ \left(\cos^{-1}\sqrt{\frac{2}{5}},\ \cos^{-1}\sqrt{\frac{1}{5}},\ \cos^{-1}\sqrt{\frac{2}{5}}\right)\right]$

(22) A boat is sailing to the east with a speed of $10\sqrt{2}$ km / hr. A man on boat feels that the wind is blowing from the south-east with a speed of 5 km / hr. Find the true velocity of the wind.

$\left[\text{Ans}: 5\sqrt{5} \text{ km/hr at an angle } \cos^{-1}\frac{3}{\sqrt{10}} \text{ with east towards north}\right]$

(23) A force of magnitude $2\sqrt{10}$ units is acting on a particle in the direction $3\bar{i} - \bar{j}$ and a force of magnitude $3\sqrt{13}$ units is acting on the same particle in the direction $2\bar{i} + 3\bar{j}$. Under the influence of these forces, the particle is displaced from A (1, 2) to B (6, 4). Find the work done. [Ans: 74 units]

(24) Prove that the diagonals of a rhombus bisect each other orthogonally.

(25) If a pair of medians of a triangle are equal, then show that the triangle is isosceles.

(26) Show that the perpendicular bisectors of sides of any triangle are concurrent.

(27) Prove that the diagonals of a rhombus are bisectors of its angles.

(28) If $\overrightarrow{AD}$ is a bisector of $\angle BAC$ in triangle ABC and if $D \in BC$, then show that $\frac{BD}{DC} = \frac{AB}{AC}$.

(29) ABCDEF is a regular hexagon. Prove that $\overrightarrow{AB} + \overrightarrow{AC} + \overrightarrow{AD} + \overrightarrow{AE} + \overrightarrow{AF} = 3\overrightarrow{AD}$.

(30) Show that centroid and in-centre of an equilateral triangle are the same. Find the incentre of the triangle with vertices (6, 4, 6), (12, 4, 0) and (4, 2, - 2).

$\left[\text{Ans : } \left(\frac{22}{3}, \frac{10}{3}, \frac{4}{3} \right) \right]$

(31) If A is (1, 2, 1) and B is (4, - 1, 2), then find S (x, y, z) such that $2\overrightarrow{AB} = \overrightarrow{AS}$.

[Ans: (7, - 4, 3)]

(32) Let A (1, 2, - 1) and B (3, 2, 2) be given points. Find in which ratios from A and at which points do the XY-, YZ- and ZX-planes divide $\overline{AB}$.

[Ans : 1 : 2, $\left(\frac{5}{3}, 2, 0\right)$; - 1 : 3, $\left(0, 2, \frac{5}{2}\right)$; $\overleftrightarrow{AB}$ is parallel to ZX - plane]

(33) Show that (6, 0, 1), (8, - 3, 7) and (2, - 5, 10) can be three vertices of some rhombus. Find the co-ordinates of the fourth vertex of this rhombus.

[Ans: (0, - 2, 4)]

(34) Show that (1, 2, 4), (- 1, 1, 1), (6, 3, 8) and (2, 1, 2) are the vertices of a trapezium. Find the area of this trapezium.

[Ans : $\frac{3}{2}\sqrt{59}$]

(35) Find the area of the parallelogram ABCD if $\overrightarrow{AC} = \bar{a}$ and $\overrightarrow{BD} = \bar{b}$.

[Ans : $\frac{1}{2}\left|\bar{a} \times \bar{b}\right|$]

(36) Find the volume of a prism having a vertex at origin and having edges $\overrightarrow{OA} = 2\bar{i} + \bar{j} + \bar{k}$, $\overrightarrow{OB} = 3\bar{i} - \bar{j} + \bar{k}$ and $\overrightarrow{OC} = -\bar{i} + \bar{j} - \bar{k}$.

[Ans: 4 cubic units]

(37) Show that (4, 5, 1), (0, - 1, - 1), (3, 9, 4) and (- 4, 4, 4) cannot be the vertices of any tetrahedron.

(38) Find the volume of the tetrahedron with vertices (4, 5, 1), (0, - 1, - 1), (3, 9, 4) and (1, 2, 3).

[Ans : $\frac{28}{3}$ cubic units]

(39) A mechanical boat is rowing towards the north with speed of 8 km / hr. If wind blows from the east with the speed of 10 km / hr, find the resulting speed of the boat and also the direction of resulting motion of the boat.

[Ans : $2\sqrt{41}$ km / hr at an angle of $\pi - \cos^{-1}\left(\frac{5}{\sqrt{41}}\right)$ with east towards north]

(40) A river flows with a speed of 5 units. A person desires to cross the river in a direction perpendicular to its flow. Find in which direction should he swim if his speed is 8 units.

[Ans : At an angle of $\pi - \cos^{-1}\left(\frac{5}{8}\right)$ with the direction of flow of the river]

(41) If speed of a particle is 5 units towards the east and $\sqrt{8}$ units towards the south-west, then find the resultant speed of the particle and its direction.

[Ans : $\sqrt{13}$ units at an angle of $\cos^{-1}\frac{3}{\sqrt{13}}$ with east towards south]

(42) A boat speeds towards the north at $6\sqrt{2}$ units. A man on the boat feels that the wind is blowing from the south-east at 5 units. Find the true velocity of the wind.

[Ans : $\sqrt{157}$ units at an angle of $\pi - \cos^{-1}\left(\frac{5}{\sqrt{314}}\right)$ with east towards north]

(43) A steamer moves to the north-east with a speed of 40 units. A passenger on the steamer feels the wind to be blowing from the north with $25\sqrt{2}$ units. Find the true velocity of the wind.

[Ans : $5\sqrt{34}$ units at an angle of $\cos^{-1}\frac{4}{\sqrt{17}}$ with east towards south]

(44) A particle is displaced from A (2, 1) to B (4, 2) when forces of magnitudes $4\sqrt{5}$ in the direction $2\bar{i} + \bar{j}$ and $6\sqrt{5}$ in the direction $\bar{i} - 2\bar{j}$ are applied. Find the work done.

[Ans: 20 units]