BOARD OF STUDIES
NEW SOUTH WALES

2008

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial x^3 is divided by $x + 3$. Calculate the remainder. **2**

(b) Differentiate $\cos^{-1}(3x)$ with respect to x. **2**

(c) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt{4 - x^2}}\, dx$. **2**

(d) Find an expression for the coefficient of x^8y^4 in the expansion of $(2x + 3y)^{12}$. **2**

(e) Evaluate $\displaystyle\int_{0}^{\frac{\pi}{4}} \cos\theta \sin^2\theta \, d\theta$. **2**

(f) Let $f(x) = \log_e\left[(x - 3)(5 - x)\right]$. **2**

What is the domain of $f(x)$?

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = \log_e x$ to evaluate $\displaystyle\int_e^{e^2} \frac{1}{x(\log_e x)^2}\,dx$. **3**

(b) A particle moves on the x-axis with velocity v. The particle is initially at rest at $x = 1$. Its acceleration is given by $\ddot{x} = x + 4$. **3**

Using the fact that $\ddot{x} = \dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$, find the speed of the particle at $x = 2$.

(c) The polynomial $p(x)$ is given by $p(x) = ax^3 + 16x^2 + cx - 120$, where a and c are constants. **3**

The three zeros of $p(x)$ are -2, 3 and α.

Find the value of α.

(d) The function $f(x) = \tan x - \log_e x$ has a zero near $x = 4$. **3**

Use one application of Newton's method to obtain another approximation to this zero. Give your answer correct to two decimal places.

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

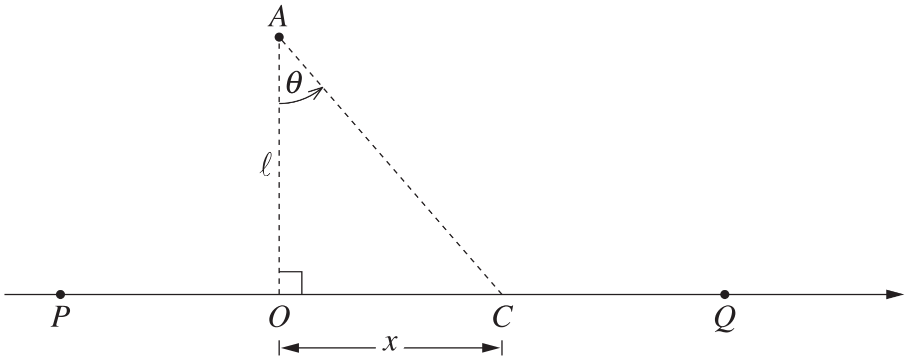
(a) (i) Sketch the graph of $y = |2x - 1|$. **1**

(ii) Hence, or otherwise, solve $|2x - 1| \le |x - 3|$. **3**

(b) Use mathematical induction to prove that, for integers $n \ge 1$, **3**

$$1 \times 3 + 2 \times 4 + 3 \times 5 + \cdots + n(n+2) = \frac{n}{6}(n+1)(2n+7).$$

(c)



A race car is travelling on the x-axis from P to Q at a constant velocity, v. A spectator is at A which is directly opposite O, and $OA = \ell$ metres. When the car is at C, its displacement from O is x metres and $\angle OAC = \theta$, with $-\frac{\pi}{2} < \theta \quad \frac{\pi}{2}$.

(i) Show that $\dfrac{d\theta}{dt} = \dfrac{v\ell}{\ell^2 + x^2}$. **2**

(ii) Let m be the maximum value of $\dfrac{d\theta}{dt}$. **1**

Find the value of m in terms of v and ℓ.

(iii) There are two values of θ for which $\dfrac{d\theta}{dt} = \dfrac{m}{4}$. **2**

Find these two values of θ.

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A turkey is taken from the refrigerator. Its temperature is 5°C when it is placed in an oven preheated to 190°C.

Its temperature, T°C, after t hours in the oven satisfies the equation

$$\frac{dT}{dt} = -k(T - 190).$$

(i) Show that $T = 190 - 185e^{-kt}$ satisfies both this equation and the initial condition. **2**

(ii) The turkey is placed into the oven at 9 am. At 10 am the turkey reaches a temperature of 29°C. The turkey will be cooked when it reaches a temperature of 80°C. **3**

At what time (to the nearest minute) will it be cooked?

(b) Barbara and John and six other people go through a doorway one at a time.

