

B O A R D O F S T U D I E S
NEW SOUTH WALES

2006

**HIGHER SCHOOL CERTIFICATE
EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Find $\int \frac{x}{\sqrt{9-4x^2}} dx$. **2**

(b) By completing the square, find $\int \frac{dx}{x^2-6x+13}$. **2**

(c) (i) Given that $\frac{16x-43}{(x-3)^2(x+2)}$ can be written as **3**

$$\frac{16x-43}{(x-3)^2(x+2)} = \frac{a}{(x-3)^2} + \frac{b}{x-3} + \frac{c}{x+2},$$

where a , b and c are real numbers, find a , b and c .

(ii) Hence find $\int \frac{16x-43}{(x-3)^2(x+2)} dx$. **2**

(d) Evaluate $\int_0^2 te^{-t} dt$. **3**

(e) Use the substitution $t = \tan \frac{\theta}{2}$ to show that **3**

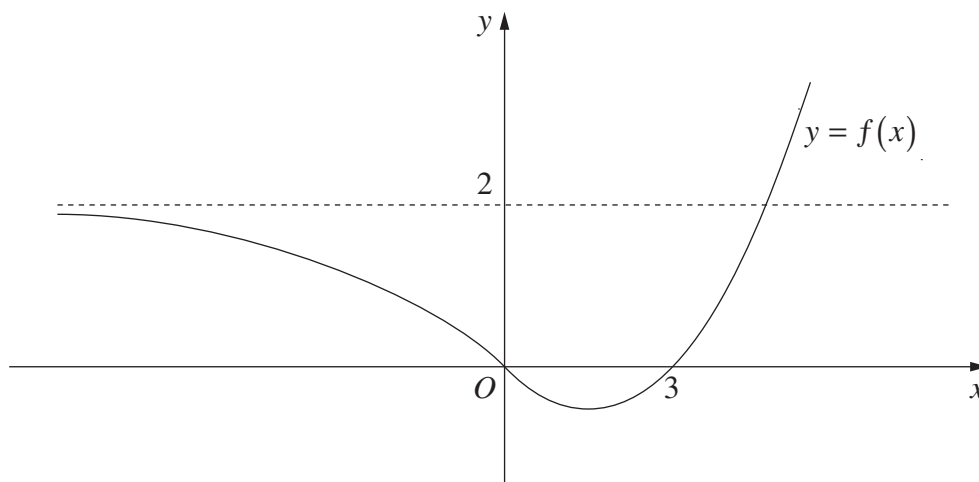
$$\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \frac{d\theta}{\sin \theta} = \frac{1}{2} \log 3.$$

Question 2 (15 marks) Use a SEPARATE writing booklet.

- (a) Let $z = 3 + i$ and $w = 2 - 5i$. Find, in the form $x + iy$,
- (i) z^2 1
 - (ii) $\bar{z}w$ 1
 - (iii) $\frac{w}{z}$. 1
- (b)
- (i) Express $\sqrt{3} - i$ in modulus-argument form. 2
 - (ii) Express $(\sqrt{3} - i)^7$ in modulus-argument form. 2
 - (iii) Hence express $(\sqrt{3} - i)^7$ in the form $x + iy$. 1
- (c) Find, in modulus-argument form, all solutions of $z^3 = -1$. 2
- (d) The equation $|z - 1 - 3i| + |z - 9 - 3i| = 10$ corresponds to an ellipse in the Argand diagram.
- (i) Write down the complex number corresponding to the centre of the ellipse. 1
 - (ii) Sketch the ellipse, and state the lengths of the major and minor axes. 3
 - (iii) Write down the range of values of $\arg(z)$ for complex numbers z corresponding to points on the ellipse. 1

Question 3 (15 marks) Use a SEPARATE writing booklet.

- (a) The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 2$.



