British Mathematical Olympiad

Round 1 : Wednesday, 14 January 1998

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions - not just answers - are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt. Do not hand in rough work.*
- *One* complete *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* left *hand corner and your name, initials and school in the top* right *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1,2,3,4,5 in order.*
- *Staple all the pages neatly together in the top* left *hand corner.*

Do not turn over until **told to do so.**

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1. A 5×5 square is divided into 25 unit squares. One of the numbers 1, 2, 3, 4, 5 is inserted into each of the unit squares in such a way that each row, each column and each of the two diagonals contains each of the five numbers once and only once. The sum of the numbers in the four squares immediately below the diagonal from top left to bottom right is called the *score*.
   Show that it is impossible for the score to be 20.
   What is the highest possible score?

2. Let $a_1 = 19$, $a_2 = 98$. For $n \geq 1$, define a_{n+2} to be the remainder of $a_n + a_{n+1}$ when it is divided by 100. What is the remainder when
   $$a_1^2 + a_2^2 + \cdots + a_{1998}^2$$
   is divided by 8?

3. ABP is an isosceles triangle with $AB = AP$ and $\angle PAB$ acute. PC is the line through P perpendicular to BP, and C is a point on this line on the same side of BP as A. (You may assume that C is not on the line AB.) D completes the parallelogram $ABCD$. PC meets DA at M.
   Prove that M is the midpoint of DA.

4. Show that there is a unique sequence of positive integers (a_n) satisfying the following conditions:
   $$a_1 = 1, \quad a_2 = 2, \quad a_4 = 12,$$
   $$a_{n+1}a_{n-1} = a_n^2 \pm 1 \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

5. In triangle ABC, D is the midpoint of AB and E is the point of trisection of BC nearer to C. Given that $\angle ADC = \angle BAE$ find $\angle BAC$.