(1) If a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ intersects the major axis in T and minor axis in T', then prove that $\frac{a^2}{CT^2} - \frac{b^2}{CT'^2} = 1$, where C is the centre of the hyperbola.

(2) Show that the angle between two asymptotes of the hyperbola $x^2 - 2y^2 = 1$ is $\tan^{-1}(2\sqrt{2})$.

(3) Prove that the product of the lengths of the perpendicular line segments from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes is $\frac{a^2b^2}{a^2 + b^2}$.

(4) Find the co-ordinates of foci, equations of directrices, eccentricity and length of the latus-rectum for the following hyperbolas:
(i) $25x^2 - 144y^2 = -3600$, (ii) $x^2 - y^2 = 16$.

$\left[\text{Ans: (i) } (0, \pm 13),\ y = \pm\frac{25}{13},\ e = \frac{13}{5},\ \frac{288}{5} \quad \text{(ii) } (\pm 4\sqrt{2}, 0),\ x = \pm 2\sqrt{2},\ e = \sqrt{2},\ 8 \right]$

(5) If the eccentricities of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$ are e_1 and e_2 respectively, then prove that $e_1^{-2} + e_2^{-2} = 1$.

(6) Prove that the equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining $P(\alpha)$ and $Q(\beta)$ is $\frac{x}{a}\cos\frac{\alpha - \beta}{2} - \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha + \beta}{2}$. If this chord passes through the focus (ae, 0), then prove that $\tan\frac{\alpha}{2}\tan\frac{\beta}{2} = \frac{1 - e}{1 + e}$

(7) If $\theta + \phi = 2\alpha$ (constant), then prove that all the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining the points $P(\theta)$ and $Q(\phi)$ pass through a fixed point.

(8) If the chord $\overline{PQ}$ of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ subtends a right angle at the centre C, then prove that $\frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$ (b > a).

(9) For a point on the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, prove that $SP \cdot S'P = CP^2 - a^2 + b^2$.

(10) Find the equation of the common tangent to the hyperbola $3x^2 - 4y^2 = 12$ and parabola $y^2 = 4x$.

[Ans: $\pm y = x + 1$]

(11) Find the condition for the line $x \cos\alpha + y \sin\alpha = p$ to be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

[Ans: $p^2 = a^2 \cos^2\alpha - b^2 \sin^2\alpha$]

(12) Find the equation of a common tangent to the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ ($a > b$).

[Ans: $x - y = \pm\sqrt{a^2 - b^2}$, $x + y = \pm\sqrt{a^2 - b^2}$]

(13) Find the equation of the hyperbola passing through the point (1, 4) and having asymptotes $y = \pm 5x$.

[Ans: $25x^2 - y^2 = 9$]

(14) Prove that the area of the triangle formed by the asymptotes and any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is ab.

(15) A line passing through the focus S and parallel to an asymptote intersects the hyperbola at the point P and the corresponding directrix at the point Q. Prove that SQ = 2 SP.

(16) K is the foot of perpendicular to an asymptote from the focus S of the rectangular hyperbola. Prove that the hyperbola bisects $\overline{SK}$.

(17) A line passing through a point P on the hyperbola and parallel to an asymptote intersects the directrix in K. Prove that PK = SP.

(18) If the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ joining the points α and β subtend the right angle at the vertex (a, 0), then prove that $a^2 + b^2 \cot\frac{\alpha}{2} \cot\frac{\beta}{2} = 0$.

(19) Find the condition that the line $lx + my + n = 0$ may be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and find the co-ordinates of the point of contact.

[Ans : $a^2l^2 - b^2m^2 = n^2$, $\left(-\frac{a^2 l}{n}, \frac{b^2 m}{n} \right)$]

(20) Prove that the segment of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ between the point of contact and its intersection with a directrix subtends a right angle at the corresponding focus.

(21) Find the equations of the tangents drawn from the point (- 2, - 1) to the hyperbola $2x^2 - 3y^2 = 6$.

[Ans: $3x - y + 5 = 0$, $x - y + 1 = 0$]

(22) If the line $y = mx + \sqrt{a^2m^2 - b^2}$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point P (α), then prove that $\sin\alpha = \frac{b}{am}$.

(23) If the lines $2y - x = 14$ and $3y - x = 9$ are tangential to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then find the values of a^2 and b^2.

[Ans: $a^2 = 288$, $b^2 = 33$]

(24) The tangent and normal at a point P on the rectangular hyperbola $x^2 - y^2 = 1$ cut off intercepts a_1, a_2 on the X-axis and b_1, b_2 on the Y-axis. Prove that $a_1 a_2 = b_1 b_2$.

(25) Prove that the locus of intersection of tangents to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, which meet at a constant angle β, is the curve

$$(x^2 + y^2 + b^2 - a^2)^2 = 4\cot^2\beta\,(a^2y^2 - b^2x^2 + a^2b^2).$$

(26) Prove that the equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which has its mid-point at (h, k) is $\frac{hx}{a^2} - \frac{ky}{b^2} = \frac{h^2}{a^2} - \frac{k^2}{b^2}$.

(27) If a rectangular hyperbola circumscribes a triangle, then prove that it also passes through the orthocentre of the triangle.

(28) If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points "t_1", "t_2", "t_3" and "t_4", then prove that

(i) product of the abscissae of the four points = the product of their ordinates = c^4,

(ii) the centre of the circle through the points "t_1", "t_2", "t_3" is

$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3}\right),\ \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3\right)\right\}$$

(29) For a rectangular hyperbola $xy = c^2$, prove that the locus of the mid-points of the chords of constant length 2d is $(x^2 + y^2)(xy - c^2) = d^2xy$.

(30) If P_1, P_2 and P_3 are three points on the rectangular hyperbola $xy = c^2$, whose abscissae are x_1, x_2 and x_3, then prove that the area of the triangle $P_1P_2P_3$ is

$$\frac{c^2}{2}\left|\frac{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}{x_1 x_2 x_3}\right|.$$