Draw separate one-third page sketches of the graphs of the following:

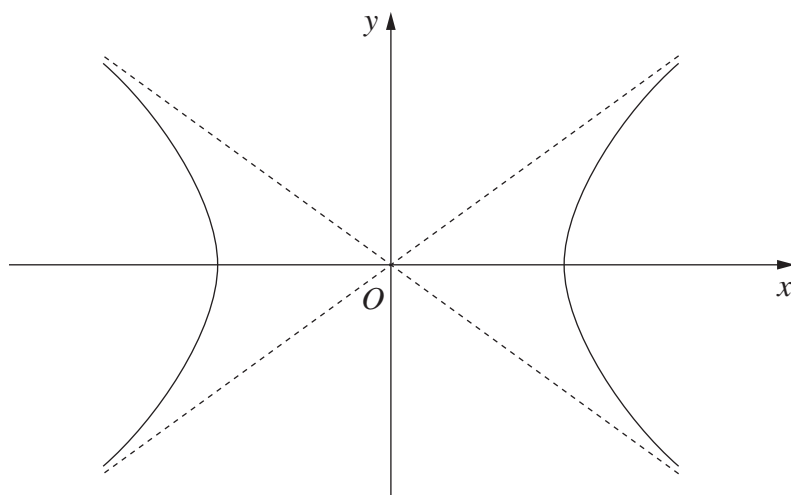
- (i) $y = (f(x))^2$ **2**
- (ii) $y = \frac{1}{f(x)}$ **2**
- (iii) $y = xf(x)$. **2**

Question 3 continues on page 5

Question 3 (continued)

(b) The diagram shows the graph of the hyperbola

$$\frac{x^2}{144} - \frac{y^2}{25} = 1.$$



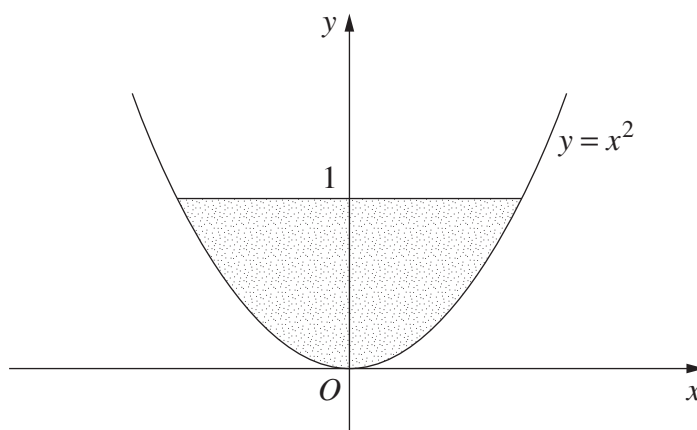
- (i) Find the coordinates of the points where the hyperbola intersects the x -axis. 1
- (ii) Find the coordinates of the foci of the hyperbola. 2
- (iii) Find the equations of the directrices and the asymptotes of the hyperbola. 2
- (c) Two of the zeros of $P(x) = x^4 - 12x^3 + 59x^2 - 138x + 130$ are $a + ib$ and $a + 2ib$, where a and b are real and $b > 0$.
- (i) Find the values of a and b . 3
- (ii) Hence, or otherwise, express $P(x)$ as the product of quadratic factors with real coefficients. 1

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

- (a) The polynomial $p(x) = ax^3 + bx + c$ has a multiple zero at 1 and has remainder 4 when divided by $x + 1$. Find a , b and c . **3**

- (b) The base of a solid is the parabolic region $x^2 \leq y \leq 1$ shaded in the diagram. **3**



Vertical cross-sections of the solid perpendicular to the y -axis are squares.

Find the volume of the solid.

