BOARD OF STUDIES
NEW SOUTH WALES

2001

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Use the table of standard integrals to find the exact value of **2**

$$\int_0^2 \frac{dx}{\sqrt{16 - x^2}} .$$

(b) Find $\frac{d}{dx}\left(x \sin^2 x\right)$. **2**

(c) Evaluate $\sum_{n=4}^{7}(2n+3)$. **1**

(d) Let A be the point $(-2, 7)$ and let B be the point $(1, 5)$. Find the coordinates of the point P which divides the interval AB externally in the ratio $1 : 2$. **2**

(e) Is $x + 3$ a factor of $x^3 - 5x + 12$? Give reasons for your answer. **2**

(f) Use the substitution $u = 1 + x$ to evaluate **3**

$$15 \int_{-1}^{0} x\sqrt{1+x}\, dx .$$

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = 3x^2 + x$. Use the definition **2**

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

to find the derivative of $f(x)$ at the point $x = a$.

(b) Find

(i) $\displaystyle\int \frac{e^x}{1+e^x}\,dx$ **1**

(ii) $\displaystyle\int_0^{\pi} \cos^2 3x\,dx$. **3**

(c) The letters A, E, I, O, and U are vowels.

(i) How many arrangements of the letters in the word ALGEBRAIC are possible? **1**

(ii) How many arrangements of the letters in the word ALGEBRAIC are possible if the vowels must occupy the 2nd, 3rd, 5th and 8th positions? **2**

(d) Find the term independent of x in the binomial expansion of **3**

$$\left(x^2 - \frac{1}{x}\right)^9 .$$

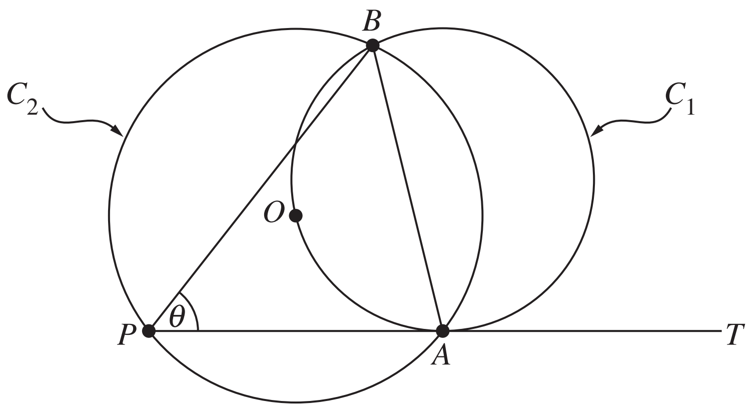
Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) The function $f(x) = \sin x + \cos x - x$ has a zero near $x = 1.2$ **3**

Use one application of Newton's method to find a second approximation to the zero. Write your answer correct to three significant figures.

(b)



Two circles, C_1 and C_2, intersect at points A and B. Circle C_1 passes through the centre O of circle C_2. The point P lies on circle C_2 so that the line PAT is tangent to circle C_1 at point A. Let $\angle APB = \theta$.

Copy or trace the diagram into your writing booklet.

(i) Find $\angle AOB$ in terms of θ. Give a reason for your answer. **1**

(ii) Explain why $\angle TAB = 2\theta$. **1**

(iii) Deduce that $PA = BA$. **2**

(c) (i) Starting from the identity $\sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \cos\theta\sin 2\theta$, and using the double angle formulae, prove the identity **2**

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

(ii) Hence solve the equation **3**

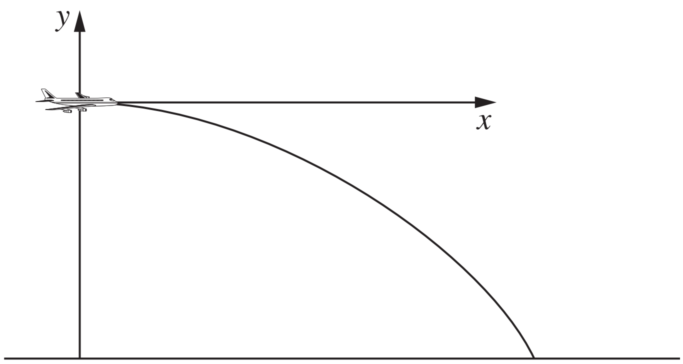
$$\sin 3\theta = 2\sin\theta \quad \text{for} \quad 0 \le \theta \le 2\pi.$$

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\dfrac{3x}{x-2} \leq 1$. **3**

(b) An aircraft flying horizontally at $V\,\mathrm{m\,s^{-1}}$ releases a bomb that hits the ground 4000 m away, measured horizontally. The bomb hits the ground at an angle of 45° to the vertical. **4**



Assume that, t seconds after release, the position of the bomb is given by

$$x = Vt, \quad y = -5t^2.$$

Find the speed V of the aircraft.

(c) A particle, whose displacement is x, moves in simple harmonic motion. **5**

Find x as a function of t if

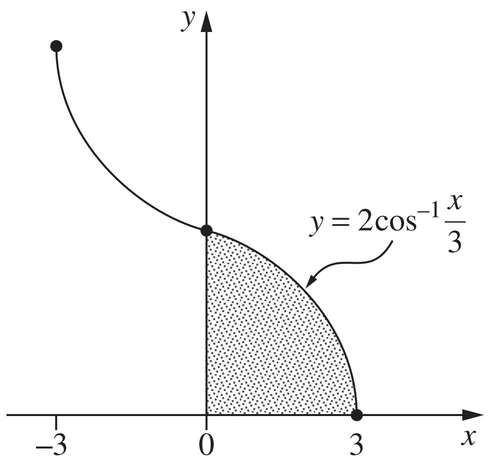
$$\ddot{x} = -4x$$

and if $x = 3$ and $\dot{x} = -6\sqrt{3}$ when $t = 0$.