(i) In how many ways can the eight people go through the doorway if John goes through the doorway after Barbara with no-one in between? **1**

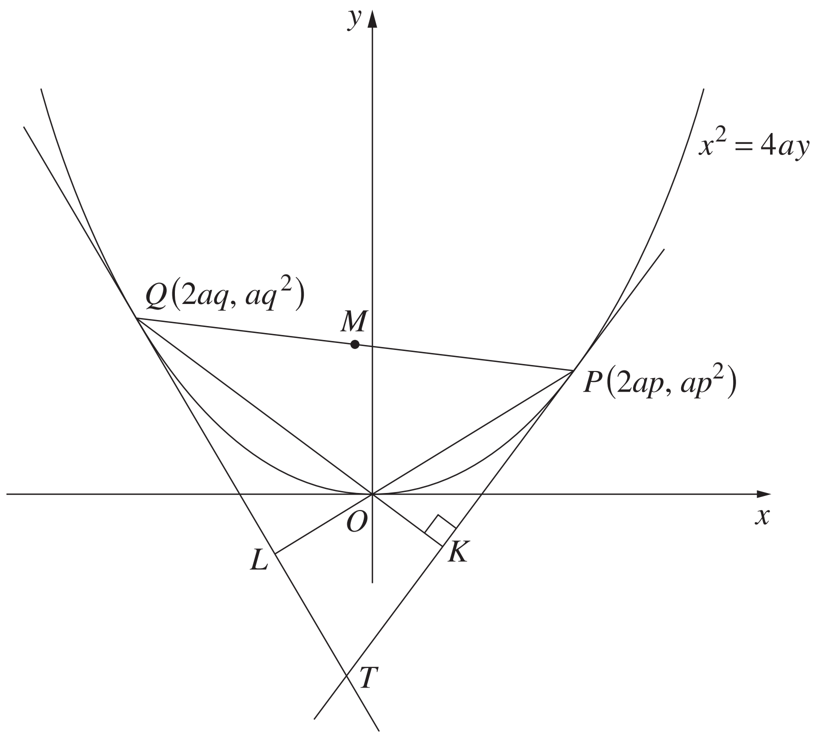
(ii) Find the number of ways in which the eight people can go through the doorway if John goes through the doorway after Barbara. **1**

Question 4 continues on page 7

Marks

Question 4 (continued)

(c)



The points $P(2ap, ap^2)$, $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents to the parabola at P and Q intersect at T. The chord QO produced meets PT at K, and $\angle PKQ$ is a right angle.

(i) Find the gradient of QO, and hence show that $pq = -2$. **2**

(ii) The chord PO produced meets QT at L. Show that $\angle PLQ$ is a right angle. **1**

(iii) Let M be the midpoint of the chord PQ. By considering the quadrilateral $PQLK$, or otherwise, show that $MK = ML$. **2**

End of Question 4

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = x - \frac{1}{2}x^2$ for $x \le 1$. This function has an inverse, $f^{-1}(x)$.

(i) Sketch the graphs of $y = f(x)$ and $y = f^{-1}(x)$ on the same set of axes. (Use the same scale on both axes.) **2**

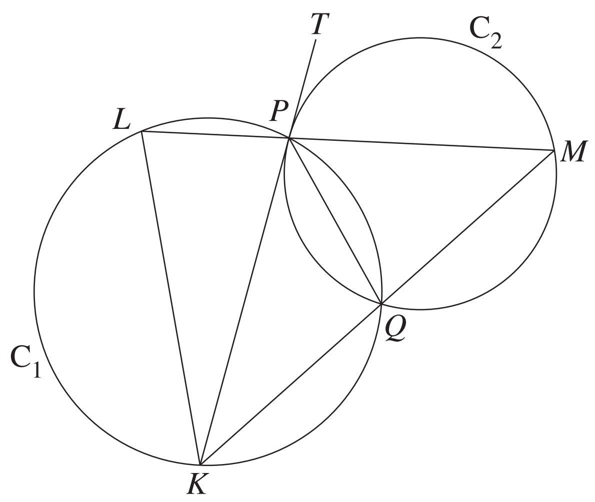
(ii) Find an expression for $f^{-1}(x)$. **3**

(iii) Evaluate $f^{-1}\left(\frac{3}{8}\right)$. **1**

(b) A particle is moving in simple harmonic motion in a straight line. Its maximum speed is 2 m s^{-1} and its maximum acceleration is 6 m s^{-2}. **3**

Find the amplitude and the period of the motion.

(c) **3**



Two circles C_1 and C_2 intersect at P and Q as shown in the diagram. The tangent TP to C_2 at P meets C_1 at K. The line KQ meets C_2 at M. The line MP meets C_1 at L.

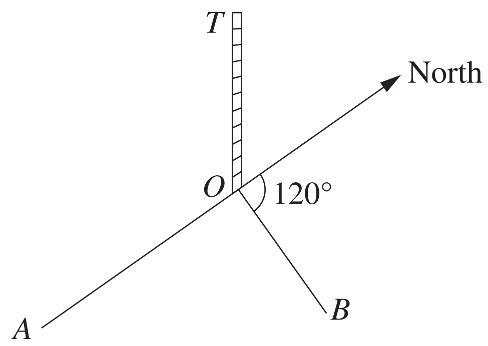
Copy or trace the diagram into your writing booklet.

Prove that ΔPKL is isosceles.

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) From a point A due south of a tower, the angle of elevation of the top of the tower T, is 23°. From another point B, on a bearing of 120° from the tower, the angle of elevation of T is 32°. The distance AB is 200 metres.



