BOARD OF STUDIES

NEW SOUTH WALES

**HIGHER SCHOOL CERTIFICATE EXAMINATION**

# 1996

# MATHEMATICS

## 4 UNIT (ADDITIONAL)

*Time allowed—Three hours*
*(Plus 5 minutes' reading time)*

**DIRECTIONS TO CANDIDATES**

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

**QUESTION 1.** Use a *separate* Writing Booklet.

**Marks**

(a) Evaluate $\int_1^3 \frac{4}{(2+x)^2}\,dx$. **2**

(b) Find $\int \sec^2\theta \tan\theta \, d\theta$. **2**

(c) Find $\int \frac{5t^2+3}{t\left(t^2+1\right)}\,dt$. **3**

(d) Using integration by parts, or otherwise, find $\int x \tan^{-1} x \, dx$. **3**

(e) Using the substitution $x = 2\sin\theta$, or otherwise, calculate **5**

$$\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx.$$

**QUESTION 2.** Use a *separate* Writing Booklet.

**Marks**

(a) Suppose that $c$ is a real number, and that $z = c - i$. **2**

Express the following in the form $x + iy$, where $x$ and $y$ are real numbers:

(i) $\overline{(iz)}$;

(ii) $\dfrac{1}{z}$.

(b) On an Argand diagram, shade the region specified by both the conditions **2**

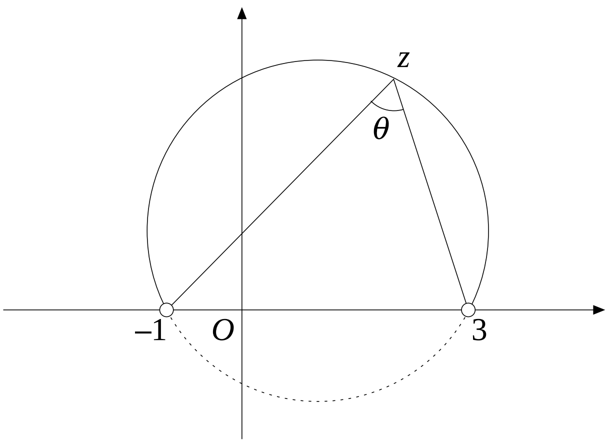
$$\operatorname{Re}(z) \le 4 \quad \text{and} \quad |z - 4 + 5i| \le 3.$$

(c) (i) Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$ for all integers $n \ge 1$. **7**

(ii) Express $w = \sqrt{3} - i$ in modulus–argument form.

(iii) Hence express $w^5$ in the form $x + iy$, where $x$ and $y$ are real numbers.

(d) **4**




The diagram shows the locus of points $z$ in the complex plane such that

$$\arg(z-3) - \arg(z+1) = \frac{\pi}{3}.$$

This locus is part of a circle. The angle between the lines from –1 to $z$ and from 3 to $z$ is $\theta$, as shown.

Copy this diagram into your Writing Booklet.

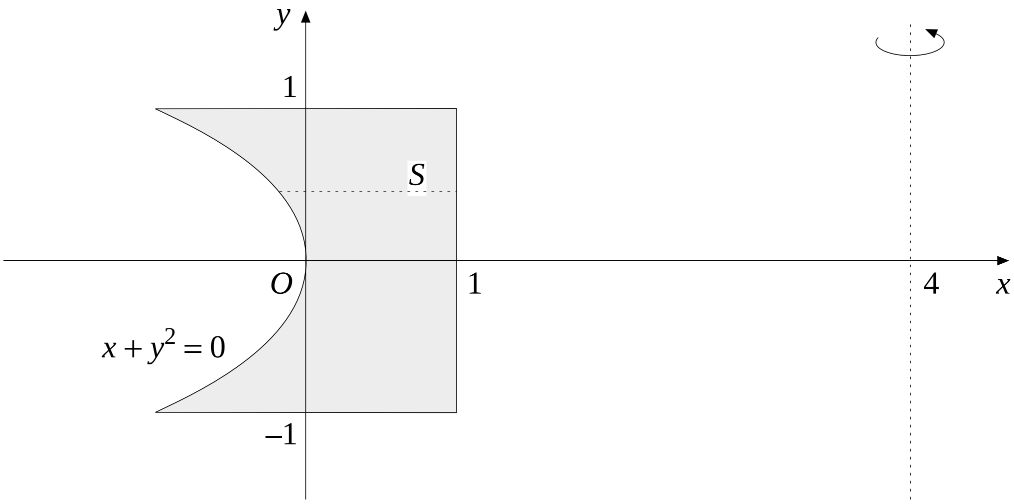
(i) Explain why $\theta = \dfrac{\pi}{3}$.

(ii) Find the centre of the circle.