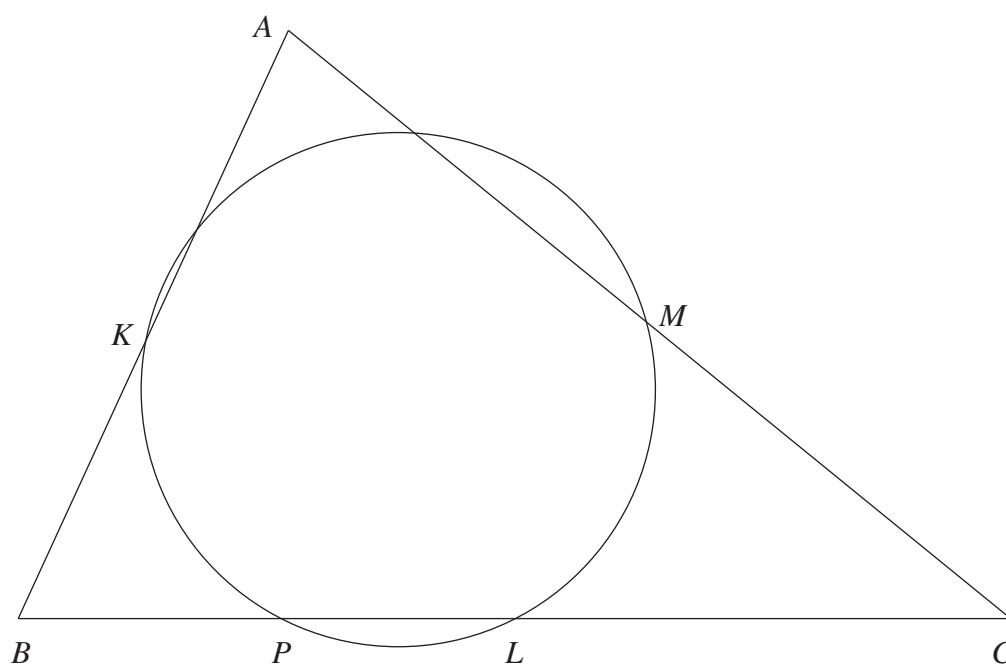
- (c) Let $P\left(p, \frac{1}{p}\right)$, $Q\left(q, \frac{1}{q}\right)$ and $R\left(r, \frac{1}{r}\right)$ be three distinct points on the hyperbola $xy = 1$.

- (i) Show that the equation of the line, ℓ , through R , perpendicular to PQ , is $y = pqx - pqr + \frac{1}{r}$. **2**
- (ii) Write down the equation of the line, m , through P , perpendicular to QR . **1**
- (iii) The lines ℓ and m intersect at T . **2**
 Show that T lies on the hyperbola.

Question 4 continues on page 7

Question 4 (continued)

(d)



In the acute-angled triangle ABC , K is the midpoint of AB , L is the midpoint of BC and M is the midpoint of CA . The circle through K , L and M also cuts BC at P as shown in the diagram.

Copy or trace the diagram into your writing booklet.

- (i) Prove that $KMLB$ is a parallelogram. 1
- (ii) Prove that $\angle KPB = \angle KML$. 1
- (iii) Prove that $AP \perp BC$. 2

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

- (a) A solid is formed by rotating the region bounded by the curve $y = x(x-1)^2$ and the line $y=0$ about the y -axis. Use the method of cylindrical shells to find the volume of this solid. **3**

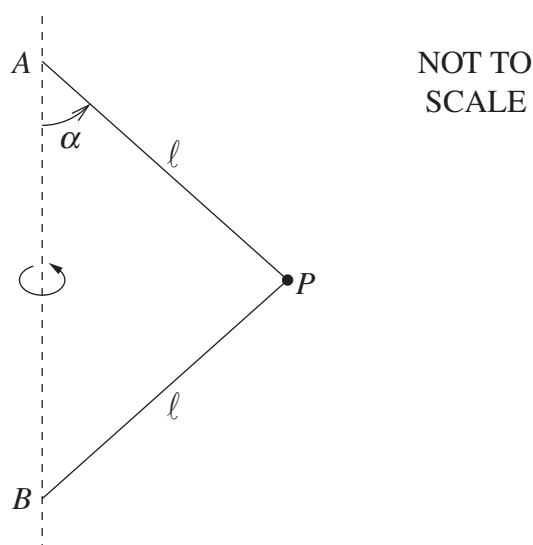
- (b) (i) Show that $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$. **1**

- (ii) Hence, or otherwise, solve the equation **3**

$$\cos \theta + \cos 2\theta + \cos 3\theta + \cos 4\theta = 0$$

for $0 \leq \theta \leq 2\pi$.

- (c) A particle, P , of mass m is attached by two strings, each of length ℓ , to two fixed points, A and B , which lie on a vertical line as shown in the diagram.