(i) Copy or trace the diagram into your writing booklet, adding the given information to your diagram. **1**

(ii) Hence find the height of the tower. **3**

(b) It can be shown that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$ for all values of θ. (Do NOT prove this.) **3**

Use this result to solve $\sin 3\theta + \sin 2\theta = \sin\theta$ for $0 \le \theta \le 2\pi$.

(c) Let p and q be positive integers with $p \le q$.

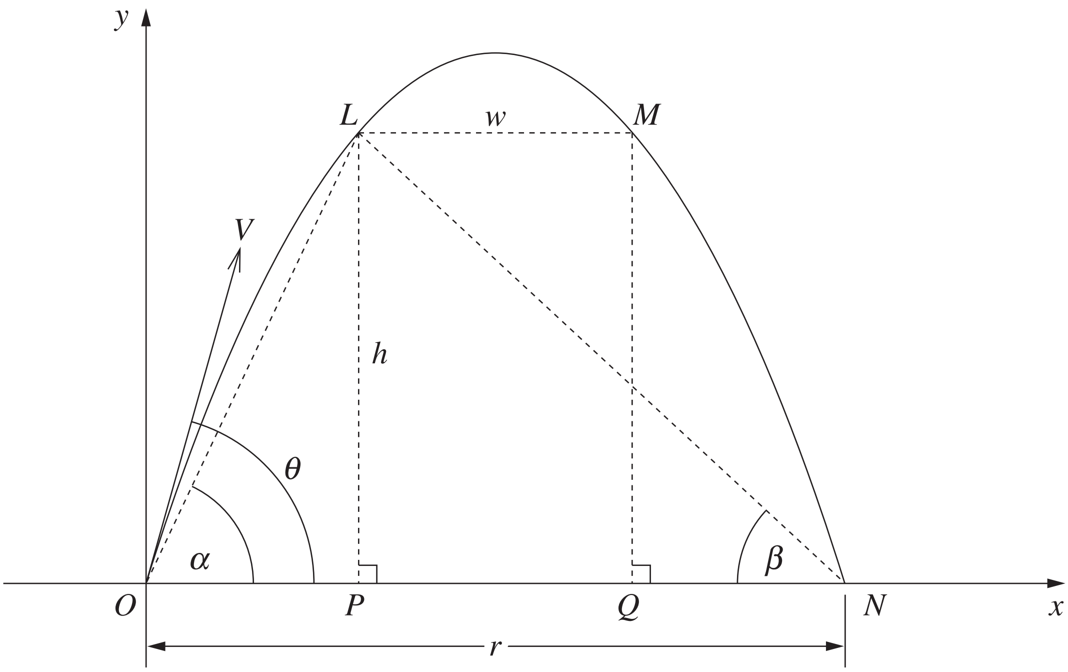
(i) Use the binomial theorem to expand $(1+x)^{p+q}$, and hence write down the term of $\dfrac{(1+x)^{p+q}}{x^q}$ which is independent of x. **2**

(ii) Given that $\dfrac{(1+x)^{p+q}}{x^q} = (1+x)^p\left(1+\dfrac{1}{x}\right)^q$, apply the binomial theorem and the result of part (i) to find a simpler expression for **3**

$$1 + \binom{p}{1}\binom{q}{1} + \binom{p}{2}\binom{q}{2} + \cdots + \binom{p}{p}\binom{q}{p}.$$

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.



A projectile is fired from O with velocity V at an angle of inclination θ across level ground. The projectile passes through the points L and M, which are both h metres above the ground, at times t_1 and t_2 respectively. The projectile returns to the ground at N.

The equations of motion of the projectile are

$$x = Vt\cos\theta \text{ and } y = Vt\sin\theta - \frac{1}{2}gt^2. \text{ (Do NOT prove this.)}$$

(a) Show that $t_1 + t_2 = \dfrac{2V}{g}\sin\theta$ AND $t_1 t_2 = \dfrac{2h}{g}$. **2**

Question 7 continues on page 11

Marks

Question 7 (continued)

Let $\angle LON = \alpha$ and $\angle LNO = \beta$. It can be shown that

$$\tan\alpha = \frac{h}{Vt_1\cos\theta} \text{ and } \tan\beta = \frac{h}{Vt_2\cos\theta}.$$ (Do NOT prove this.)

(b) Show that $\tan\alpha + \tan\beta = \tan\theta$. **2**

(c) Show that $\tan\alpha\tan\beta = \dfrac{gh}{2V^2\cos^2\theta}$. **1**

Let $ON = r$ and $LM = w$.

(d) Show that $r = h(\cot\alpha + \cot\beta)$ and $w = h(\cot\beta - \cot\alpha)$. **2**

Let the gradient of the parabola at L be $\tan\phi$.

(e) Show that $\tan\phi = \tan\alpha - \tan\beta$. **3**

(f) Show that $\dfrac{w}{\tan\phi} = \dfrac{r}{\tan\theta}$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$