**QUESTION 3.** Use a *separate* Writing Booklet.

**Marks**

(a) **5**




The shaded region is bounded by the lines $x = 1$, $y = 1$, and $y = -1$ and by the curve $x + y^2 = 0$. The region is rotated through 360° about the line $x = 4$ to form a solid. When the region is rotated, the line segment $S$ at height $y$ sweeps out an annulus.

(i) Show that the area of the annulus at height $y$ is equal to

$$\pi\left(y^4 + 8y^2 + 7\right).$$

(ii) Hence find the volume of the solid.

(b) (i) Show that $\displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx = \frac{1}{2k+1}$, where $k$ is a positive integer. **6**

(ii) By writing $(\cos x)^{2n} = \left(1 - \sin^2 x\right)^n$, show that

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2n+1} dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k}.$$

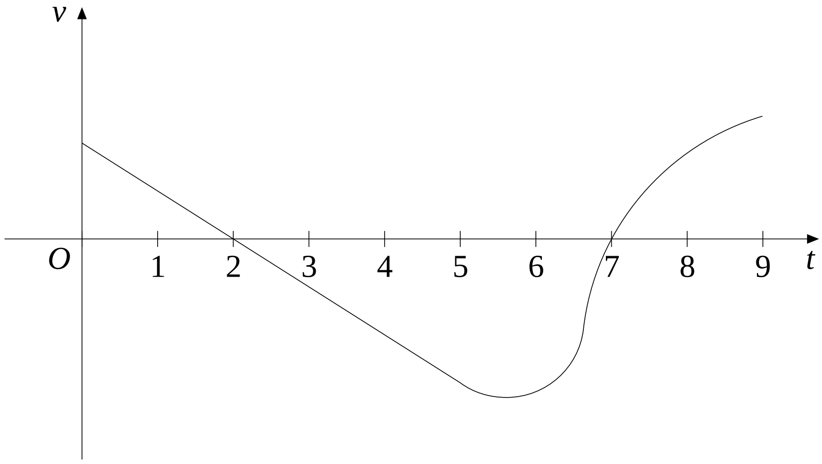
(iii) Hence, or otherwise, evaluate

$$\int_0^{\frac{\pi}{2}} \cos^5 x \, dx.$$

QUESTION 3. (Continued)

**Marks**

(c) **4**




A particle moves along the $x$ axis. At time $t = 0$, the particle is at $x = 0$. Its velocity $v$ at time $t$ is shown on the graph.

Trace or copy this graph into your Writing Booklet.

(i) At what time is the acceleration greatest? Explain your answer.

(ii) At what time does the particle first return to $x = 0$? Explain your answer.

(iii) Sketch the displacement graph for the particle from $t = 0$ to $t = 9$.

**QUESTION 4.** Use a *separate* Writing Booklet.

**Marks**

(a) By differentiating both sides of the formula **3**

$$1 + x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1},$$

find an expression for

$$1 + 2 \times 2 + 3 \times 4 + 4 \times 8 + \cdots + n2^{n-1}.$$

(b) (i) On the same set of axes, sketch and label clearly the graphs of the functions **6**

$$y = x^{\frac{1}{3}} \text{ and } y = e^x.$$

(ii) Hence, on a different set of axes, without using calculus, sketch and label clearly the graph of the function

$$y = x^{\frac{1}{3}} e^x.$$

(iii) Use your sketch to determine for which values of $m$ the equation $x^{\frac{1}{3}} e^x = mx + 1$ has exactly one solution.

(c) Consider a lotto-style game with a barrel containing twenty similar balls numbered 1 to 20. In each game, four balls are drawn, without replacement, from the twenty balls in the barrel. **6**

The probability that any particular number is drawn in any game is 0·2.

(i) Find the probability that the number 20 is drawn in *exactly* two of the next five games played.

(ii) Find the probability that the number 20 is drawn in *at least* two of the next five games played.

Let $j$ be an integer, with $4 \le j \le 20$.

(iii) Write down the probability that, in any one game, all four selected numbers are less than or equal to $j$.

(iv) Show that the probability that, in any one game, $j$ is the *largest* of the four numbers drawn is

$$\frac{\binom{j-1}{3}}{\binom{20}{4}}.$$

**QUESTION 5.** Use a *separate* Writing Booklet.

**Marks**

(a) For any non-zero real number $t$, the point $\left(t, \frac{1}{t}\right)$ lies on the graph of $y = \frac{1}{x}$. **8**

(i) Show that $3xy = 4$ is the equation of the locus of the midpoint of the straight line joining $\left(t, \frac{1}{t}\right)$ and $\left(3t, \frac{1}{3t}\right)$, as $t$ varies.

(ii) Show that the line joining $\left(t, \frac{1}{t}\right)$ and $\left(3t, \frac{1}{3t}\right)$ is tangent to the locus in part (i).