The system revolves with constant angular velocity ω about AB . The string AP makes an angle α with the vertical. The tension in the string AP is T_1 and the tension in the string BP is T_2 where $T_1 \geq 0$ and $T_2 \geq 0$. The particle is also subject to a downward force, mg , due to gravity.

- (i) Resolve the forces on P in the horizontal and vertical directions. **2**
- (ii) If $T_2 = 0$, find the value of ω in terms of ℓ , g and α . **1**

Question 5 continues on page 9

Question 5 (continued)

- (d) In a chess match between the Home team and the Away team, a game is played on each of board 1, board 2, board 3 and board 4.

On each board, the probability that the Home team wins is 0.2, the probability of a draw is 0.6 and the probability that the Home team loses is 0.2.

The results are recorded by listing the outcomes of the games for the Home team in board order. For example, if the Home team wins on board 1, draws on board 2, loses on board 3 and draws on board 4, the result is recorded as WDL D.

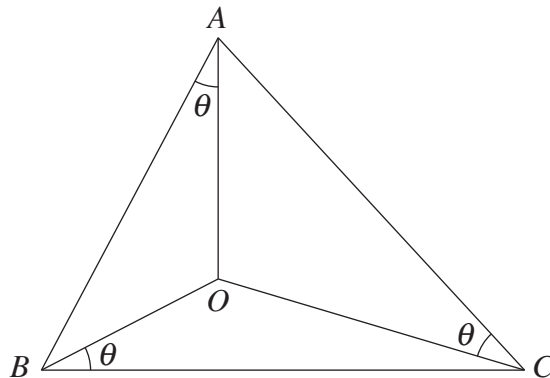
- (i) How many different recordings are possible? **1**
- (ii) Calculate the probability of the result which is recorded as WDL D. **1**
- (iii) Teams score 1 point for each game won, $\frac{1}{2}$ a point for each game drawn and 0 points for each game lost. **3**

What is the probability that the Home team scores more points than the Away team?

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

- (a) In $\triangle ABC$, $\angle CAB = \alpha$, $\angle ABC = \beta$ and $\angle BCA = \gamma$. The point O is chosen inside $\triangle ABC$ so that $\angle OAB = \angle OBC = \angle OCA = \theta$, as shown in the diagram.



- (i) Show that $\frac{OA}{OB} = \frac{\sin(\beta - \theta)}{\sin \theta}$. 1
- (ii) Hence show that $\sin^3 \theta = \sin(\alpha - \theta) \sin(\beta - \theta) \sin(\gamma - \theta)$. 2
- (iii) Prove the identity $\cot x - \cot y = \frac{\sin(y - x)}{\sin x \sin y}$. 1
- (iv) Hence show that 1
 $(\cot \theta - \cot \alpha)(\cot \theta - \cot \beta)(\cot \theta - \cot \gamma) = \operatorname{cosec} \alpha \operatorname{cosec} \beta \operatorname{cosec} \gamma$.
- (v) Hence find the value of θ when $\triangle ABC$ is an isosceles right triangle. 2

Question 6 continues on page 11

Question 6 (continued)

- (b) In an alien universe, the gravitational attraction between two bodies is proportional to x^{-3} , where x is the distance between their centres.

A particle is projected upward from the surface of a planet with velocity u at time $t=0$. Its distance x from the centre of the planet satisfies the equation

$$\ddot{x} = -\frac{k}{x^3}.$$

- (i) Show that $k = gR^3$, where g is the magnitude of the acceleration due to gravity at the surface of the planet and R is the radius of the planet. **1**

- (ii) Show that v , the velocity of the particle, is given by **3**

$$v^2 = \frac{gR^3}{x^2} - (gR - u^2).$$

- (iii) It can be shown that $x = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$. **2**

(Do NOT prove this.)

Show that if $u \geq \sqrt{gR}$ the particle will not return to the planet.

- (iv) If $u < \sqrt{gR}$ the particle reaches a point whose distance from the centre of the planet is D , and then falls back.

- (1) Use the formula in part (ii) to find D in terms of u , R and g . **1**

- (2) Use the formula in part (iii) to find the time taken for the particle to return to the surface of the planet in terms of u , R and g . **1**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The curves $y = \cos x$ and $y = \tan x$ intersect at a point P whose x -coordinate is α .