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)



The sketch shows the graph of the curve $y = f(x)$ where $f(x) = 2\cos^{-1}\frac{x}{3}$. The area under the curve for $0 \le x \le 3$ is shaded.

(i) Find the y intercept. **1**

(ii) Determine the inverse function $y = f^{-1}(x)$, and write down the domain D of this inverse function. **2**

(iii) Calculate the area of the shaded region. **2**

(b) By using the binomial expansion, show that **3**

$$(q+p)^n - (q-p)^n = 2\binom{n}{1}q^{n-1}p + 2\binom{n}{3}q^{n-3}p^3 + \cdots$$

What is the last term in the expansion when n is odd? What is the last term in the expansion when n is even?

(c) A fair six-sided die is randomly tossed n times.

(i) Suppose $0 \le r \le n$. What is the probability that exactly r 'sixes' appear in the uppermost position? **2**

(ii) By using the result of part (b), or otherwise, show that the probability that an odd number of 'sixes' appears is **2**

$$\frac{1}{2}\left\{1 - \left(\frac{2}{3}\right)^n\right\}.$$

Marks

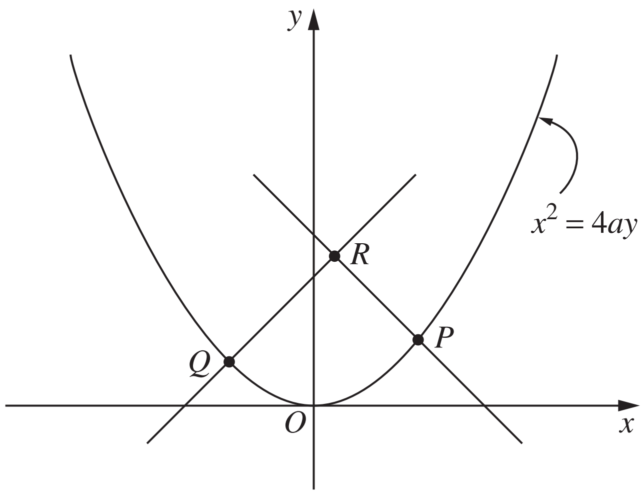
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Prove by induction that **3**

$$n^3 + (n+1)^3 + (n+2)^3$$

is divisible by 9 for $n = 1, 2, 3, \ldots$

(b)



Consider the variable point $P(2at, at^2)$ on the parabola $x^2 = 4ay$.

(i) Prove that the equation of the normal at P is $x + ty = at^3 + 2at$. **2**

(ii) Find the coordinates of the point Q on the parabola such that the normal at Q is perpendicular to the normal at P. **1**

(iii) Show that the two normals of part (ii) intersect at the point R, whose coordinates are **4**

$$x = a\left(t - \frac{1}{t}\right), \quad y = a\left(t^2 + 1 + \frac{1}{t^2}\right).$$

(iv) Find the equation in Cartesian form of the locus of the point R given in part (iii). **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) A particle moves in a straight line so that its acceleration is given by

$$\frac{dv}{dt} = x - 1$$

where v is its velocity and x is its displacement from the origin.

Initially, the particle is at the origin and has velocity $v = 1$.

(i) Show that $v^2 = (x-1)^2$. **2**

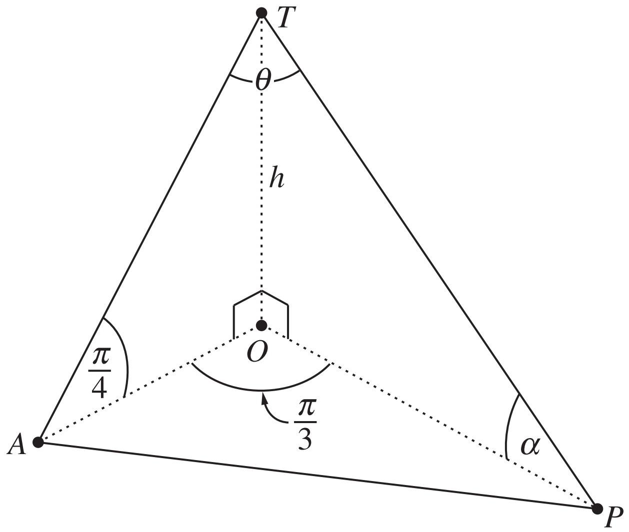
(ii) By finding an expression for $\frac{dt}{dx}$, or otherwise, find x as a function of t. **2**

Question 7 continues on page 9

Marks

Question 7 (continued)

(b)



Consider the diagram, which shows a vertical tower OT of height h metres, a fixed point A, and a variable point P that is constrained to move so that angle AOP is $\frac{\pi}{3}$ radians. The angle of elevation of T from A is $\frac{\pi}{4}$ radians.

Let the angle of elevation of T from P be α radians and let angle ATP be θ radians.

(i) By considering triangle AOP, show that **1**

$$AP^2 = h^2 + h^2\cot^2\alpha - h^2\cot\alpha.$$

(ii) By finding a second expression for AP^2, deduce that **3**

$$\cos\theta = \frac{1}{\sqrt{2}}\sin\alpha + \frac{1}{2\sqrt{2}}\cos\alpha.$$

(iii) Sketch a graph of θ for $0 < \alpha < \frac{\pi}{2}$, identifying and classifying any turning points. Discuss the behaviour of θ as $\alpha \to 0$ and as $\alpha \to \frac{\pi}{2}$. **4**

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$