(iii) Show that the equation of the normal to $y = \frac{1}{x}$ at the point $\left(t, \frac{1}{t}\right)$ may be written in the form

$$t^4 - t^3 x + ty - 1 = 0.$$

(iv) $R(0, h)$ is a point on the $y$ axis. Show that there are exactly two points on the hyperbola $y = \frac{1}{x}$ with normals that pass through $R$.

(b) Consider the polynomial equation **7**

$$x^4 + ax^3 + bx^2 + cx + d = 0,$$

where $a$, $b$, $c$, and $d$ are all integers. Suppose the equation has a root of the form $ki$, where $k$ is real, and $k \neq 0$.

(i) State why the conjugate $-ki$ is also a root.

(ii) Show that $c = k^2 a$.

(iii) Show that $c^2 + a^2 d = abc$.

(iv) If 2 is also a root of the equation, and $b = 0$, show that $c$ is even.

**QUESTION 6.** Use a *separate* Writing Booklet.

**Marks**

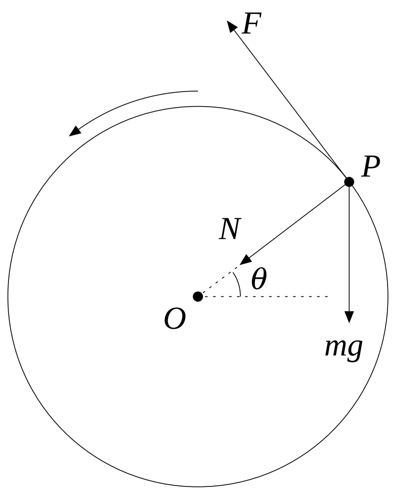
(a) Solve $3x^2 - 2x - 2 \le |3x|$. **3**

(b) A circular drum is rotating with uniform angular velocity round a horizontal axis. A particle $P$ is rotating in a vertical circle, without slipping, on the inside of the drum. **3**

The radius of the drum is $r$ metres and its angular velocity is $\omega$ radians/second. Acceleration due to gravity is $g$ metres/second$^2$, and the mass of $P$ is $m$ kilograms.

The centre of the drum is $O$, and $OP$ makes an angle $\theta$ to the horizontal.

The drum exerts a normal force $N$ on $P$, as well as a frictional force $F$, acting tangentially to the drum, as shown in the diagram.

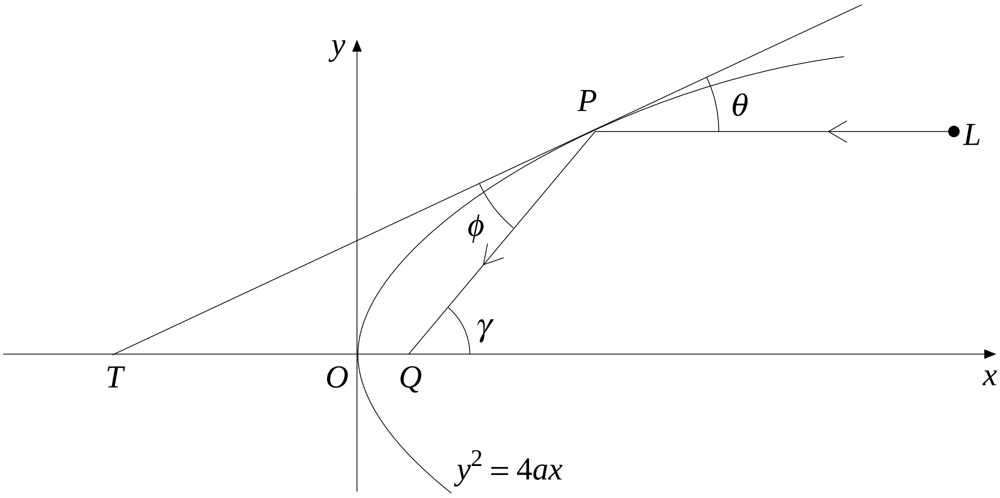



By resolving forces perpendicular to and parallel to $OP$, find an expression for $\dfrac{F}{N}$ in terms of the data.

QUESTION 6. (Continued)

**Marks**

(c) **9**




A mirror is described by the parabola $y^2 = 4ax$, where $a$ is a constant. Light travels parallel to the $x$ axis from a point, $L$, towards the mirror. The light travels along the line $y = 2at$, and meets the mirror at $P\left(at^2, 2at\right)$. The reflected light meets the $x$ axis at $Q$. The tangent to the parabola at $P$ meets the $x$ axis at $T$.

The light is reflected at $P$ by the mirror in such a way that $\theta$, the angle of incidence between the light ray and the tangent, is equal to $\phi$, the angle of reflection between the tangent and the reflected ray, i.e., $\theta = \phi$. The reflected ray makes an angle $\gamma$ with the $x$ axis.