(i) Show that the curves intersect at right angles at P . 3

(ii) Show that $\sec^2 \alpha = \frac{1 + \sqrt{5}}{2}$. 2

(b) (i) Let $I_n = \int_0^x \sec^n t \, dt$, where $0 \leq x < \frac{\pi}{2}$. Show that 3

$$I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}.$$

(ii) Hence find the exact value of 2

$$\int_0^{\frac{\pi}{3}} \sec^4 t \, dt.$$

Question 7 continues on page 13

Question 7 (continued)

(c) The sequence $\{x_n\}$ is given by

$$x_1 = 1 \quad \text{and} \quad x_{n+1} = \frac{4 + x_n}{1 + x_n} \quad \text{for } n \geq 1.$$

(i) Prove by induction that for $n \geq 1$ **4**

$$x_n = 2 \left(\frac{1 + \alpha^n}{1 - \alpha^n} \right),$$

where $\alpha = -\frac{1}{3}$.

(ii) Hence find the limiting value of x_n as $n \rightarrow \infty$. **1**

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose $0 \leq t \leq \frac{1}{\sqrt{2}}$.

(i) Show that $0 \leq \frac{2t^2}{1-t^2} \leq 4t^2$. **2**

(ii) Hence show that $0 \leq \frac{1}{1+t} + \frac{1}{1-t} - 2 \leq 4t^2$. **1**

(iii) By integrating the expressions in the inequality in part (ii) with respect to t from $t=0$ to $t=x$ (where $0 \leq x \leq \frac{1}{\sqrt{2}}$), show that **2**

$$0 \leq \log_e \left(\frac{1+x}{1-x} \right) - 2x \leq \frac{4x^3}{3}.$$

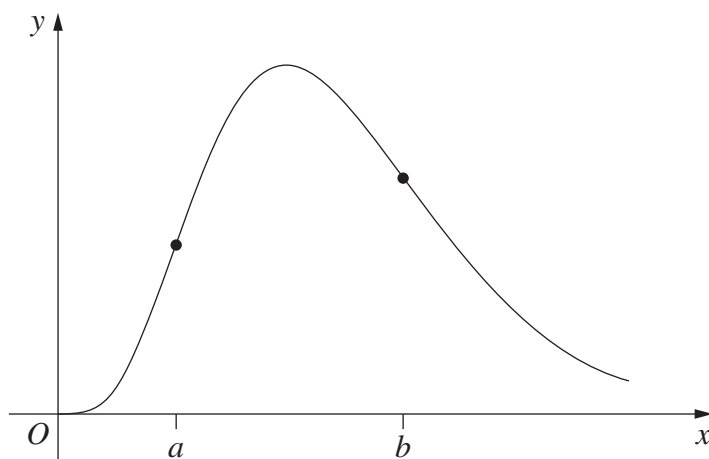
(iv) Hence show that for $0 \leq x \leq \frac{1}{\sqrt{2}}$ **1**

$$1 \leq \left(\frac{1+x}{1-x} \right) e^{-2x} \leq e^{\frac{4x^3}{3}}.$$

Question 8 continues on page 15

Question 8 (continued)

(b) For $x > 0$, let $f(x) = x^n e^{-x}$, where n is an integer and $n \geq 2$.



(i) The two points of inflexion of $f(x)$ occur at $x = a$ and $x = b$, where $0 < a < b$. Find a and b in terms of n . 4

(ii) Show that 2

$$\frac{f(b)}{f(a)} = \left(\frac{1 + \frac{1}{\sqrt{n}}}{1 - \frac{1}{\sqrt{n}}} \right)^n e^{-2\sqrt{n}}.$$

(iii) Using the result of part (a) (iv), show that 2

$$1 \leq \frac{f(b)}{f(a)} \leq e^{\frac{4}{3\sqrt{n}}}.$$

(iv) What can be said about the ratio $\frac{f(b)}{f(a)}$ as $n \rightarrow \infty$? 1

End of paper

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln \left(x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln \left(x + \sqrt{x^2 + a^2} \right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$