(i) Show that $\gamma = 2\theta$.

(ii) By considering the gradient of $PT$, show that $\tan\theta = \dfrac{1}{t}$.

(iii) Hence show that the equation of the line $PQ$ is

$$2tx = \left(t^2 - 1\right)y + 2at.$$

(iv) Show that $Q$ is the point $(a, 0)$, the focus of the parabola.

(v) Use the focus–directrix definition of the parabola to show that the path $LPQ$ is the shortest path from $L$ to $Q$ via the parabola.

**QUESTION 7.** Use a *separate* Writing Booklet.

**Marks**

(a) A particle is moving along the $x$ axis. Its acceleration is given by **6**

$$\frac{d^2x}{dt^2} = \frac{5-2x}{x^3}$$

and the particle starts from rest at the point $x = 1$.

(i) Show that the particle starts moving in the positive $x$ direction.

(ii) Let $v$ be the velocity of the particle. Show that

$$v = \frac{\sqrt{x^2+4x-5}}{x} \quad \text{for } x \geq 1.$$

(iii) Describe the behaviour of the velocity of the particle after the particle passes $x = \frac{5}{2}$.

(b) (i) Let $f(x) = \ln x - ax + b$, for $x > 0$, where $a$ and $b$ are real numbers and $a > 0$. Show that $y = f(x)$ has a single turning point which is a maximum. **9**

(ii) The graphs of $y = \ln x$ and $y = ax - b$ intersect at points $A$ and $B$. Using the result of part (i), or otherwise, show that the chord $AB$ lies below the curve $y = \ln x$.

(iii) Using integration by parts, or otherwise, show that

$$\int_1^k \ln x \, dx = k \ln k - k + 1.$$

(iv) Use the trapezoidal rule on the intervals with integer endpoints 1, 2, 3, ... , $k$ to show that

$$\int_1^k \ln x \, dx$$

is approximately equal to $\frac{1}{2}\ln k + \ln[(k-1)!]$.

(v) Hence deduce that

$$k! < e\sqrt{k}\left(\frac{k}{e}\right)^k.$$

**QUESTION 8.** Use a *separate* Writing Booklet.

**Marks**

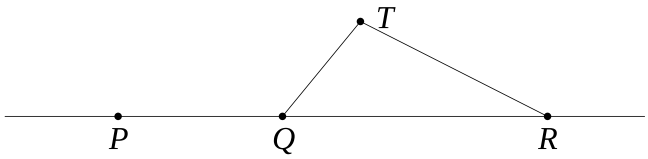
(a) Let $w = \cos\dfrac{2\pi}{9} + i\sin\dfrac{2\pi}{9}$. **7**

(i) Show that $w^k$ is a solution of $z^9 - 1 = 0$, where $k$ is an integer.

(ii) Prove that $w + w^2 + w^3 + w^4 + w^5 + w^6 + w^7 + w^8 = -1$.

(iii) Hence show that $\cos\left(\dfrac{\pi}{9}\right)\cos\left(\dfrac{2\pi}{9}\right)\cos\left(\dfrac{4\pi}{9}\right) = \dfrac{1}{8}$.

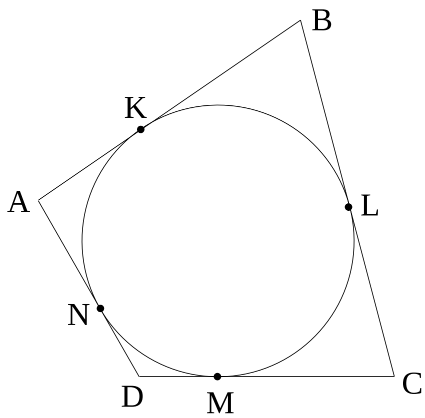
(b) **1**




The points $P$, $Q$, $R$ lie on a straight line, in that order, and $T$ is any point not on the line. Using the fact that $PR - PQ = QR$, show that

$$QT - QP > RT - RP.$$

(c) (i) **7**




*ABCD* is a quadrilateral, and the sides of *ABCD* are tangent to a circle at points *K*, *L*, *M*, and *N*, as in the diagram. Show that

$$AB + CD = AD + BC.$$

(ii) *ABCD* is a quadrilateral, with all angles less than 180°. Let *X* be the point of intersection of the angle bisectors of $\angle ABC$ and of $\angle BCD$. Prove that *X* is the centre of a circle to which *AB*, *BC*, and *CD* are tangent.

(iii) *ABCD* is a quadrilateral, with all angles less than 180°. Given that

$$AB + CD = AD + BC,$$

show that there exists a circle to which all sides of *ABCD* are tangent.

You may use the result of part (b).

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$